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A GENERIC RESULT ON WEYL TENSOR

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Abstract. Let M be a connected compact C^∞ manifold of dimension $n \geq 4$ without boundary. Let $\mathcal{M}^k$ be the set of all C^k Riemannian metrics on M. Any $g \in \mathcal{M}^k$ determines the Weyl tensor

$$\mathcal{W}^g\colon M \to \mathbb{R}^{4n}, \qquad \mathcal{W}^g(\xi) := (W^g_{ijkl}(\xi))_{i,j,k,l=1,\dots,n}.$$

We prove that the set

$$\mathcal{A} := \big\{g \in \mathcal{M}^k : |\mathcal{W}^g(\xi)| + |D\mathcal{W}^g(\xi)| + |D^2\mathcal{W}^g(\xi)| > 0 \text{ for any } \xi \in M\big\}$$

is an open dense subset of $\mathcal{M}^k$.

1. Introduction

Let M be a connected compact C^∞ manifold of dimension $n \geq 4$ without boundary. Let $\mathcal{M}^k$ be the set of all C^k Riemannian metrics on M. Any $g \in \mathcal{M}^k$ determines the Weyl tensor

$$\mathcal{W}^g\colon M \to \mathbb{R}^{4n}, \qquad \mathcal{W}^g(\xi) := (W^g_{ijkl}(\xi))_{i,j,k,l=1,\dots,n}.$$

Our goal is to prove that, for a generic Riemannian metric g, it holds true that if Weyl tensor and its first derivative vanish at a point $\xi \in M$ then the second derivative at ξ is not zero. More precisely, we prove that

Theorem 1.1. *The set*

$$\mathcal{A} := \Big\{g \in \mathcal{M}^k : \min_{\xi \in M}(|\mathcal{W}^g(\xi)| + |D\mathcal{W}^g(\xi)| + |D^2\mathcal{W}^g(\xi)|) > 0\Big\}$$

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is an open dense subset of $\mathcal{M}^k$.

Our result is motivated by the study of the compactness of the set of solutions of the Yamabe equation. Yamabe asked the question if there exists a metric $\widetilde{g}$ conformal to g with constant scalar curvature. If $\widetilde{g} = u^{\frac{4}{n-2}} g$, the problem is equivalent to finding a positive solution u to the equation

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \kappa u^{(n+2)/(n-2)} \quad \text{in } M, \tag{1.1}$$

for some constant κ. R_g is the scalar curvature of g and $4(n-1)/(n-2)\kappa$ is nothing but the scalar curvature of $\widetilde{g}$. Yamabe problem has been completely solved in the works of Yamabe [19], Aubin [1], Schoen [12] and Trudinger [17].

In particular, the solution is unique in the case of negative scalar curvature and (up to a constant factor) in the case of zero scalar curvature. The uniqueness fails in general in the case of positive scalar curvature. Indeed, Schoen in [13], [15] and Pollack in [9] proved the existence of a large number of high energy solutions of (1.1) with high Morse index for some suitable manifolds. Therefore, the structure of the set of solutions to (1.1) becomes an interesting and intriguing problem. Schoen in [14], [15] asks the question about the compactness of the full set of positive solutions to (1.1).

Compactness of solutions is equivalent for finding an upper bound for the $C^{2,\alpha}$-norm of solutions to (1.1). The compactness does not hold in the case of the round sphere $\mathbb{S}^n$ as Obata shows in [8]. Brendle in [2] and Brendle and Marques in [3] build examples of manifolds with dimension $n \geq 25$ for which compactness is not true.

On the other hand, the compactness issue is proved by Khuri, Marques and Schoen [4] for manifolds of dimension $n \leq 24$ which satisfy the Positive Mass Theorem. For a long time, the Positive Mass Theorem have been established for spaces of dimension $n \leq 7$ (Schoen and Yau [16]) and for spin manifolds (Witten [20]). Very recently, Lohkamp in [5] seems to have proved that it holds in general manifolds.

The study of compactness is strictly related to the blow-up analysis of solutions to (1.1). In particular, Schoen conjectured that the possible blow-up points must be points where Weyl's tensor and its derivatives up to order $[(n-6)/2]$ vanishes. We refer to the survey [6] by Marques for a complete list of contributions to these problems. In particular, Khuri, Marques and Schoen proved that compactness does hold, without assuming the Positive Mass Theorem, provided $6 \leq n \leq 24$ and

$$\min_{\xi \in M} \sum_{k=0}^{[(n-6)/2]} \left| D^k \mathcal{W}^g(\xi) \right|^2 > 0.$$

Combining this result with Theorem 1.1 we get

Corollary 1.2. *Let $10 \leq n \leq 24$. The set*

$$\mathcal{C} := \{g \in \mathcal{M}^k : \textit{Yamabe problem (1.1) is compact}\}$$

is an open dense subset of $\mathcal{M}^k$.

The proof of Theorem 1.1 relies on the transversality argument described in Section 2. The key transversality condition (namely (b) in Theorem 2.1) is proved in Section 3.

2. Formulation of the problem and proof of the main result

We denote by $\mathcal{S}^k$ the space of all C^k symmetric covariant 2−tensors on M. $\mathcal{S}^k$ is a Banach space equipped with the norm $\|\cdot\|_k$ defined in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha\in L}$ of M such that the closure of V_α is contained U_α, where $\{U_\alpha, \psi_\alpha\}$ is the open coordinate neighbourhood. If $h \in \mathcal{S}^k$ we denote by h_{ij} the components of h with respect to the coordinates $(x_1, \ldots, x_n)$ on V_α. We define

$$\|h\|_k := \sum_{\alpha\in L} \sum_{|\beta|\leq k} \sum_{i,j=1}^{n} \sup_{\psi_\alpha(V_\alpha)} \frac{\partial^\beta h_{ij}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}.$$

The set $\mathcal{M}^k$ of all C^k Riemannian metrics on M is an open set of $\mathcal{S}^k$.

In the following we will assume $k \geq 4$.

Given $\widehat{g} \in \mathcal{M}^k$, it is possible to define an atlas on M whose charts are $(B_{\widehat{g}}(\xi, R), \varphi^{-1})$ where $\varphi\colon B(0,R) \to B_{\widehat{g}}(\xi, R)$. Here $B_{\widehat{g}}(\xi, R) \subset M$ is the ball centered at ξ with radius R given by the metric $\widehat{g}$ and $B(0,R) \subset \mathbb{R}^n$ is the ball centered at 0 with radius R in the euclidean space $\mathbb{R}^n$. Let $\mathcal{B}_\rho := \{h \in \mathcal{S}^k : \|h\|_k < \rho\}$ the ball centered at 0 with radius ρ in $\mathcal{S}^k$.

For any $\xi \in M$ and $h \in \mathcal{B}_\rho$, with ρ small enough so that $\widehat{g} + h \in \mathcal{M}^k$, we consider Weyl's curvature tensor $\mathcal{W}^{\widehat{g}+h}(\xi)$ of $(M, \widehat{g} + h)$ at the point $\xi \in M$ whose components are $W^{\widehat{g}+h}_{abcd}(\xi)$.

Here and in the following we use the Einstein summation convention, i.e. when an index variable appears twice in a single term, once in an upper (superscript) and/or in a lower (subscript) position, it implies that we are summing over all of its possible values.

Given $\xi_0 \in M$ and the chart $\left(B_{\widehat{g}}(\xi_0, R), \varphi^{-1}\right)$ we set

$$\widetilde{\mathcal{W}}^{\widehat{g}+h}(x) := \mathcal{W}^{\widehat{g}+h}(\varphi(x)) \quad \text{if } x \in B(0,R) \text{ and } h \in \mathcal{B}_\rho.$$

Now, for any choice of indices i, j, k, l with $i \neq j$ and $k \neq l$, we introduce the C^1−map $F\colon \mathcal{B}_\rho \times B(0,R) \subset \mathcal{S}^k \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(h,x) = F_{ijkl}(h,x) := \nabla_x \widetilde{W}^{\widehat{g}+h}_{ijkl}(x). \tag{2.1}$$

We observe that $W^{\widehat{g}+h}_{iikl} \equiv W^{\widehat{g}+h}_{ijkk} \equiv 0$ in M. We shall apply to the map F an abstract transversality theorem (see [10], [11], [18]). We recall it (see Theorem 1.1 in [11]) in the following.

Theorem 2.1. *Let X, Y, Z be three Banach spaces and $U \subset X$, $V \subset Y$ open subsets. Let $F\colon U \times V \to Z$ be a $C^\alpha-$map with $\alpha \geq 1$. Assume that*

(a) *for any $y \in V$, $F(\,\cdot\,, y)\colon U \to Z$ is a Fredholm map of index l with $l \leq \alpha$;*
(b) *0 is a regular value of F, i.e. the operator $F'(x_0, y_0)\colon X \times Y \to Z$ is onto at any point (x_0, y_0) such that $F(x_0, y_0) = 0$;*
(c) *the map $\pi \circ i\colon F^{-1}(0) \to Y$ is σ-proper, i.e. $F^{-1}(0) = \bigcup_{s=1}^{+\infty} C_s$, where C_s is a closed set and the restriction $\pi \circ i_{|C_s}$ is proper for any s. Here $i\colon F^{-1}(0) \to X \times Y$ is the embedding and $\pi\colon X \times Y \to Y$ is the projection.*

Then the set $\Theta := \{y \in V : 0$ is a regular value of $F(\,\cdot\,, y)\}$ is a residual subset of V, i.e. $V \setminus \Theta$ is a countable union of close subsets without interior points. In particular, Θ is a dense subset of V.

By Theorem 2.1 we obtain the following result, which is crucial to deduce Theorem 1.1. Let $\mathcal{D} := \{g \in \mathcal{M}^k : \mathcal{W}^g \not\equiv 0 \text{ on } M\}$.

Theorem 2.2. *For any $\widehat{g} \in D$ there exist indices i, j, k, l such that W_{ijkl} does not vanish identically on M. Then the set*

$$\mathcal{D}_{ijkl} := \big\{h \in \mathcal{B}_\rho : \text{all the critical points } \xi \text{ of } W_{ijkl}^{\widehat{g}+h} \text{ are nondegenerate}\big\}$$

is a residual (hence dense) subset of the ball $\mathcal{B}_\rho$ in $\mathcal{S}^k$.

Proof. We are going to apply Theorem 2.1 to the map F defined in (2.1). In this case we have $X = \mathbb{R}^n$, $Z = \mathbb{R}^{4n^2}$ and $Y = \mathcal{S}^k$. We choose $z_0 = 0$. Since X is a finite dimensional space, it is easy to check that for any $h \in \mathcal{B}_\rho$ the map $x \to F(h, x)$ is Fredholm of index 0 and so assumption(a) holds.

Assumption (b) is verified in Lemma 3.1.

In order to prove (c) we set

$$F^{-1}(0) = \bigcup_{s=1}^{+\infty} C_s \quad \text{and} \quad C_s = \big(\overline{B(0, R - 1/s)} \times \overline{\mathcal{B}_{\rho-1/s}}\big) \cap F^{-1}(0).$$

The map $\pi \circ i\colon C_s \to \mathcal{S}^k$ is proper because the set $\overline{\mathcal{B}_{\rho-1/s}} \subset \mathcal{S}^k$ is closed and the set $\overline{B(0, R - 1/s)} \subset \mathbb{R}^n$ is compact.

Finally, we are in position to apply Theorem 2.1 and we get that the set

$$\begin{aligned}
\mathcal{D}_{ijkl}(\xi_0) :=& \big\{h \in \mathcal{B}_\rho : F'_x(h, x)\colon \mathbb{R}^n \to \mathbb{R}^n \\
&\quad \text{is invertible at any point } (h, x) \text{ such that } F(h, x) = 0\big\} \\
=& \big\{h \in \mathcal{B}_\rho : \text{all the critical points of } W_{ijkl}^{\widehat{g}+h} \text{ in } B_{\widehat{g}}(\xi_0, R) \\
&\quad \text{are non degenerate}\big\}
\end{aligned} \tag{2.2}$$

is a residual subset of $\mathcal{B}_\rho$.

Now, since M is compact, there exists a finite covering $\big\{B_{\widehat{g}}(\xi_t,R)\big\}_{t=1,\dots,\nu}$ of M, where $\xi_1,\dots,\xi_\nu \in M$. For any index t there exists a residual subset $\mathcal{D}_{ijkl}(\xi_t)$ (see (2.2)). Let

$$\mathcal{D}_{ijkl} := \bigcap_{t=1,\dots,\nu} \mathcal{D}_{ijkl}(\xi_t).$$

It is immediate that $\mathcal{D}_{ijkl}$ is a dense subset of $\mathcal{B}_\rho$ such that the critical points of $W^{\widehat{g}+h}_{ijkl}$ in M are non degenerate for any $h \in \mathcal{D}_{ijkl}$. □

Proof of Theorem 1.1. It is clear that $\mathcal{A}$ is an open set. The density follows by Theorem 2.2. If $\widehat{g} \in \mathcal{D}$ there exist indices i,j,k,l with $i \neq j$ and $l \neq k$ sucht that $W^{\widehat{g}}_{ijkl}$ is not identically equal to zero on M. By Theorem 2.2, for any $h \in \mathcal{D}_{ijkl}$, we have

$$\big|\nabla W^{\widehat{g}+h}_{ijkl}(\xi)\big| + \big|\nabla^2 W^{\widehat{g}+h}_{ijkl}(\xi)\big| > 0 \quad \text{for any } \xi \in M.$$

Moreover, by Lemma 2.3, the set $\mathcal{M}^k \setminus D$ is a closed subset without interior points. □

Lemma 2.3. *The set* $\mathcal{M}^k \setminus D = \big\{g \in \mathcal{M}^k : |\mathcal{W}^g(\xi)| \neq 0$ *for any* $\xi \in M\big\}$ *is a closed subset without interior points.*

Proof. If $W^g_{ijkl}(\xi) = 0$ with $i \neq j$ and $k \neq l$ then $D_h W^g_{ijkl}(\xi)[h] \neq 0$ if we choose $h \in \mathcal{S}^k$ such that the map $z \to h_{ab}\big(\exp_\xi(z)\big)$, with its first derivative, is vanishing at the point 0, for any indices a and b' i.e. $h_{ab}(0) = 0$ and $\partial_c h_{ab}(0) = 0$ for any a, b, c. Indeed, by (3.11), together with (3.1), (3.7) and the derivative of Christoffel symbols, we get

$$\begin{aligned} D_h W^g_{ijkl}(\xi)[h] &= D_h R_{ijkl}(\xi)[h] = D_h R^s_{ikl}(\xi)[h] g_{js} + R^s_{ikl}(\xi) h_{sj} \\ &= D_h \partial_k \Gamma^j_{li}(\xi)[h] - D_h \partial_l \Gamma^j_{ki}(\xi)[h] \\ &= \frac{1}{2}\, \partial_k G_{lij}(h,\xi) - \frac{1}{2}\, \partial_l G_{kij}(h,\xi) \\ &= \frac{1}{2}\big(\partial^2_{ki} h_{lj} - \partial^2_{kj} h_{li}\big) - \frac{1}{2}\big(\partial^2_{li} h_{kj} - \partial^2_{lj} h_{ki}\big) \end{aligned}$$

and, if we choose $h_{ab} \equiv 0$ if $(a,b) \neq (l,j)$ and $h_{lj}(x) = x_k x_i$, we get

$$D_h W^g_{ijkl}(\xi)[h] = \frac{1}{2}\, \partial^2_{ki} h_{lj} \neq 0.$$ □

3. The transversality condition

3.1. Notation. Let us recall the definition of the Weyl tensor $\mathcal{W}^g(\xi)$ of the metric g at the point ξ in local chart. We denote by g^{ij} the inverse matrix of g_{ij} and by δ_{ij} the Kronecker symbol.

Let $\xi_0 \in M$ be fixed. Given a coordinate system, the Weyl tensor in a point $\xi(x)$ belonging to $B_g(\xi_0, R)$ can be expressed as follows:

$$W^g_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}),$$

where R_{ijkl} is the Riemann curvature tensor, R_{ij} is the Ricci tensor and R is the scal curvature. We agree that all the previous functions are evaluated at the point x. Namely, the Riemann curvature tensor reads as

$$\begin{aligned} R_{ijkl} &= R_{ijkl}(g,x) = R^h_{ikl}g_{hj}, \\ R^l_{kij} &= \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{is}\Gamma^s jk, -\Gamma^l_{js}\Gamma^s_{ik}, \end{aligned} \tag{3.1}$$

the Ricci tensor reads as $R_{ij} = R_{ij}(g,x) = g^{kl}R_{ikjl}$ and the scalar curvature reads as

$$R = R(g,x) = g^{ij}R_{ij}. \tag{3.2}$$

Here Γ^l_{ij} are the Christoffel symbols

$$\Gamma^l_{ij} = \Gamma^l_{ij}(g,x) = \frac{1}{2}g^{lm}G_{ijk} \tag{3.3}$$

where $G_{ijk} = G_{ijk}(g,x) := \big(\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}\big)$.

Given the metric $g = \widehat{g} + h$ with $h \in \mathcal{B}_\rho$ and a point $\xi \in B_g(\xi_0, R)$, let us consider the local normal coordinates on the Riemannian manifold (M, g) given by the exponential map $\exp_\xi(z)$. Therefore, the metric g in normal coordinates satisfies

$$g^{ij}(0) = g_{ij}(0) = \delta_{ij} \quad \text{and} \quad \partial_k g^{ij}(0) = \partial_k g_{ij}(0) = 0,$$

which implies $\Gamma^k_{ij}(g,0) = 0$ for any indexes i, j and k.

In particular, the functions G_{ijk} defined in (3.3) have the following property

$$\partial^2_{\alpha\beta}G_{ijk}(h,0) = \partial^3_{\alpha\beta i}h_{kj}(0) + \partial^3_{\alpha\beta j}h_{ki}(0) - \partial^3_{\alpha\beta k}h_{ij}(0). \tag{3.4}$$

Moreover, we always choose $h \in \mathcal{S}^k$ such that the map $z \to h_{ij}(\exp_\xi(z))$, with its first and second derivatives, is vanishing at the point 0, for any indexes i and j, i.e.

$$h_{ij}(0) = 0, \quad \partial_k h_{ij}(0) = 0 \quad \text{and} \quad \partial^2_{kl}h_{ij}(0) = 0 \quad \text{for any } i,j,k,l. \tag{3.5}$$

3.2. Calculus. All the derivatives have been already computed in [7]. For sake of completeness, we recall their expressions.

3.2.1. The derivative of Christoffel symbols. By (3.3) a straightforward computation gives

$$\begin{aligned}
\partial_\alpha \Gamma^l_{ij}(g,x) &= \frac{1}{2}\,\partial_\alpha g^{lk} G_{ijk}(g,x) + \frac{1}{2}\, g^{lk}\partial_\alpha G_{ijk}(g,x),\\
\partial^2_{\alpha\beta}\Gamma^l_{ij}(g,x) &= \frac{1}{2}\,\partial^2_{\alpha\beta} g^{lk} G_{ijk}(g,x) + \frac{1}{2}\, g^{lk}\partial^2_{\alpha\beta} G_{ijk}(g,x)\\
&\quad + \partial_\alpha g^{lk}\partial_\beta G_{ijk}(g,x) + \partial_\beta g^{lk}\partial_\alpha G_{ijk}(g,x),\\
D_g\Gamma^l_{ij}(g,x)[h] &= \frac{1}{2}\, g^{lk} G_{ijk}(h,x) - \frac{1}{2}\, g^{ls}h_{st}g^{tk} G_{ijk}(g,x),\\
\partial_\alpha D_g\Gamma^l_{ij}(g,x)[h] &= \frac{1}{2}\,\partial_\alpha g^{lk} G_{ijk}(h,x) + \frac{1}{2}\, g^{lk}\partial_\alpha G_{ijk}(h,x)\\
&\quad - \frac{1}{2}\partial_\alpha\big(g^{ls}h_{st}g^{tk}\big) G_{ijk}(g,x) - \frac{1}{2} g^{ls}h_{st}g^{tk}\partial_\alpha G_{ijk}(g,x)\\
\partial^2_{\alpha\beta} D_g\Gamma^l_{ij}(g,x)[h] &= \frac{1}{2}\,\partial^2_{\alpha\beta} g^{lk} G_{ijk}(h,x) + \frac{1}{2}\,\partial_\alpha g^{lk}\partial_\beta G_{ijk}(h,x)\\
&\quad + \frac{1}{2}\,\partial_\beta g^{lk}\partial_\alpha G_{ijk}(h,x) + \frac{1}{2}\, g^{lk}\partial^2_{\alpha\beta} G_{ijk}(h,x)\\
&\quad - \frac{1}{2}\,\partial^2_{\alpha\beta}\big(g^{ls}h_{st}g^{tk}\big) G_{ijk}(g,x) - \frac{1}{2}\,\partial_\alpha\big(g^{ls}h_{st}g^{tk}\big)\partial_\beta G_{ijk}(g,x)\\
&\quad - \frac{1}{2}\, g^{ls}h_{st}g^{tk}\partial^2_{\alpha\beta} G_{ijk}(g,x) - \frac{1}{2}\,\partial_\beta\big(g^{ls}h_{st}g^{tk}\big)\partial_\alpha G_{ijk}(g,x).
\end{aligned}$$

In particular, if we assume (3.5) we get

$$\begin{gathered}
D_h\Gamma^l_{ij}(g,x)[h]_{|_{x=0}} = 0, \qquad \partial_\alpha D_h\Gamma^l_{ij}(g,x)[h]_{|_{x=0}} = 0,\\
\partial^2_{\alpha\beta}\Gamma^l_{ij}(g,x)[h]_{|_{x=0}} = \frac{1}{2}\,\partial^2_{\alpha\beta} G_{ijl}(h,0).
\end{gathered}\tag{3.6}$$

3.2.2. The derivative of the Riemann tensor. By (3.1) a straightforward computation gives

$$\begin{aligned}
D_h R^i_{jkl}(g,x)[h] &= D_h\partial_k\Gamma^i_{lj}(g,x)[h] - D_h\partial_l\Gamma^i_{kj}(g,x)[h]\\
&\quad + D_h\Gamma^i_{ks}(g,x)[h]\Gamma^s_{lj} + \Gamma^i_{ks} D_h\Gamma^s_{lj}(g,x)[h]\\
&\quad - D_h\Gamma^i_{ls}(g,x)[h]\Gamma^s_{kj} - \Gamma^i_{ls} D_h\Gamma^s_{kj}(g,x)[h]
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
\partial_\alpha D_h R^i_{jkl}(g,x)[h] &= D_h\partial^2_{\alpha k}\Gamma^i_{lj}(g,x)[h] - D_h\partial^2_{\alpha l}\Gamma^i_{kj}(g,x)[h]\\
&\quad + D_h\partial_\alpha\Gamma^i_{ks}(g,x)[h]\Gamma^s_{lj} + D_h\Gamma^i_{ks}(g,x)[h]\partial_\alpha\Gamma^s_{lj}\\
&\quad + \partial_\alpha\Gamma^i_{ks} D_h\Gamma^s_{lj}(g,x)[h] + \Gamma^i_{ks} D_h\partial_\alpha\Gamma^s_{lj}(g,x)[h]\\
&\quad - D_h\partial_\alpha\Gamma^i_{ls}(g,x)[h]\Gamma^s_{kj} - D_h\Gamma^i_{ls}(g,x)[h]\partial_\alpha\Gamma^s_{kj}\\
&\quad - \partial_\alpha\Gamma^i_{ls} D_h\Gamma^s_{kj}(g,x)[h] - \Gamma^i_{ls} D_h\partial_\alpha\Gamma^s_{kj}(g,x)[h].
\end{aligned}$$

If we assume (3.5), by (3.6) we get

$$
\begin{aligned}
D_h R^i_{jkl}(g,x)[h]_{|x=0} &= 0,\\
\partial_\alpha D_h R^i_{jkl}(g,x)[h]_{|x=0} &= \frac{1}{2}\,\partial^2_{\alpha k} G_{lji}(h,0) - \frac{1}{2}\,\partial^2_{\alpha l} G_{kji}(h,0).
\end{aligned}
\tag{3.8}
$$

Again, by (3.1) $R_{ijkl} = g_{js} R^s_{ikl}$, a straightforward computation leads to

$$
\begin{aligned}
D_h R_{ijkl}(g,x)[h] &= h_{js} R^s_{ikl} + g_{js} D_h R^s_{ikl}(g,x)[h],\\
\partial_\alpha D_h R_{ijkl}(g,x)[h] &= \partial_\alpha h_{ij} R^s_{ikl} + h_{js}\partial_\alpha R^s_{ikl}\\
&\quad + \partial_\alpha g_{js} D_h R^s_{ikl}(g,x)[h] + g_{js} D_h \partial_\alpha R^s_{ikl}(g,x)[h].
\end{aligned}
$$

In particular, if we assume (3.5), by (3.8) we get

$$
D_h R_{ijkl}(g,x)[h]_{|x=0} = 0,
$$

and

$$
\partial_\alpha D_h R_{ijkl}(g,x)[h]_{|x=0} = \frac{1}{2}(\partial^2_{\alpha k} G_{ilj}(h,0) - \partial^2_{\alpha l} G_{ikj}(h,0)).
$$

3.2.3. The derivative of the Ricci tensor. By (3.1) $R_{ij} = g^{kl} R_{ikjl}$. A straightforward computation gives

$$
\begin{aligned}
D_h R_{ij}(g,x)[h] &= h^{kl} R_{ikjl} + g^{kl} D_h R_{ikjl}(g,x)[h],\\
\partial_\alpha D_h R_{ij}(g,x)[h] &= \partial_\alpha h^{kl} R_{ikjl} + h^{kl}\partial_\alpha R_{ikjl}\\
&\quad + \partial_\alpha g^{kl} D_h R_{ikjl}(g,x)[h] + g^{kl} D_h \partial_\alpha R_{ikjl}(g,x)[h].
\end{aligned}
$$

In particular, if we assume (3.5), by (3.9) we get

$$
\begin{aligned}
D_h R_{ij}(g,x)[h]_{|x=0} &= 0,\\
\partial_\alpha D_h R_{ij}(g,x)[h]_{|x=0} &= \frac{1}{2}\big(\partial^2_{\alpha j} G_{ill}(h,0) - \partial^2_{\alpha l} G_{ijl}(h,0)\big).
\end{aligned}
\tag{3.9}
$$

3.2.4. The derivative of the scalar curvature. By (3.2) $R = g^{ij} R_{ij}$ and a straightforward computation gives

$$
\begin{aligned}
D_h R(g,x)[h] &= h^{ij} R_{ij} + g^{ij} D_h R_{ij}(g,x)[h],\\
\partial_\alpha D_h R(g,x)[h] &= \partial_\alpha h^{ij} R_{ij} + h^{ij}\partial_\alpha R_{ij}\\
&\quad + \partial_\alpha g^{ij} D_h R_{ij}(g,x)[h] + g^{ij} D_h \partial_\alpha R_{ij}(g,x)[h].
\end{aligned}
$$

In particular, if we assume (3.5), by (3.9) we get

$$
\begin{aligned}
D_h R(g,x)[h]_{|x=0} &= 0,\\
\partial_\alpha D_h R(g,x)[h]_{|x=0} &= \frac{1}{2}\big(\partial^2_{\alpha i} G_{ill}(h,0) - \partial^2_{\alpha l} G_{iil}(h,0)\big).
\end{aligned}
\tag{3.10}
$$

3.3. The derivative of the Weyl's tensor. Let us recall that

$$W^g_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}).$$

A straightforward computation shows that

$$\begin{aligned} D_h W^g_{ijkl}(g,x)[h] = {} & D_h R_{ijkl}(g,x)[h] \\ & - \frac{1}{n-2}(R_{ik}h_{jl} - R_{il}h_{jk} + R_{jl}h_{ik} - R_{jk}h_{il}) \\ & - \frac{1}{n-2}(D_h R_{ik}(g,x)[h]g_{jl} - D_h R_{il}(g,x)[h]g_{jk} \\ & + D_h R_{jl}(g,x)[h]g_{ik} - D_h R_{jk}(g,x)[h]g_{il}) \\ & + \frac{R}{(n-1)(n-2)}(h_{jl}g_{ik} + g_{jl}h_{ik} - h_{jk}g_{il} - g_{jk}h_{il}) \\ & + \frac{1}{(n-1)(n-2)}D_h R(g,x)[h](g_{jl}g_{ik} - g_{jk}g_{il}) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \partial_\alpha D_h W^g_{ijkl}(g,x)[h] = {} & \partial_\alpha D_h R_{ijkl}(g,x)[h] \\ & - \frac{1}{n-2}(\partial_\alpha R_{ik}h_{jl} - \partial_\alpha R_{il}h_{jk} + \partial_\alpha R_{jl}h_{ik} - \partial_\alpha R_{jk}h_{il}) \\ & - \frac{1}{n-2}(R_{ik}\partial_\alpha h_{jl} - R_{il}\partial_\alpha h_{jk} + R_{jl}\partial_\alpha h_{ik} - R_{jk}\partial_\alpha h_{il}) \\ & - \frac{1}{n-2}(D_h\partial_\alpha R_{ik}(g,x)[h]g_{jl} - D_h\partial_\alpha R_{il}(g,x)[h]g_{jk} \\ & + D_h\partial_\alpha R_{jl}(g,x)[h]g_{ik} - D_h\partial_\alpha R_{jk}(g,x)[h]g_{il}) \\ & - \frac{1}{n-2}(D_h R_{ik}(g,x)[h]\partial_\alpha g_{jl} - D_h R_{il}(g,x)[h]\partial_\alpha g_{jk} \\ & + D_h R_{jl}(g,x)[h]\partial_\alpha g_{ik} - D_h R_{jk}(g,x)[h]\partial_\alpha g_{il}) \\ & + \frac{R}{(n-1)(n-2)}\,\partial_\alpha(h_{jl}g_{ik} + g_{jl}h_{ik} - h_{jk}g_{il} - g_{jk}h_{il}) \\ & + \frac{1}{(n-1)(n-2)}\partial_\alpha R(h_{jl}g_{ik} + g_{jl}h_{ik} - h_{jk}g_{il} - g_{jk}h_{il}) \\ & + \frac{1}{(n-1)(n-2)}\,D_h R(g,x)[h]\partial_\alpha(g_{jl}g_{ik} - g_{jk}g_{il}) \\ & + \frac{1}{(n-1)(n-2)}\,D_h\partial_\alpha R(g,x)[h](g_{jl}g_{ik} - g_{jk}g_{il}). \end{aligned}$$

In particular, if we assume (3.5), by (3.9) and (3.10) we get

$$D_h W^g_{ijkl}(g,x)[h]_{|x=0} = 0$$

and

$$\begin{aligned}\partial_\alpha D_h W^g_{ijkl}(g,x)[h]_{|x=0} &= \frac{1}{2}\big[\partial^3_{\alpha ki}h_{lj}(0) - \partial^3_{\alpha jk}h_{il}(0) - \partial^3_{\alpha li}h_{kj}(0) + \partial^3_{\alpha lj}h_{ik}(0)\big]\\ &- \frac{1}{2(n-2)}\big\{\big[\partial^3_{\alpha ki}h_{ss}(0) - \partial^3_{\alpha ks}h_{is}(0) - \partial^3_{\alpha si}h_{sk}(0) + \partial^3_{\alpha ss}h_{ik}(0)\big]\delta_{jl}\\ &- \big[\partial^3_{\alpha li}h_{ss}(0) - \partial^3_{\alpha ls}h_{is}(0) - \partial^3_{\alpha si}h_{sl}(0) + \partial^3_{\alpha ss}h_{il}(0)\big]\delta_{jk}\\ &+ \big[\partial^3_{\alpha lj}h_{ss}(0) - \partial^3_{\alpha ls}h_{js}(0) - \partial^3_{\alpha sj}h_{ls}(0) + \partial^3_{\alpha ss}h_{jl}(0)\big]\delta_{ik}\\ &- \big[\partial^3_{\alpha kj}h_{ss}(0) - \partial^3_{\alpha ks}h_{js}(0) - \partial^3_{\alpha sj}h_{ks}(0) + \partial^3_{\alpha ss}h_{jk}(0)\big]\delta_{il}\big\}\\ &+ \frac{1}{(n-1)(n-2)}\big[\partial^3_{\alpha tt}h_{ss}(0) - \partial^3_{\alpha st}h_{st}(0)\big](\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il})\end{aligned}$$

where we used (3.4), i.e.

$$\partial^2_{\alpha\beta}G_{ijk}(h,0) = \partial^3_{\alpha\beta i}h_{kj}(0) + \partial^3_{\alpha\beta j}h_{ki}(0) - \partial^3_{\alpha\beta k}h_{ij}(0).$$

3.4. The transversality condition: proof.

LEMMA 3.1. *The map $(h,x) \to F'_h(\widetilde{h},\widetilde{x})[h] + F'_x(\widetilde{h},\widetilde{x})x$ is onto on $\mathbb{R}^n$ for any $(\widetilde{h},\widetilde{x})$ such that $F(\widetilde{h},\widetilde{x}) = 0$.*

PROOF. Let $\widehat{g}+h$ with $h \in \mathcal{B}_\rho \subset \mathcal{S}^k$ with $k \geq 4$. The function $F(h,x) = \nabla_x \widetilde{W}^{\widehat{g}+h}_{ijkl}(x)$ defined in (2.1) is of class C^2. Let $(\widetilde{h},\widetilde{x})$ such that $F(\widetilde{h},\widetilde{x}) = 0$.

We shall prove that the map $F'_h(\widetilde{h},\widetilde{x})\colon \mathcal{S}^k \to \mathbb{R}^n$ defined by

$$F'_h(\widetilde{h},\widetilde{x})[h] = \Big(D_h\partial_1 \widetilde{W}^{\widehat{g}+\widetilde{h}}_{ijkl}(\widetilde{x})[h], \dots, D_h\partial_n \widetilde{W}^{\widehat{g}+\widetilde{h}}_{ijkl}(\widetilde{x})[h]\Big)$$

is onto.

We point out that the ontoness of the map $h \to F'_h(\widetilde{h},\widetilde{x})[h]$ is invariant with respect to a change of variable $x = \psi(z)$, where ψ is a diffeomorphism. Therefore, we compute $D_h\partial_\alpha \widetilde{W}^{\widehat{g}+\widetilde{h}}_{ijkl}(\widetilde{x})[h]$ by choosing the normal coordinates on the Riemannian manifold $(M,\widehat{g}+\widetilde{h})$ given by the exponential map $\exp_{\widetilde{\xi}}(z)$, where $\widetilde{\xi}$ corresponds to $\widetilde{x}$.

We choose $h \in \mathcal{S}^k$ such that the map $z \to h_{ij}(\exp_{\widetilde{\xi}}(z))$, with its first and second derivatives, is vanishing at the point 0, for any indexes i and j, so that condition (3.5) holds. Therefore, we are lead to prove that the map $T\colon \mathcal{S}^k \to \mathbb{R}^n$ whose components T_α, $\alpha = 1,\dots,n$, are defined by

$$\begin{aligned}T_\alpha(h) :=& \frac{1}{2}\big[\partial^3_{\alpha ki}h_{lj}(0) - \partial^3_{\alpha jk}h_{il}(0) - \partial^3_{\alpha li}h_{kj}(0) + \partial^3_{\alpha lj}h_{ik}(0)\big]\\ &- \frac{1}{2(n-2)}\big\{\big[\partial^3_{\alpha ki}h_{ss}(0) - \partial^3_{\alpha ks}h_{is}(0) - \partial^3_{\alpha si}h_{sk}(0) + \partial^3_{\alpha ss}h_{ik}(0)\big]\delta_{jl}\\ &- \big[\partial^3_{\alpha li}h_{ss}(0) - \partial^3_{\alpha ls}h_{is}(0) - \partial^3_{\alpha si}h_{sl}(0) + \partial^3_{\alpha\beta s}h_{il}(0)\big]\delta_{jk}\\ &+ \big[\partial^3_{\alpha lj}h_{ss}(0) - \partial^3_{\alpha ls}h_{js}(0) - \partial^3_{\alpha sj}h_{ls}(0) + \partial^3_{\alpha ss}h_{jl}(0)\big]\delta_{ik}\end{aligned}$$

$$- \big[\partial^3_{\alpha kj} h_{ss}(0) - \partial^3_{\alpha ks} h_{js}(0) - \partial^3_{\alpha sj} h_{ks}(0) + \partial^3_{\alpha ss} h_{jk}(0)\big]\delta_{il}\big\}$$
$$+ \frac{1}{(n-1)(n-2)}\big[\partial^3_{\alpha tt} h_{ss}(0) - \partial^3_{\alpha st} h_{st}(0)\big](\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il})$$

is onto.

- *All the indices i, j, k, l are different.*

  The operator $T = (T_1, \ldots, T_n)$ reduces to

$$T_\alpha(h) = \frac{1}{2}\big[\partial^3_{\alpha ki} h_{lj}(0) - \partial^3_{\alpha jk} h_{il}(0) - \partial^3_{\alpha li} h_{kj}(0) + \partial^3_{\alpha lj} h_{ik}(0)\big], \quad \alpha = 1, \ldots, n.$$

For any $\ell = 1, \ldots, n$ we choose $h^{(\ell)} \in \mathcal{S}^k$ defined by

$$h^{(\ell)}_{lj}(x) = x_\ell x_i x_k \quad \text{and} \quad h^{(\ell)}_{ab}(x) = 0 \quad \text{if } (a,b) \neq (l,j).$$

Therefore

$$T_\alpha(h^{(\ell)}) = \frac{1}{2}\,\partial^3_{\alpha ki} h^{(\ell)}_{lj}(0) \quad \text{and} \quad T(h^{(\ell)}) = c(0, \ldots, \underset{\substack{\uparrow \\ \ell\text{-th}}}{1}, \ldots, 0).$$

for some positive constant c. That proves that T is onto.

- *$i = k$ and the three indices i, j, l are different, i.e. $i \neq j$, $i \neq l$ and $j \neq l$.*

  The operator $T = (T_1, \ldots, T_n)$ reduces to

$$T_\alpha(h) := \frac{1}{2}\big[\partial^3_{\alpha ii} h_{lj}(0) - \partial^3_{\alpha ji} h_{il}(0) - \partial^3_{\alpha li} h_{ij}(0) + \partial^3_{\alpha lj} h_{ii}(0)\big]$$
$$- \frac{1}{2(n-2)}\big[\partial^3_{\alpha lj} h_{ss}(0) - \partial^3_{\alpha ls} h_{js}(0) - \partial^3_{\alpha sj} h_{ls}(0) + \partial^3_{\alpha ss} h_{jl}(0)\big].$$

For any $\ell = 1, \ldots, n$ we choose $h^{(\ell)} \in \mathcal{S}^k$ defined by

$$h^{(\ell)}_{ii}(x) = x_\ell x_j x_l \quad \text{and} \quad h^{(\ell)}_{ab}(x) = 0 \quad \text{if } (a,b) \neq (i,i).$$

Therefore

$$T_\alpha(h^{(\ell)}) = \frac{n-3}{2(n-2)}\,\partial^3_{\alpha lj} h^{(\ell)}_{ii}(0) \quad \text{and} \quad T(h^{(\ell)}) = c(0 \ldots, \underset{\substack{\uparrow \\ \ell\text{-th}}}{1}, \ldots, 0)$$

for some positive constant c. That proves that T is onto.

- *$i = k$, $j = l$ and $i \neq j$.*

  The operator $T = (T_1, \ldots, T_n)$ reduces to

$$T_\alpha(h) := \frac{1}{2}\big[\partial^3_{\alpha ii} h_{jj}(0) - 2\partial^3_{\alpha ji} h_{ij}(0) + \partial^3_{\alpha jj} h_{ii}(0)\big]$$
$$- \frac{1}{2(n-2)}\big[\partial^3_{\alpha ii} h_{ss}(0) - 2\partial^3_{\alpha is} h_{is}(0) + \partial^3_{\alpha ss} h_{ii}(0)$$
$$+ \partial^3_{\alpha jj} h_{ss}(0) - 2\partial^3_{\alpha js} h_{js}(0) + \partial^3_{\alpha ss} h_{jj}(0)\big]$$
$$+ \frac{1}{(n-1)(n-2)}\left[\partial^3_{\alpha tt} h_{ss}(0) - \partial^3_{\alpha st} h_{st}(0)\right].$$

For any $\ell = 1, \dots, n$, $\ell \neq i$ and $\ell \neq j$ we choose $h^{(\ell)} \in \mathcal{S}^k$ defined by

$$h^{(\ell)}_{ij}(x) = x_\ell x_i x_j \quad \text{and} \quad h^{(\ell)}_{ab}(x) = 0 \quad \text{if } (a,b) \neq (i,j),$$

if $\ell = i$ we choose

$$h^{(i)}_{ij}(x) = x_i^2 x_j \quad \text{and} \quad h^{(i)}_{ab}(x) = 0 \quad \text{if } (a,b) \neq (i,j)$$

and if $\ell = j$ we choose

$$h^{(j)}_{ij}(x) = x_i x_j^2 \quad \text{and} \quad h^{(j)}_{ab}(x) = 0 \quad \text{if } (a,b) \neq (i,j).$$

Therefore

$$T_\alpha(h^{(\ell)}) = -\frac{n^2 - 5n + 5}{(n-1)(n-2)}\, \partial^3_{\alpha ij} h^{(\ell)}_{ij}(0) \quad \text{and} \quad T(h^{(\ell)}) = c(0, \dots, \underset{\ell\text{-th}}{\underset{\uparrow}{1}}, \dots, 0)$$

for some negative constant c. That proves that T is onto. $\square$

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