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GLOBAL EXISTENCE FOR REACTION-DIFFUSION SYSTEMS MODELING IONS ELECTRO-MIGRATION THROUGH BIOLOGICAL MEMBRANES WITH MASS CONTROL AND CRITICAL GROWTH WITH RESPECT TO THE GRADIENT

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Abstract. This paper studies the existence of global weak solutions for reaction-diffusion systems depending on two main assumptions: the non-negative of solutions and the total mass of components are preserved with time, the non-linearities have critical growth with respect to the gradient. This work is a generalization of the work developed by Alaa and Lefraich [2] without the presence of the gradient in the kinetic reaction terms.

1. Introduction

Some classes of models of ions migration through biological membranes are studied by Alaa and Lefraich [2]. Such migrations take place for most living cells and biochemical processes. As the motion of ions is due to diffusion and the electrical field, and they undergo reactions, the ions concentrations satisfy the Nernst–Planck equations, including kinetic reaction terms in the general form and the potential is given by Poisson equation. The equations of that model can

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be written as follows:

$$
(1.1)\quad
\begin{cases}
\dfrac{\partial C_i}{\partial t} - d_i \Delta C_i - m_i \operatorname{div}(C_i \nabla \phi) = f_i(t,x,C,\nabla C) & \text{in } Q_T,\\
-\varepsilon \Delta \phi = \sum_{i=1}^{N_s} z_i C_i \Big/ \Big(1 + \varepsilon \sum_{i=1}^{N_s} C_i\Big) - g & \text{in } Q_T,\\
 & \text{for } i = 1, \ldots, N_s,\\
d_i \dfrac{\partial C_i}{\partial \nu} + m_i C_i \dfrac{\partial \phi}{\partial \nu} = 0 & \text{in } \Sigma_T,\\
\phi(t,x) = 0 & \text{in } \Sigma_T,\\
C_i(0,x) = C_{i,0}(x) & \text{on } \Omega,\\
\phi(0,x) = \phi_0(x) & \text{on } \Omega,
\end{cases}
$$

where Ω is an open bounded subset of $\mathbb{R}^N$, ($N = 2$ or 3), with smooth boundary $\partial\Omega$, $Q_T = \,]0,T[\, \times \Omega$ and $\Sigma_T = \,]0,T[\, \times \partial\Omega$, where ($T > 0$), for each i, C_i is the concentration of the A_i species which has diffusion coefficient d_i, mobility m_i, and valency z_i. ϕ is the electrical potential, g is the fixed charges concentration, and f_i are the reaction terms. Given a normed vector space X and its continuous dual X', $\sigma(X, X')$ is called the weak topology on X and $\sigma(X', X)$ the weak* topology on X', in this paper, $X = L^1(Q_T)$ and $X' = L^\infty(Q_T)$.

In this study, we suppose the following assumptions:

- $(\mathrm{H}_f)_1$ The non-negative solution is preserved with time, which is ensured by $f_i(\widehat{C}_i) \geq 0$, where $\widehat{C}_i = (t, x, C_1, \ldots, C_{i-1}, 0, C_{i+1}, \ldots, C_{N_s}, p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{N_s})$, for all $1 \leq i \leq N_s$, $(C,p) \in (\mathbb{R}^+)^{N_s} \times \mathbb{R}^{NN_s}$ and, for almost every $(t,x) \in Q_T$, $C_{i,0} \geq 0$, for all $1 \leq i \leq N_s$.
- $(\mathrm{H}_f)_2$ The total mass of the components $C_1, \ldots, C_{N_s}$ is controlled with time, which for all $(C,p) \in (\mathbb{R}^+)^{N_s} \times \mathbb{R}^{NN_s}$ and almost every $(t,x) \in Q_T$, is ensured by

$$
\sum_{1 \leq i \leq r} f_i(t,x,C,p) \leq 0, \quad \text{for all } 1 \leq r \leq N_s.
$$

- $(\mathrm{H}_f)_3$ $f_i \colon Q_T \times \mathbb{R}^{N_s} \times \mathbb{R}^{N_s N} \to \mathbb{R}$ are measurable, for all $1 \leq i \leq N_s$.
- $(\mathrm{H}_f)_4$ $f_i \colon \mathbb{R}^{N_s} \times \mathbb{R}^{N_s N} \to \mathbb{R}$ are locally Lipschitz continuous. Namely

$$
\sum_{1 \leq i \leq N_s} \big| f_i(t,x,C,p) - f_i\big(t,x,\widehat{C},\widehat{p}\big) \big|
\leq K(r) \bigg[\sum_{1 \leq i \leq N_s} \big| C_i - \widehat{C}_i \big| + \sum_{1 \leq i \leq N_s} \big\| p_i - \widehat{p}_i \big\| \bigg]
$$

for all $0 \leq |C_i|, |\widehat{C}_i|, \|p_i\|, \|\widehat{p}_i\| \leq r$.

$(\mathrm{H}_f)_5$ $|f_1(t,x,C,\nabla C)| \le h_1(|C_1|)\Big[F_1(t,x) + \|\nabla C_1\|^2 + \sum_{2\le j\le N_s} \|\nabla C_j\|^{\alpha_j}\Big]$
where $h_1\colon [0,+\infty[\,\to [0,+\infty[$ is non-decreasing $F_1 \in L^1(Q_T)$ and $1 \le \alpha_j < 2$.

$(\mathrm{H}_f)_6$ $|f_i(t,x,C,\nabla C)| \le h_i\Big(\sum_{j=1}^{i} |C_j|\Big)\Big[F_i(t,x) + \sum_{1\le j\le N_s} \|\nabla C_j\|^2\Big]$
where $h_i\colon [0,+\infty[\,\to [0,+\infty[$ is non-decreasing, $F_i \in L^1(Q_T)$ and $2 \le i < N_s$.

$(\mathrm{H}_C)_7$ For all $i = 1,\dots,N_s$, $C_{i,0} \in L^2(\Omega)$ and satisfy $C_{i,0} \ge 0$.

The present paper proves the existence of solutions for the reaction-diffusion systems of the type (1.1). This is done in three steps: the first step is to truncate the equation. It shows that the problem obtained has a solution. In the second step, appropriate estimates are established on the approximate solutions. Finally, the convergence of the approximate system has been shown. Recently, the new technique was used in this study, in fact, our approach for $f = f_i(t,x,C,\nabla C)$ is a generalization of the work $f = f_i(C)$ presented by Alaa and Lefraich [2].

2. Statement of the result

Definition 2.1. The pair $(C,\phi) = (C_1,,\dots,C_{N_s},\phi)$ is a weak solution of (1.1) if and only if, for every $1 \le i \le N_s$,

$$(2.1)\quad \begin{cases} C \in C([0,T],(L^2(\Omega))^{N_s}) \cap L^2(0,T,(H^1(\Omega))^{N_s}), \\ \phi \in L^\infty(0,T,H^1_0(\Omega)), \quad f_i(t,x,C,\nabla C) \in L^1(Q_T) \\ \qquad\qquad\qquad \text{for all } v \in C^1(Q_T) \text{ such that } v(T,\cdot\,) = 0, \\ \displaystyle\int_{Q_T} \Big[-C_i\frac{\partial v}{\partial t} + d_i \nabla C_i \nabla v + m_i C_i \nabla\phi\nabla v\Big]\,dt\,dx \\ \qquad \displaystyle -\int_\Omega C_{i,0}(x)\,v(0,x) = \int_{Q_T} f_i(t,x,C,\nabla C)\,v \quad \text{for all } \Psi \in H^1_0(\Omega), \\ \displaystyle\varepsilon\int_\Omega \nabla\phi\nabla\Psi = \int_\Omega \Big(\sum_{i=1}^{N_s} z_i C_i \Big/ \Big(1+\varepsilon\sum_{i=1}^{N_s} C_i\Big) - g\Big)\Psi \qquad \text{for a.e. } t>0, \\ \phi(0,x) = \phi_0(x) \qquad\qquad \text{on } \Omega. \end{cases}$$

Our main result in this paper is the following existence theorem.

Theorem 2.2. *Assume that conditions* $(\mathrm{H}_f)_1$–$(\mathrm{H}_f)_6$ *and* $(\mathrm{H}_C)_7$ *are satisfied. Then there exists a global weak solution* (C,ϕ) *of system* (1.1) *such that*

$$C \ge 0 \quad \text{in } Q_T \quad \text{and} \quad \phi \in L^\infty(0,T;W^{1,\infty}(\Omega)).$$

The proof of Theorem 2.2 is divided into three steps.

2.1. Approximating scheme. To every function f_i we associate the function $\widehat{f}_i$, such that $\widehat{f}_i(t,x,C,\nabla C) = f_i(t,x,C^+,\nabla C)$, where $C^+ = \max(C,0)$, and we consider the truncation $\eta_n \in C_0^\infty(\mathbb{R}^{N_s})$, that satisfies the following conditions

$$\begin{cases} 0 \le \eta_n \le 1, \\ \eta_n(r) = 1 & \text{if } |r| \le n, \\ \eta_n = 0 & \text{if } |r| \ge n+1. \end{cases}$$

For every $C = (C_1,\ldots,C_{N_s}) \in \mathbb{R}^{N_s}$, let

$$f_{i,n}(t,x,C,\nabla C) = \eta_n\bigg[\sum_{1\le j\le N_s}(|C_j| + \|\nabla C_j\|)\widehat{f}_i(t,x,C,\nabla C)\bigg]. \tag{2.2}$$

Note in passing that $f_{i,n}$, as defined, is globally Lipschitz. Indeed, $f_{i,n}$ is locally Lipschitz from $(\mathrm{H}_f)_4$ and, as it is bounded by definition, then $f_{i,n}$ is globally Lipschitz.

Furthermore, for every $\psi \in C\big([0,T];H^1(\Omega)\big)$ and $t \in [0,T]$, let us consider the bilinear form $a^i_\psi(t,\,\cdot\,,\,\cdot\,)$ defined on $H^1(\Omega)\times H^1(\Omega)$ by

$$a^i_\psi(t,u,v) = \int_\Omega (d_i\nabla u\nabla v + m_i u\nabla\psi\nabla v + \lambda uv),$$

where λ is a strictly constant the value of which will be established later. Finally, for $v \in L^2(Q_T)$ we introduce its time regularization

$$v^{(n)}(t,x) = \int_\Omega nv(s,x)\exp(n(s-t))\,ds.$$

In view of Boccarde, Murat [14], we deduce that

$$\begin{cases} v^{(n)} \in C([0,T];L^2(\Omega)), \\ v^{(n)} \to v \quad \text{in } L^2(Q_T), \\ \sup\limits_{0<t<T}\|v^{(n)}(t,\,\cdot\,)\|_{L^1(\Omega)} \le \sup\limits_{0<t<T}\|v(t,\,\cdot\,)\|_{L^1(\Omega)}. \end{cases} \tag{2.3}$$

Now, let us consider the following truncated system

$$\begin{cases} C_n \in W(0,T), \quad \phi_n \in L^\infty(0,T,H^1_0(\Omega)), \\[1ex] \displaystyle\int_\Omega \frac{\partial C_{i,n}}{\partial t}v + a^i_{\phi_n}(t,C_{i,n},v) \\ \displaystyle\qquad = \int_\Omega f_{i,n}(t,x,C_n,\nabla C_n)v + \lambda\int_\Omega C_{i,n}v \quad \text{for all } v \in H^1(\Omega), \\[1ex] \displaystyle\varepsilon\int_\Omega \nabla\phi_n\nabla\psi = \int_\Omega\bigg(\sum_{i=1}^{N_s} z_i C^{(n)}_{i,n}\Big/\Big(1+\varepsilon\sum_{i=1}^{N_s}C^{(n)}_{i,n}\Big) - g\bigg)\psi \\ \qquad\qquad\qquad\qquad\qquad\text{for a.e. } t>0,\ \text{for all } \psi \in H^1_0(\Omega), \\[1ex] \phi_n(0,x) = \phi_0(x), \quad C_{i,n}(0,x) = C_{i,0}(x), \end{cases} \tag{2.4}$$

where $W(0,T)$ is the Hilbert space defined by

$$W(0,T)=\Big\{u\in L^2(0,T,(H^1(\Omega))^{N_s}):\frac{\partial u}{\partial t}\in L^2(0,T,(H^{-1}(\Omega))^{N_s})\Big\}.$$

Let us note that every $v\in W(0,T)$ is almost everywhere equal to a continuous function from $[0,T]$ in $(L^2(\Omega))^{N_s}$. Furthermore,

$$W(0,T)\subset C([0,T];(L^2(\Omega))^{N_s})$$

and the embedding is continuous (see for more details Dautray and Lions [18]). So for $C_n\in W(0,T)$, the expression $C_{i,n}$ take for $t=0$ the value $C_{i,0}$ which make sense, with an application $C_n\in W(0,T)\mapsto C_n(0)\in(L^2(\Omega))^{N_s}$ continuous. Concerning our problem (2.4), we have the following result.

Theorem 2.3. *Under the hypothesis* $(\mathrm{H}_f)_1$–$(\mathrm{H}_f)_6$ *and* $(\mathrm{H}_C)_7$, *the problem* (2.4) *admits a weak solution* $(C_n,\phi_n)\in W(0,T)\times L^\infty(0,T;W^{1,\infty}(\Omega))$ *such that* $C_n\geq 0$ *in* Q_T.

Proof. Considering $\overline{C}_n=C_ne^{-\lambda t}$, then $\overline{C}_n$ satisfies

$$(2.5)\quad\begin{cases}\displaystyle\int_\Omega\frac{\partial\overline{C}_{i,n}}{\partial t}\varphi+a^i_{\phi_n}(t,\overline{C}_{i,n},\varphi)=\int_\Omega S_{i,n}(t,\overline{C}_n,\nabla\overline{C}_n)\varphi\\ \qquad\text{for all }\varphi\in H^1(\Omega),\\ \qquad\text{with }S_{i,n}(t,\overline{C}_n,\nabla\overline{C}_n)=f_{i,n}(t,x,\overline{C}_ne^{\lambda t},\nabla\overline{C}_ne^{\lambda t}).e^{-\lambda t},\\ \displaystyle\varepsilon\int_\Omega\nabla\phi_n\nabla\psi=\int_\Omega\bigg(\sum_{i=1}^{N_s}z_i\overline{C}^{(n)}_{i,n}e^{\lambda t}\Big/\Big(1+\varepsilon\sum_{i=1}^{N_s}\overline{C}^{(n)}_{i,n}e^{\lambda t}\Big)-g\bigg)\psi\\ \qquad\qquad\text{for a.e. }t>0,\ \text{for all }\psi\in H^1_0(\Omega),\\ \phi_n(0,x)=\phi_0(x),\quad C_{i,n}(0,x)=C_{i,0}(x).\end{cases}$$

To prove the existence of a solution of (2.4), it suffices to prove the existence of a solution for the problem (2.5). We get this result by using Schauder fixed point theorem. Let

$$W_0(0,T)=\Big\{v\in W(0,T):C_0=v(0)\ \text{ and }\sup_{0<t<T}\|\overline{v}(t)\|_{(L^2(\Omega))^{N_s}}+\|\overline{v}\|_{L^2(0,T,(H^1(\Omega))^{N_s})}\leq C\Big\}.$$

We construct the following mapping:

$$\pounds_n\colon W_0(0,T)\to W_0(0,T),\qquad v\mapsto\pounds_n(v)=\overline{C}_n$$

where, for all $t\in\,]0,T[$, ϕ_n is the unique solution of the elliptic problem

$$(2.6)\quad\begin{cases}\displaystyle-\varepsilon\Delta\phi_n=\sum_{i=1}^{N_s}z_iv_i^{(n)}e^{\lambda t}\Big/\Big(1+\varepsilon\sum_{i=1}^{N_s}v_i^{(n)}e^{\lambda t}\Big)-g & \text{on }Q_T=\,]0,T[\,\times\Omega,\\ \phi_n(t,x)=0 & \text{in }]0,T[\,\times\partial\Omega,\end{cases}$$

and $\overline{C}_n = \pounds_n(v)$ satisfies the following parabolic system

$$
\begin{cases}
\displaystyle\int_\Omega \frac{\partial \overline{C}_{i,n}}{\partial t}\,\varphi + a^i_{\phi_n}(t,\overline{C}_{i,n},\varphi) = \int_\Omega S_{i,n}(t,v,\nabla v)\,\varphi \quad \text{for all } \varphi \in H^1(\Omega),\\
\overline{C}_{i,n}(0,x) = C_{i,0}(x).
\end{cases} \tag{2.7}
$$

In order to prove that this setting is correct, we use the Lax-Milgram theorem in order to prove that (2.7) has a unique solution (which actually shows that $\pounds_n$ is a well-defined map). Then by the use of the Schauder fixed point theorem, we prove that $\pounds_n$ has a fixed point which is the solution of (2.5). This concludes the proof. □

Concerning the solution of the problem (2.6), we make the following remark.

Remark 2.4. The solution of (2.6) satisfy

$$
\phi_n(t,x) = \int_\Omega H(x,s)\,\theta^n_\varepsilon(t,s)\,ds
$$

where

$$
\theta^n_\varepsilon(t,s) = \sum_{i=1}^{N_s} z_i v_i^{(n)}(t,s)e^{\lambda t} \Big/ \bigg(1 + \varepsilon \sum_{i=1}^{N_s} v_i^{(n)}(t,s)e^{\lambda t}\bigg) - g(s), \quad \text{for } s \in \Omega,
$$

and H is the Green function. We have $v^{(n)} \in C([0,T];L^2(\Omega))$, $\phi_n(t,\,\cdot\,) \in H^2(\Omega)$ for every $t \in \,]0,T[$ and $\phi_n \in L^\infty(0,T;W^{1,\infty}(\Omega))$.

Now, let us make some assumptions on the bilinear form $a^i_{\phi_n}$. We have the following result.

Lemma 2.5. *$a^i_{\phi_n}$ is a continuous and coercive bilinear form on $H^1(\Omega) \times H^1(\Omega)$, i.e. for every $t \in [0,T]$:*

(a) *there exists a positive constant C, which depends only on d_i, m_i, λ and $\|\nabla\phi_n\|_{L^\infty}$, such that*

$$
\big|a^i_{\phi_n}(t,u,\widehat{u})\big| \le C\|u\|_{H^1(\Omega)}\big\|\widehat{u}\big\|_{H^1(\Omega)};
$$

(b) *there exist constants α_0 and λ^* strictly positive such that*

$$
a^i_{\phi_n}(t,u,u) \ge \alpha_0\|u\|^2_{H^1(\Omega)} \quad \textit{for every } \lambda \ge \lambda^*.
$$

Proof. (a) It is obvious that $a^i_{\phi_n}$ is a bilinear form on $H^1(\Omega)\times H^1(\Omega)$. By using the Hölder inequality and the fact that $\phi_n \in L^\infty(0,T;W^{1,\infty}(\Omega))$, we can prove that $a^i_{\phi_n}$ is continuous on $H^1(\Omega)\times H^1(\Omega)$.

(b) Let us put $d_0 = \min_{1\le i\le N_s}(d_i)$. Then

$$
a^i_{\phi_n}(t,u,u) \ge d_0\|\nabla u\|^2_{L^2(\Omega)} + m_i\int_\Omega u\nabla\phi_n\nabla u + \lambda\|u\|^2_{L^2(\Omega)}.
$$

On the other hand, by the use of the Hölder and Young inequalities, we can show that

$$a^i_{\phi_n}(t,u,u) \geq \left(d_0 - \frac{\alpha}{2}\right)\|\nabla u\|^2_{L^2(\Omega)} + \left(\lambda - \frac{1}{2\alpha}(M\|\nabla\phi_n\|_{L^\infty})^2\right)\|u\|^2_{L^2(\Omega)}.$$

Then, by choosing $\alpha = d_0$ and $\lambda \geq \lambda^* = 2\alpha(M\|\nabla\phi_n\|_{L^\infty})^2/2$ we deduce that

$$a^i_{\phi_n}(t,u,u) \geq \alpha_0\|u\|^2_{H^1(\Omega)} \quad \text{with } \alpha_0 = \min\left(\frac{d_0}{2}, \lambda - \lambda^*\right).$$

This implies that the bilinear form $a^i_{\phi_n}$ is coercive on $H^1(\Omega)\times H^1(\Omega)$. □

The second member $S_{i,n}(t,v,\nabla v)$ is fixed in $L^2(Q_T)$, allowing to apply the Lax–Milgram theorem (see Dautray and Lions [18]), and then to conclude that for every $v \in W_0(0,T)$, the problem (2.7) admits a unique solution $\overline{C}_n$. Now, it is time to prove that $\pounds_n$ admits a fixed point. To this end, we prove through the following lemma, that the hypothesis for the Schauder fixed point theorem are satisfied. Thus, we have

Lemma 2.6. *$\pounds_n$ is a continuous and compact operator on $W_0(0,T)$.*

Proof. Let $(v_m)_{m\geq 0}$ a sequence in $W_0(0,T)$ such that

$$v_m \to v \quad \text{in } W_0(0,T) \text{ when } m \to \infty.$$

Let us show that $\pounds_n(v_m) \to \pounds_n(v)$ in $W_0(0,T)$ when $m \to \infty$. Let $\overline{C}^m_n = \left(\overline{C}^m_{i,n}\right)_{1\leq i\leq N_s} = \pounds_n(v_m)$ the solution of

$$\begin{cases} \overline{C}^m_{i,n} \in C([0,T], L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)), \\ \dfrac{\partial \overline{C}^m_{i,n}}{\partial t} - d_i\Delta\overline{C}^m_{i,n} - m_i\operatorname{div}\left(\overline{C}^m_{i,n}\nabla\phi^m_n\right) = S^m_i(t,v_m,\nabla v_m), \\ d_i\,\dfrac{\partial \overline{C}^m_{i,n}}{\partial\nu} + m_i\overline{C}^m_{i,n}\,\dfrac{\partial\phi^m_n}{\partial\nu} = 0, \\ \overline{C}^m_{i,n}(0,x) = C_{i,0}(x). \end{cases}$$

As ϕ^m_n is uniformly bounded in $L^\infty(0,T,W^{1,\infty}(\Omega))$, we conclude the existence of $\phi_n \in L^\infty(0,T,W^{1,\infty}(\Omega))$ such that:

$$\nabla\phi^m_n \to \nabla\phi_n \quad \text{for the topology } \sigma\big(L^\infty(Q_T), L^1(Q_T)\big).$$

According to the classical results on compactness, from the sequence $\left(\overline{C}^m_{i,n}\right)_m$ we can extract a subsequence (still denoted $\left(\overline{C}^m_{i,n}\right)_m$) such that

$$\overline{C}^m_{i,n} \rightharpoonup \overline{C}_{i,n} \quad \text{weakly in } L^2\big(0,T,H^1(\Omega)\big)$$

which gives the following two convergences:

$$\begin{aligned} \overline{C}^m_{i,n} &\to \overline{C}_{i,n} && \text{strongly in } L^2\big(0,T,L^2(\Omega)\big) \text{ and almost everywhere in } (Q_T), \\ \nabla\overline{C}^m_{i,n} &\rightharpoonup \nabla\overline{C}_{i,n} && \text{weakly in } L^2\big(0,T,L^2(\Omega)\big). \end{aligned}$$

Next we deduce that

$$\overline{C}_{i,n}^{m} \to \overline{C}_{i,n} \quad \text{in } L^1(Q_T), \text{ and } D'(Q_T),$$

$$\bigg[\bigg|\int_{Q_T}(\overline{C}_{i,n}^{m}-\overline{C}_{i,n})\varphi\bigg| \le \big\|\overline{C}_{i,n}^{m}-\overline{C}_{i,n}\big\|_{L^2(0,T,L^2(\Omega))}\|\varphi\|_{L^2}\bigg].$$

Now let us prove that

$$\overline{C}_{i,n}^{m}\nabla\phi_n^m \to \overline{C}_{i,n}\nabla\phi_n \quad \text{in } D'(Q_T).$$

To this end we need to prove that

$$\overline{C}_{i,n}^{m}\nabla\phi_n^m \to \overline{C}_{i,n}\nabla\phi_n \quad \text{in the topology } \sigma\big(L^1(Q_T), L^\infty(Q_T)\big).$$

Let $\Psi \in L^\infty(Q_T)$. We have

$$\begin{aligned}&\int_0^T\int_\Omega \Psi\big(\overline{C}_{i,n}^{m}\nabla\phi_n^m - \overline{C}_{i,n}\nabla\phi_n\big)\,dx\,dt\\ &\quad= \int_0^T\int_\Omega \Psi\nabla\phi_n^m\big(\overline{C}_{i,n}^{m}-\overline{C}_{i,n}\big)\,dx\,dt + \int_0^T\int_\Omega \Psi\overline{C}_{i,n}\big(\nabla\phi_n^m-\nabla\phi_n\big)\,dx\,dt.\end{aligned}$$

Concerning the first term of the last equality, we have

$$\bigg|\int_0^T\int_\Omega \Psi\nabla\phi_n^m\big(\overline{C}_{i,n}^{m}-\overline{C}_{i,n}\big)\,dx\,dt\bigg| \le \|\Psi\|_{L^\infty}\|\nabla\phi_n^m\|_{L^\infty}\big\|\overline{C}_{i,n}^{m}-\overline{C}_{i,n}\big\|_{L^1(Q_T)}$$

as $\big\|\overline{C}_{i,n}^{m}-\overline{C}_{i,n}\big\|_{L^1(Q_T)} \to 0$ when $m\to\infty$. We deduce that

$$\int_0^T\int_\Omega \Psi\nabla\phi_n^m\big(\overline{C}_{i,n}^{m}-\overline{C}_{i,n}\big)\,dx\,dt \to 0 \quad \text{when } m\to\infty.$$

The second term tends to zero also since $\Psi\overline{C}_{i,n}\in L^1(Q_T)$ and $\nabla\phi_n^m$ converge to $\nabla\phi_n$ for the topology $\sigma\big(L^\infty(Q_T), L^1(Q_T)\big)$. Using

$$\big\|S_i^n(t,v_m,\nabla v_m) - S_i^n(t,v,\nabla v)\big\|_{L^2(Q_T)} \le \|v_m - v\|_{L^2(Q_T)} + \|\nabla v_m - \nabla v\|_{L^2(Q_T)},$$

we conclude that

$$\frac{\partial\overline{C}_{i,n}^{m}}{\partial t} - d_i\Delta\overline{C}_{i,n}^{m} - m_i\operatorname{div}\big(\overline{C}_{i,n}^{m}\nabla\phi_n^m\big) = S_i^n(t,v_m,\nabla v_m)$$

converge in $D'(Q_T)$ to

$$\frac{\partial\overline{C}_{i,n}}{\partial t} - d_i\Delta\overline{C}_{i,n} - m_i\operatorname{div}\big(\overline{C}_{i,n}\nabla\phi_n\big) = S_i^n(t,v,\nabla v).$$

Consequently, $\overline{C}_{i,n} = \pounds_n(v)$ then $\lim\limits_{m\to\infty}(v_m) = \overline{C}_{i,n} = \pounds_n(v)$.

It remains to prove that $\big(\overline{C}_{i,n}\big)\in W_0(0,T)$. We have

$$\text{(2.8)}\quad \begin{cases}\displaystyle\int_\Omega \frac{\partial\overline{C}_{i,n}}{\partial t}\varphi + a_{\phi_n}^i\big(t,\overline{C}_{i,n},\varphi\big) = \int_\Omega S_i^n(t,v,\nabla v)\varphi \quad \text{for all } \varphi\in H^1(\Omega),\\ \overline{C}_{i,n}(0,x) = C_{i,0}(x).\end{cases}$$

As $S_{i,n}(t,v,\nabla v) \in L^\infty(Q_T)$, it implies that the first term of (2.8) have meaning. We take $\varphi = \overline{C}_{i,n}$ and show that

$$\big\|\overline{C}_{i,n}(t)\big\|^2_{L^2(\Omega)} + \alpha_0 \int_0^t \big\|\overline{C}_{i,n}(s)\big\|^2_{H^1(\Omega)}\,ds \le \frac{C_{n,T}}{\alpha_0} + \|C_{i,0}\|^2_{L^2(\Omega)},$$
$$\sup_{0<t<T} \big\|\overline{C}_n(t)\big\|_{(L^2(\Omega))^{N_s}} + \big\|\overline{C}\big\|_{L^2(0,T,(H^1(\Omega))^{N_s})} \le C,$$

where

$$C = T\bigg(\frac{C_{n,T}}{\alpha_0} + \|C_0\|_{(L^2(\Omega))^{N_s}}\bigg).$$

Furthermore

$$\int_\Omega \frac{\partial \overline{C}^m_{i,n}}{\partial t}\,\varphi + a^i_{\phi_n}\big(t,\overline{C}^m_{i,n},\varphi\big) = \int_\Omega S^n_i(t,v_m,\nabla v_m)\,\varphi,$$
$$\bigg|\int_\Omega \frac{\partial \overline{C}^m_{i,n}}{\partial t}\,\varphi\bigg| \le C\big\|\overline{C}^m_{i,n}\big\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)} + \|S^n_i(t,v_m,\nabla v_m)\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)},$$
$$S^n_i(t,v,\nabla v) = e^{-\lambda t} f_{i,n}\big(\overline{v}e^{\lambda t},\nabla\overline{v}e^{\lambda t}\big),$$
$$f_{i,n}(u,p) = \psi_n\bigg(\sum_{1\le j\le N_s}|u_j| + \|p_j\|\bigg) f_i(u,p),$$
$$\big\|S^n_i(t,v,\nabla v)\big\|_{L^2(Q_T)} \le C_{n,T}, \quad \text{for } 0<t<T.$$

Hence

$$\bigg|\int_\Omega \frac{\partial \overline{C}^m_{i,n}}{\partial t}\,\varphi\bigg| \le \big(C\big\|\overline{C}^m_{i,n}\big\|_{H^1(\Omega)}\| + C_{n,T}\big)\|\varphi\|_{H^1(\Omega)},$$
$$\bigg\|\frac{\partial \overline{C}^m_{i,n}}{\partial t}\bigg\|_{H^{-1}(\Omega)} = \sup_{\varphi\in H^1(\Omega);\,\varphi\ne 0} \frac{1}{\|\varphi\|_{H^1(\Omega)}} \int_\Omega \frac{\partial \overline{C}^m_{i,n}}{\partial t}\,\varphi$$
$$\le C\big\|\overline{C}^m_{i,n}\big\|_{H^1(\Omega)} + C_{n,T},$$
$$\int_0^T \bigg\|\frac{\partial \overline{C}^m_{i,n}}{\partial t}\bigg\|^2_{H^{-1}(\Omega)} \le 2\int_0^T \big(C\big\|\overline{C}^m_{i,n}\big\|^2_{H^1(\Omega)} + C_{n,T}\big) \le C.$$

Then

$$\int_0^T \bigg\|\frac{\partial \overline{C}^m_{i,n}}{\partial t}\bigg\|_{L^2(0,T;H^{-1}(\Omega)^{N_s})} \le C.$$

This allows us to deduce $\big\|\partial\overline{C}^m_{i,n}/\partial t\big\|_{H^{-1}(\Omega)}$ is bounded in $L^2\big(0,T;H^{-1}(\Omega)^{N_s}\big)$. We know that we have the injections, $L^2\big(0,T;H^1(\Omega)^{N_s}\big) \subset L^2(Q_T)^{N_s}$ and $L^2\big(0,T;H^{-1}(\Omega)^{N_s}\big) \subset L^2(Q_T)^{N_s}$ are compact, consequently we conclude that $\pounds_n$ is a compact operator. □

Finally, the operator $\pounds_n$ admits a fixed point C_n such that (C_n,ϕ_n) is the solution we are seeking. Now, we have to prove the positivity of C_n. To this aim we introduce a mapping $Z_n = (Z_{i,n})_{1\le i\le N_s}$ defined by

$$Z_{i,n} = C_{i,n} \exp\left(\frac{m_i}{d_i}(\phi_n)\right) \quad \text{for } i = 1,\dots,N_s.$$

Moreover, we consider

$$p_{i,n} = \exp\left(\frac{m_i}{d_i}(\phi_n)\right) \quad \text{and} \quad q_{i,n} = \frac{1}{p_{i,n}}.$$

For every $v \in C^1(Q_T)$, $Z_{i,n}$ satisfies

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \frac{\partial(q_{i,n}Z_{i,n})}{\partial t} v + d_i \int_0^T \int_\Omega q_{i,n} \nabla Z_{i,n} \nabla v \\ \displaystyle\qquad\qquad = \int_0^T \int_\Omega f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)v, \\ Z_{i,n}(0,x) = Z_{i,0}(x)p_{i,n}(0,x) \quad \text{for all } x\in\Omega. \end{cases} \tag{2.9}$$

Let us introduce the function sign^- defined on $\mathbb{R}$ by

$$\text{sign}^- r = \begin{cases} -1 & \text{if } r<0, \\ 0 & \text{if } r \ge 0. \end{cases}$$

As sign^- is an increasing function, letting $\varepsilon > 0$ we may consider a convex function $\jmath_\varepsilon(s) \in C^2(\mathbb{R})$ such that $\jmath'_\varepsilon(r) \to \text{sign}^-(r)$ when $\varepsilon \to 0$ ([1]).

Let us take $\jmath'_\varepsilon(Z_{i,n})$ as a test function in (2.9). We then have

$$\begin{aligned} \int_0^t \int_\Omega \frac{\partial(q_{i,n}Z_{i,n})}{\partial t} \jmath'_\varepsilon(Z_{i,n}) = &- d_i \int_0^t \int_\Omega q_{i,n} \nabla Z_{i,n} \nabla(\jmath'_\varepsilon(Z_{i,n})) \\ &+ \int_0^t \int_\Omega f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)\jmath'_\varepsilon(Z_{i,n}). \end{aligned} \tag{2.10}$$

Let us note by I_1 and I_2, respectively, the two terms in the right side of equality (2.10). Using the convexity of $\jmath_\varepsilon$, we deduce that the first integral

$$I_1 = -d_i \int_0^t \int_\Omega q_{i,n} |\nabla Z_{i,n}|^2 \jmath''_\varepsilon(Z_{i,n}) \le 0.$$

([1]) A typical example of $\jmath_\varepsilon(s)$ can be given by

$$\jmath_\varepsilon(s) = \begin{cases} \displaystyle -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\exp\left(-\varepsilon \int_0^s \frac{t}{t-\varepsilon}\,dt\right) & \text{if } s<0, \\ 0 & \text{if } s \ge 0. \end{cases}$$

Concerning the second integral I_2, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2 &= \lim_{\varepsilon\to 0}\int_0^t\int_\Omega f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)\jmath'_\varepsilon(Z_{i,n})\\ &= \lim_{\varepsilon\to 0}\int_{(0,t)\times[Z_{i,n}>0]} f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)\jmath'_\varepsilon(Z_{i,n})\\ &\quad+\lim_{\varepsilon\to 0}\int_{(0,t)\times[Z_{i,n}<0]} f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)\jmath'_\varepsilon(Z_{i,n})\\ &= \lim_{\varepsilon\to 0}\int_{(0,t)\times[Z_{i,n}<0]} f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)\jmath'_\varepsilon(Z_{i,n}).\end{aligned}$$

By (2.2) we have

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2 &= -\int_{(0,t)\times[Z_{i,n}<0]} f_i^n(t,x,q_nZ_n,\nabla q_nZ_n)\\ &= -\int_{(0,t)\times[Z_{i,n}<0]} \eta_n(q_nZ_n)\widehat{f}_i(t,x,q_nZ_n,\nabla q_nZ_n)\\ &= -\int_{(0,t)\times[Z_{i,n}<0]} \eta_n(q_nZ_n)f_i\big(t,x,(q_nZ_n)^+,\nabla(q_nZ_n)^+\big).\end{aligned}$$

By $(\mathrm{H}_f)_2$, $\lim\limits_{\varepsilon\to 0} I_2\le 0$. Then

$$\lim_{\varepsilon\to 0}\int_0^t\int_\Omega \frac{\partial(q_{i,n}Z_{i,n})}{\partial t}\,\jmath'_\varepsilon(Z_{i,n})\le 0.$$

We have $q_{i,n}\ge 0$ and

$$\begin{aligned}\Lambda &= \int_0^t\int_\Omega \frac{\partial(q_{i,n}Z_{i,n})}{\partial t}\,\mathrm{sign}^-(Z_{i,n})\le 0,\\ \Lambda &= \int_0^t\int_\Omega \left(\frac{\partial[(q_{i,n}Z_{i,n})^+]}{\partial t}-\frac{\partial[(q_{i,n}Z_{i,n})^-]}{\partial t}\right)\mathrm{sign}^-(Z_{i,n})\le 0,\\ \Lambda &= -\int_0^t\int_\Omega \frac{\partial[(q_{i,n}Z_{i,n})^-]}{\partial t}\,\mathrm{sign}^-(Z_{i,n})\le 0,\\ \Lambda &= \int_0^t\int_\Omega \frac{\partial[(q_{i,n}Z_{i,n})^-]}{\partial t}\,\mathrm{sign}^-(Z_{i,n})\le 0,\end{aligned}$$

which implies

$$\int_\Omega (q_{i,n},Z_{i,n})^-(t,x)\le\int_\Omega (q_{i,n},Z_{i,n})^-(0,x).$$

As $(q_{i,n},Z_{i,n})(0,x)\ge 0$ for almost every $x\in\Omega$, we deduce that

$$\int_\Omega (q_{i,n},Z_{i,n})^-(t,x)\le 0.$$

Finally, $(q_{i,n},Z_{i,n})^-=0$ and then $Z_{i,n}\ge 0$ which allows us to conclude that $C_{i,n}\ge 0$ for every $i=1,\dots,N_s$.

2.2. A priori estimates.

Lemma 2.7. *There exists a constant C depending only on $\|C_0\|_{(L^2(\Omega))^{N_s}}$ such that*:

(a) $\displaystyle\sup_{0<t<T}\int_\Omega\sum_{i=1}^{N_s}|C_{i,n}(t,x)|\le C,$

(b) $\|\phi_n\|_{L^\infty(0,T,W^{1,\infty}(\Omega))}\le C.$

Proof. (a) Let us remember that (C_n,ϕ_n) is a solution of the approximate problem that satisfies:

$$\int_\Omega\frac{\partial C_{i,n}}{\partial t}\varphi+d_i\int_\Omega\nabla C_{i,n}\nabla\varphi+\int_\Omega m_iC_{i,n}\nabla\phi_n\nabla\varphi =\int_\Omega f_{i,n}(t,x,C_n,\nabla C_n)\varphi. \tag{2.11}$$

Taking $\varphi=1$ we obtain

$$\sum_{i=1}^{N_s}\int_\Omega\frac{\partial}{\partial t}C_{i,n}=\int_\Omega\sum_{i=1}^{N_s}f_{i,n}(t,x,C_n,\nabla C_n).$$

Hypothesis $(\mathrm{H}_f)_2$ implies

$$\sum_{i=1}^{N_s}\int_0^t\frac{d}{dt}\bigg[\int_\Omega C_{i,n}\bigg]\le 0,$$

$$\sum_{i=1}^{N_s}\int_\Omega C_{i,n}(t,x)\,dx-\sum_{i=1}^{N_s}\int_\Omega C_{i,n}(0,x)\,dx\le 0.$$

We have

$$\sum_{i=1}^{N_s}\int_\Omega|C_{i,n}(t,x)|\,dx\le C\bigg(\sum_{i=1}^{N_s}\|C_{i,0}\|_{L^2(\Omega)}\bigg).$$

Consequently,

$$\sup_{0<t<T}\int_\Omega\sum_{i=1}^{N_s}|C_{i,n}(t,x)|\,dx\le C.$$

(b) Let us remember that ϕ_n satisfy:

$$\begin{cases}-\varepsilon\Delta\phi_n(t,x)=\dfrac{\displaystyle\sum_{i=1}^{N_s}z_i\,C_{i,n}^{(n)}(t,x)\,e^{\lambda t}}{1+\varepsilon\displaystyle\sum_{i=1}^{N_s}C_{i,n}^{(n)}(t,x)e^{\lambda t}}-g(x) & \text{in } Q_T,\\[2ex] \phi_n(t,x)=0 & \text{in }]0,T[\,\times\partial\Omega.\end{cases}$$

Then

$$\phi_n(t,x) = \int_\Omega H(x,s)\left(\frac{\displaystyle\sum_{i=1}^{N_s} z_i\, C_{i,n}^{(n)}(t,s)\, e^{\lambda t}}{1+\varepsilon\displaystyle\sum_{i=1}^{N_s} C_{i,n}^{(n)}(t,s)\, e^{\lambda t}} - g(s)\right) ds.$$

We have

$$\left\|\sum_{i=1}^{N_s} z_i\, C_{i,n}^{(n)}\, e^{\lambda t}\Big/\left(1+\varepsilon\sum_{i=1}^{N_s} C_{i,n}^{(n)}\, e^{\lambda t}\right) - g\right\|_{L^\infty(Q_T)} \le C$$

and $\|\phi_n\|_{L^\infty(0,T,W^{1,\infty}(\Omega))} \le C$.

Considering the equations satisfied by $C_{i,n}, 1 \le i \le N_s$, we can write

$$-f_{i,n} = -\frac{\partial C_{i,n}}{\partial t} + d_i \Delta C_{i,n} + m_i\, \mathrm{div}(C_{i,n}\nabla\phi).$$

Integrating on Q_T, the positivity of the solutions yield, by putting $\varphi = 1$ in (2.11), that for all $1 \le i \le N_s$,

$$\begin{aligned}-\int_{Q_T} f_{i,n} &= -\int_{Q_T}\frac{\partial C_{i,n}}{\partial t} = -\int_0^T \frac{d}{dt}\int_\Omega C_{i,n}\\ &= -\int_\Omega C_{i,n}(T,x)\,dx + \int_\Omega C_{i,n}(0,x)\,dx \le \int_\Omega C_{i,n}(0,x)\,dx,\end{aligned}$$

because $C_{i,n}(T,x) \ge 0$. Hence, by hypothesis $(\mathrm{H}_f)_2$,

$$\int_{Q_T} |f_{1,n}(t,x,C_n,\nabla C_n)| \le \|C_{1,0}\|_{L^1(\Omega)}.$$

Similarly, we get by hypothesis $(\mathrm{H}_f)_2$, that for all $2 \le j \le N_s$,

$$\begin{aligned}\int_{Q_T}\Big|\sum_{1\le i\le j} f_{i,n}(t,x,C_n,\nabla C_n)\Big| &= \int_{Q_T}\Big(-\sum_{1\le i\le j} f_{i,n}\Big)\\ &\le \sum_{1\le i\le j}\int_\Omega C_{i,0}(x) = \sum_{1\le i\le j}\|C_{i,0}\|_{L_1(\Omega)}.\end{aligned}$$

Then

$$\int_{Q_T} |f_{j,n}(t,x,C_n,\nabla C_n)| \le R_1 = \sum_{1\le i\le j}(j-i+1)\|C_{i,0}\|_{L_1(\Omega)}. \qquad \square$$

Given a real positive number k, we set $T_k(s) = \max\{-k, \min(k,s)\}$ and $G_k(s) = s - T_k(s)$ and observe that for $0 \le s \le k$, $T_k(s) = s$ and $T_k(s) = k$ for $s > k$.

Lemma 2.8.

(a) *There exists a constant* R_2 *depending on* k *and* $\sum_{1\le i\le N_s}\|C_{i,0}\|_{L^1(\Omega)}$ *such that, for all* $1\le j\le N_s$,

$$\int_{Q_T}|\nabla T_k(C_{j,n})|^2\le R_2.$$

(b) *There exists a constant* R_3, *depending on* $\sum_{1\le j\le r}\|C_{j,0}\|_{L^2(\Omega)}$, *such that, for all* $2\le r\le N_s$,

$$\int_{Q_T}\bigg|\nabla T_k\bigg(\sum_{1\le j\le r}C_{j,n}\bigg)\bigg|^2\le R_3.$$

Proof. (a) We take $\varphi=T_k(C_{i,n})$ in (2.11), so

$$\int_\Omega\frac{\partial C_{i,n}}{\partial t}\,T_k(C_{i,n})+d_i\int_\Omega\nabla C_{i,n}\nabla T_k(C_{i,n})\\
+\int_\Omega m_iT_k(C_{i,n})\nabla\phi_n\nabla T_k(C_{i,n})=\int_\Omega f_{i,n}(t,x,C_n,\nabla C_n)T_k(C_{i,n}).$$

We set

$$S_k(r)=\int_0^r T_k(s)\,ds,$$

since $S_k(C_{j,n}(T))\ge 0$ and, for all $r\ge 0$, $|S_k(r)|\le k^2/2+k(r-k)^+$. We have

$$\int_\Omega\int_0^t\frac{\partial C_{i,n}}{\partial t}\,T_k(C_{i,n})+d_i\int_{Q_t}|\nabla T_k(C_{i,n})|^2\\
\le -m_i\int_{Q_t}T_k(C_{i,n})\nabla\phi_n\nabla T_k(C_{i,n})+k\int_{Q_t}|f_{i,n}|,$$

$$\int_\Omega S_k(C_{i,n}(t))+d_i\int_{Q_t}|\nabla T_k(C_{i,n})|^2\\
\le k\int_{Q_t}|f_{i,n}|+\int_\Omega S_k(C_{i,n}(0))+m_i\bigg|\int_{Q_t}T_k(C_{i,n})\nabla\phi_n\nabla T_k(C_{i,n})\bigg|,$$

where

$$m_i\bigg|\int_{Q_t}T_k(C_{i,n})\nabla\phi_n\nabla T_k(C_{i,n})\bigg|\le m_i\int_{Q_t}T_k(C_{i,n})\|\nabla\phi_n\|\|\nabla T_k(C_{i,n})\|\\
\le\varepsilon\int_{Q_t}|\nabla T_k(C_{i,n})|^2+\frac{Cm_i^2}{4\varepsilon}\int_{Q_t}|T_k(C_{i,n})|^2,$$

as $\|\nabla\phi_n\|_{L^\infty(Q_T)}\le C$. Then

$$(d_i-\varepsilon)\int_{Q_t}|\nabla T_k(C_{i,n})|^2\le kR_1+\frac{Cm_i^2}{4\varepsilon}\,k^2|Q_t|+\int_\Omega\bigg(\frac{k^2}{2}+k(C_{i,n}(0)-k)^+\bigg),$$

if $0 < \varepsilon < \min\limits_{1\le i\le N_s} d_i$, implies

$$\int_{Q_t} |\nabla T_k(C_{i,n})|^2 \le C(k, \|C_{i,n}(0)\|_{L^1(\Omega)}) = R_2.$$

(b) We denote $C_{r,n} = \sum\limits_{j=1}^{r} c_{j,n}$, $2 \le r \le N_s$,

$$\frac{\partial c_{j,n}}{\partial t} - d_j \Delta c_{j,n} - m_j \operatorname{div}(c_{j,n}\nabla\phi_n) = f_{j,n} \quad \text{for } 1 \le j \le N_s,$$

$$\frac{\partial}{\partial t}\sum_{j=1}^{r} c_{j,n} - \sum_{j=1}^{r} d_j\Delta c_{j,n} - \sum_{j=1}^{r} m_j \operatorname{div}(c_{j,n}\nabla\phi_n) = \sum_{j=1}^{r} f_{j,n} \quad \text{for } 1 \le j \le N_s,$$

$$\frac{\partial}{\partial t}\sum_{j=1}^{r} c_{j,n} - d_r\Delta\sum_{j=1}^{r} c_{j,n} + dr\sum_{j=1}^{r}\Delta c_{j,n} \\ - m_r \operatorname{div}\Big(\Big(\sum_{j=1}^{r} c_{j,n}\Big)\nabla\phi_n\Big) + m_r\sum_{j=1}^{r}\operatorname{div}(c_{j,n}\nabla\phi_n) = \sum_{j=1}^{r} f_{j,n},$$

$$\frac{\partial}{\partial t}\sum_{j=1}^{r} c_{j,n} - d_r\Delta\sum_{j=1}^{r} c_{j,n} + \sum_{j=1}^{r-1}(d_r - d_j)\Delta c_{j,n} \\ - m_r \operatorname{div}\Big(\Big(\sum_{j=1}^{r} c_{j,n}\Big)\nabla\phi_n\Big) + \sum_{j=1}^{r-1}(m_r - m_j)\operatorname{div}(c_{j,n}\nabla\phi_n) = \sum_{j=1}^{r} f_{j,n}.$$

We obtain

$$\frac{\partial C_{r,n}}{\partial t} - d_r\Delta C_{r,n} - m_r\operatorname{div}(C_{r,n}\nabla\phi_n) + \sum_{j=1}^{r-1}(d_r - d_j)\Delta c_{j,n} \\ + \sum_{j=1}^{r-1}(m_r - m_j)\operatorname{div}(c_{j,n}\nabla\phi_n) = \sum_{j=1}^{r} f_{j,n} \le 0.$$

Now, we multiply by $T_k(C_{r,n})$ and integrate on Q_T, we obtain

$$\int_{Q_T} T_k(C_{r,n})\frac{\partial C_{r,n}}{\partial t} + d_r\int_{Q_T}|\nabla T_k(C_{r,n})|^2 \\ + \sum_{j=1}^{r-1}(d_j - d_r)\int_{Q_T}\nabla T_k(C_{r,n})\nabla T_k(c_{j,n}) + \sum_{j=1}^{r} m_j\int_{Q_T} c_{j,n}\nabla\phi_n\nabla T_k(C_{r,n}) \le 0,$$

$$\int_\Omega S_k(C_{r,n}(T,x))dx - \int_\Omega S_k(C_{r,n}(0,x))\,dx + d_r\int_{Q_T}|\nabla T_k(C_{r,n})|^2 \\ + \sum_{j=1}^{r-1}(d_j - d_r)\int_{Q_T}\nabla T_k(C_{r,n})\nabla T_k(c_{j,n}) + \sum_{j=1}^{r} m_j\int_{Q_T} c_{j,n}\nabla\phi_n\nabla T_k(C_{r,n}) \le 0,$$

where

$$S_k(r) = \int_0^r T_k(s)\,ds, \qquad \frac{d}{dt}S_k(C_{r,n}) = T_k(C_{r,n})\,\frac{\partial C_{r,n}}{\partial t},$$

then

$$d_r \int_{Q_T} |\nabla T_k(C_{r,n})|^2 \le \sum_{j=1}^{r-1} |d_j - d_r| \big(|\nabla T_k(C_{r,n})|^2\big)^{1/2} \big(|\nabla T_k(c_{j,n})|^2\big)^{1/2}$$
$$+ \|\nabla\phi_n\|_{L^\infty} \bigg(\int_{Q_T} \bigg(\sum_{j=1}^{r} m_j c_{j,n}\bigg)^2\bigg)^{1/2} \big(|\nabla T_k(C_{r,n})|^2\big)^{1/2} + \int_\Omega S_k(C_{r,n}(0)),$$

for all $\varepsilon_1 > 0$ and for all $\varepsilon_2 > 0$, using the Young inequality, we have

$$d_r \int_{Q_T} |\nabla T_k(C_{r,n})|^2 \le \varepsilon_1 \int_{Q_T} |\nabla T_k(C_{r,n})|^2 + C_{\varepsilon_1} \sum_{j=1}^{r-1} |d_j - d_r| \int_{Q_T} |\nabla T_k(c_{j,n})|^2$$
$$+ \varepsilon_2 \int_{Q_T} |\nabla T_k(C_{r,n})|^2 + C_{\varepsilon_2} \|\nabla\phi_n\|_{L^\infty} \int_{Q_T} \bigg(\sum_{j=1}^{r} m_j c_{j,n}\bigg)^2 + \int_\Omega S_k(C_{r,n}(0)).$$

Using the fact that $\|\nabla\phi_n\|_{L^\infty(Q_T)} \le C$, we have

$$\int_{Q_T} |\nabla T_k(c_{j,n})|^2 \le R_2 \quad \text{and} \quad \int_{Q_T} (\sum_{j=1}^{r} m_j c_{j,n})^2 \le C.$$

Because $\|c_{j,n}\|_{L^2(0,T;H^1(\Omega))} \le C$ and, if $\varepsilon_1 + \varepsilon_2 = d_r/2$, which implies

$$\frac{d_r}{2} \int_{Q_T} |\nabla T_k(C_{r,n})|^2 \le R_2 C_{\varepsilon_1} \sum_{j=1}^{r-1} |d_j - d_r| + C + \int_\Omega S_k(C_{r,n}(0))$$

and

$$\int_{Q_T} \bigg|\nabla T_k\bigg(\sum_{1\le j\le r} C_{j,n}\bigg)\bigg|^2 \le R_3, \qquad R_3 = R_3\bigg(\sum_{j=1}^{r} \|C_{j,0}\|_{L^2(\Omega)})\bigg),$$

for $2 \le r \le N_s$. □

Lemma 2.9. *There exists a constant R_4 depending on $\sum_{1\le j\le N_s} \|c_{j,0}\|_{L^2(\Omega)}$, such that*

$$\int_{Q_T} |f_{i,n}(t,x,C_n,\nabla C_n)| \bigg(\sum_{1\le r\le N_s} (N_s - r + 1) C_{r,n}\bigg) \le R_4 \quad \textit{for all } 1 \le j \le N_s.$$

Proof. We have

$$\frac{\partial}{\partial t} c_{j,n} - d_j \Delta c_{j,n} - m_j \operatorname{div}(c_{j,n}\nabla\phi_n) = f_{j,n} \quad \text{for } 1 \le j \le N_s. \tag{2.12}$$

Set, for all $2 \le r \le N_s$,

$$\theta_n = \sum_{r=1}^{N_s} (N_s - r + 1) C_{r,n}, \qquad Z_n = \sum_{r=1}^{N_s} (N_s - r + 1) d_r C_{r,n},$$
$$\xi_n = \sum_{r=1}^{N_s} (N_s - r + 1) m_r C_{r,n}, \qquad R_{r,n} = -\sum_{r=1}^{N_s} f_{j,n}.$$

By hypothesis $(\mathrm{H}_f)_2$ we have

$$R_{r,n} = -\sum_{r=1}^{N_s} f_{j,n} \ge 0 \quad \text{for all } 2 \le r \le N_s.$$

Combining equations (2.12), we have

$$\frac{\partial}{\partial t}\theta_n - \Delta Z_n - \operatorname{div}(\xi_n \nabla \phi_n) + |f_{1,n}| + \sum_{r=2}^{N_s} R_{r,n} = 0.$$

Multiplying by θ_n and integrating on Q_T yield

$$\frac{1}{2}\int_\Omega \theta_n^2(T) + \int_{Q_T} \nabla Z_n \nabla \theta_n + \int_{Q_T} \theta_n |f_{1,n}| + \sum_{r=2}^{N_s} \int_{Q_T} \theta_n \Big| \sum_{j=1}^{r} f_{j,n} \Big| \le \frac{1}{2}\int_\Omega \theta_n^2(0) + \Big| \int_{Q_T} \xi_n \nabla \phi_n \nabla \theta_n \Big|.$$

Then

$$\int_{Q_T} \theta_n |f_{1,n}| + \sum_{r=2}^{N_s} \int_{Q_T} \theta_n \Big| \sum_{j=1}^{r} f_{j,n} \Big| \le \frac{1}{2}\int_\Omega \theta_n^2(0) + \int_{Q_T} |\nabla Z_n||\nabla \theta_n| + \|\nabla \phi_n\|_{L^\infty(Q_T)} \Big(\int_{Q_T} \xi_n^2 \Big)^{1/2} \Big(\int_{Q_T} (\nabla \theta_n)^2 \Big)^{1/2}.$$

Using the Young inequality, we conclude that

$$\int_{Q_T} \theta_n |f_{1,n}| + \sum_{r=2}^{N_s} \int_{Q_T} \theta_n \Big| \sum_{j=1}^{r} f_{j,n} \Big| \le \frac{1}{2}\int_\Omega \theta_n^2(0) + \frac{1}{2}\int_{Q_T} \big[|\nabla Z_n|^2 + |\nabla \theta_n|^2 \big] + \frac{1}{2}\|\nabla \phi_n\|_{L^\infty(Q_T)} \Big[\int_{Q_T} |\xi_n|^2 + \int_{Q_T} |\nabla \theta_n|^2 \Big],$$

$$\int_{Q_T} \theta_n |f_{1,n}| + \sum_{r=2}^{N_s} \int_{Q_T} \theta_n \Big| \sum_{j=1}^{r} f_{j,n} \Big| \le C.$$

Then

$$\int_{Q_T} \theta_n |f_{1,n}| \le C \quad \text{and} \quad \sum_{r=2}^{N_s} \int_{Q_T} \theta_n \Big| \sum_{j=1}^{r} f_{j,n} \Big| \le C \quad \text{for all } 2 \le r \le N_s.$$

We have

$$\int_{Q_T} \theta_n |f_{2,n}| \leq \int_{Q_T} \theta_n |f_{1,n} + f_{2,n}| + \int_{Q_T} \theta_n |f_{1,n}| \leq \widehat{C}$$

and, for all $2 \leq k \leq N_s$, we have

$$\int_{Q_T} \theta_n |f_{k,n}| \leq \int_{Q_T} \theta_n \bigg| \sum_{1 \leq j \leq k} f_{j,n} \bigg| + \int_{Q_T} \theta_n \bigg| \sum_{1 \leq j \leq k-1} f_{j,n} \bigg| \leq \widehat{C},$$

which gives us the result

$$\int_{Q_T} |f_{i,n}(t,x,C_n,\nabla C_n)| \bigg(\sum_{1 \leq r \leq N_s} (N_s - r + 1) C_{r,n} \bigg) \leq R_4$$

for all $1 \leq j \leq N_s$. □

2.3. Convergence. The point is to show that the solutions (C_n, ϕ_n) of the problem (2.4) converge to a solution (C, ϕ) to (1.1). By Lemmas 2.7 and 2.8 and according to a result of Barras, Hassan and Veron [10], the applications:

$$(C^n_{i,0}, f_{i,n}) \to C_{i,n}, \quad \text{for } 1 \leq i \leq N_s$$

are compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1(0,T; W^{1,1}_0)$. Therefore, we can deduce the existence of a subsequence denoted (C_n, ϕ_n), such that, for all $i = 1, \ldots, N_s$,

$$C_{i,n} \to C_i \quad \text{strongly in } L^1(Q_T), \tag{2.13}$$

$$C_{i,n} \to C_i \quad \text{almost everywhere in } Q_T, \tag{2.14}$$

as ϕ_n is uniformly bounded in $L^\infty(0,T,W^{1,\infty}(\Omega))$. We conclude the existence of $\phi \in L^\infty(0,T,W^{1,\infty}(\Omega))$ such that:

$$\nabla \phi_n \to \nabla \phi \quad \text{for the topology } \sigma(L^\infty(Q_T), L^1(Q_T)).$$

Let us prove now that $C_{i,n} \nabla \phi_n \to C_i \nabla \phi$ in $D'(Q_T)$. To this end we need to prove that

$$C_{i,n} \nabla \phi_n \to C_i \nabla \phi \quad \text{for the topology } \sigma(L^1(Q_T), L^\infty(Q_T)).$$

Let $\Psi \in L^\infty(Q_T)$, we have

$$\int_0^T \int_\Omega \Psi (C_{i,n} \nabla \phi_n - C_i \nabla \phi) \, dx \, dt = \int_0^T \int_\Omega \Psi \nabla \phi_n (C_{i,n} - C_i) \, dx \, dt + \int_0^T \int_\Omega \Psi C_i (\nabla \phi_n - \nabla \phi) \, dx \, dt.$$

Concerning the first term of the last equality, we have

$$\bigg| \int_0^T \int_\Omega \Psi \nabla \phi_n (C_{i,n} - C_i) \, dx \, dt \bigg| \leq \|\Psi\|_{L^\infty} \|\nabla \phi_n\|_{L^\infty} \|C_{i,n} - C_i\|_{L^1(Q_T)},$$

as $\|C_{i,n} - C_i\|_{L^1(Q_T)} \to 0$. We deduce that

$$\int_0^T \int_\Omega \Psi \nabla \phi_n (C_{i,n} - C_i) \, dx \, dt \to 0.$$

The second term tends to zero, too, because $\Psi C_i \in L^1(Q_T)$ and $\nabla\phi_n$ converge to $\nabla\phi$ in the topology $\sigma(L^\infty(Q_T), L^1(Q_T))$. Consequently,

$$\frac{\partial C_{i,n}}{\partial t} - d_i \Delta C_{i,n} - m_i \operatorname{div}(C_{i,n}\nabla\phi_n)$$

converge in $D'(Q_T)$ to

$$\frac{\partial C_i}{\partial t} - d_i \Delta C_i - m_i \operatorname{div}(C_i \nabla\phi), \qquad \phi_n(t,x) = \int_\Omega H(x,s)\theta_\varepsilon^n(t,s),$$

where

$$\theta_\varepsilon^n(t,x) = \left(\sum_{i=1}^{N_s} z_i C_{i,n}^{(n)}(t,x) \Big/ \left(1 + \varepsilon \sum_{i=1}^{N_s} C_{i,n}^{(n)}(t,x)\right) - g(x)\right).$$

By (2.12)–(2.14) and (2.3) we have

$$-\varepsilon\Delta\phi_n(t,\,\cdot\,) \to -\varepsilon\Delta\phi = \sum_{i=1}^{N_s} z_i C_i \Big/ \left(1 + \varepsilon\sum_{i=1}^{N_s} C_i\right) - g \quad \text{strongly in } L^1(\Omega).$$

Since $f_{1,n}, \dots, f_{N_s,n}$ is continuous, we have

$$f_{i,n}(t,x,C_n,\nabla C_n) \to f_i(t,x,C,\nabla C) \quad \text{almost everywhere in } Q_T, 1 \le i \le N_s.$$

This is not sufficient to ensure that (C,ϕ) is a solution of (1.1). In fact, we have to prove that the previous convergence takes place in $L^1(Q_T)$. In view of the Vitali theorem, in order to show that

$$f_{i,n}(t,x,C_n,\nabla C_n) \to f_i(t,x,C,\nabla C), \quad \text{for } 1 \le i \le m \text{ in } L^1(Q_T)$$

one needs to prove that $f_{i,n}(t,x,C_n,\nabla C_n)$ is equi-integrable in $L^1(Q_T)$. We have the following lemma.

Lemma 2.10. *For every $i = 1, \dots, N_s$, $f_{i,n}(t,x,C_n,\nabla C_n)$ are equi-integrable in $L^1(Q_T)$.*

The proof of this lemma requires the following result based on some properties of two time-regularization denoted by C_γ and C_σ $(\gamma,\sigma > 0)$, where $C \in L^2(0,T,H_0^1(\Omega))$is such that $C(0) = C_0 \in L^2(\Omega)$. Let $\gamma > 0$. We define

$$C_\gamma(t) = \exp(-\gamma t)C_0^\gamma + \gamma\int_0^t C(s)e^{\gamma(s-t)}\,ds$$

where C_0^γ is the solution of the elliptic problem

$$\begin{cases} C_0^\gamma - \dfrac{1}{\gamma}\Delta C_0^\gamma = C_0, \\ C_0^\gamma \in H_0^1(\Omega) \cap H^2(\Omega). \end{cases}$$

We can see that $(C_\gamma)_t = \gamma(C - C_\gamma)$ and $C_0^\gamma \to C_0$ in $L^2(\Omega)$, and we have the following lemma.

Lemma 2.11. *Let $C \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T],L^2(\Omega))$ such that $C(0) = C_0 \in L^2(\Omega)$. Then*

(a) *$C_\gamma \to C$ strongly in $L^2(0,T,H_0^1(\Omega))$.*
(b) *$|T_k(C)_\gamma(t)| \le k$.*

Proof. (a) This immediately follows from the result of Boccardo, Murat and Puel [14, Lemma 2.2, p. 377].

(b) By definition of regularization in time of $T_k(C)$, we have

$$T_k(C_\gamma)(t) = \exp(-\gamma t)T_k(C_0)^\gamma + \gamma \int_0^t (T_kC)(s)e^{\gamma(s-t)}\,ds$$

but we know that $|T_k(C)| \le k$, so it comes

$$\begin{aligned}|T_k(C_\gamma)(t)| &= ke^{(-\gamma t)} + \gamma \int_0^t ke^{\gamma(s-t)}\,ds \\ &\le ke^{-\gamma t} + k\gamma e^{-\gamma t}\left[\frac{e^{\gamma s}}{\gamma}\right]_0^T = ke^{-\gamma t} + k(1-e^{-\gamma t}) = k.\end{aligned}$$

Hence the result. □

By $w(\varepsilon)$ we denote a quantity such that $w(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $w^\sigma(\varepsilon)$ a quantity such that $w^\sigma(\varepsilon) \to 0$ for every fixed $\sigma > 0$ as $\varepsilon \to 0$.

Lemma 2.12. *Let $(C_n)_n$ be a sequence in $L^2(0,T;H_0^1(\Omega)) \cap C([0,T])$ such that $C_n(0) = C_0^n \in L^2(\Omega)$ and $(C_n)_t = \rho_{1,n} + \rho_{2,n}$ with $\rho_{1,n} \in L^2(0,T;H^{-1}(\Omega))$ and $\rho_{2,n} \in L^1(Q_T)$. Moreover, assume that C_n converges to C in $L^2(Q_T)$ and C_0^n converges to $C(0)$ in $L^2(\Omega)$. Let ψ be a function in $C^1([0,T])$ such that $\psi \ge 0$, $\psi' \le 0$, $\psi(T) = 0$ (2). Let φ be a Lipschitz increasing function in $C^0(\mathbb{R})$ such that $\varphi(0) = 0$ (3). Then, for all $k, \gamma > 0$,*

$$\begin{aligned}\langle \rho_{1,n}, \psi\varphi(T_k(C_n) - T_k(C_m)_\gamma)\rangle &+ \int_{Q_T} \rho_{2,n}\psi\varphi(T_k(C_n) - T_k(C_m)_\gamma) \\ &\ge w^{\gamma,n}\left(\frac{1}{m}\right) + w^\gamma\left(\frac{1}{n}\right) + \int_\Omega \psi(0)\phi(T_k(C) - T_k(C)_\gamma)(0)\,dx \\ &\quad - \int_\Omega (C)(0)\psi(0)\varphi(T_k(C) - T_k(C)_\gamma)(0),\end{aligned}$$

where $\phi(t) = \int_0^t \varphi(s)ds$ and $G_k(s) = s - T_k(s)$.

For the proof see [3, Lemma 7, p. 544].

Lemma 2.13. *Suppose that $C_{j,n}, C_j, 1 \le j \le N_s$ are as above.*

(2) For example $\psi(t) = e^{(T-t)} - 1$ if $0 \le t \le T$.
(3) For example $\varphi(s) = s/(1+|s|)$ answer the question $1 \ge \varphi'(s) = 1/(1+|s|)^2 > 0$.

(a) *If*

$$|f_{1,n}| \leq h_1(|C_{1,n}|)\Big[F_1(t,x) + \|\nabla C_{1,n}\|^2 + \sum_{2\leq j\leq N_s} \|\nabla C_j\|^{\alpha_j}\Big],$$

where $h_1\colon [0,+\infty[\ \to [0,+\infty[$ *is non-decreasing,* $F_1 \in L^1(Q_T)$ *and* $1 \leq \alpha_j < 2$. *Then, for each fixed* k,

$$\lim_{n\to\infty} \int_{Q_T} |\nabla T_k(C_{1,n}) - \nabla T_k(C_1)|^2 \chi_{\Big[E_n = \sum_{1\leq j\leq N_s} c_{j,n} \leq k\Big]} = 0.$$

(b) *If*

$$|f_{i,n}(t,x,C,\nabla C)| \leq h_i\bigg(\sum_{j=1}^{i} |C_j|\bigg)\Big[F_i(t,x) + \sum_{1\leq j\leq N_s} \|\nabla C_j\|^2\Big],$$

where $h_i\colon [0,+\infty[\ \to [0,+\infty[$ *is non-decreasing,* $F_i \in L^1(Q_T)$ *and* $2 \leq i < N_s$. *Then, for each fixed* k *and for all* $2 \leq i \leq N_s$,

$$\lim_{n\to\infty} \int_{Q_T} \bigg|\nabla T_k\bigg(\sum_{1\leq j\leq i} C_{j,n}\bigg) - \nabla T_k\bigg(\sum_{1\leq j\leq i} C_j\bigg)\bigg|^2 \chi_{\Big[E_n = \sum_{1\leq j\leq N_s} c_{j,n} \leq k\Big]} = 0.$$

Proof. (a) This is a direct consequence of the resulting output established in proof of Lemma 6 in [3, p. 548].

(b) Let k and $\gamma > 0$, let $\ell \in \mathbb{N}$, and choose Ψ as in previous. Let $\varphi(s) = se^{\mu s^2}$, with μ to be fixed later. Consider the following equation satisfied by $C_{1,n} + C_{2,n}$.

$$\frac{\partial}{\partial t}[C_{1,n} + C_{2,n}] - d_1 \Delta C_{1,n} - d_2 \Delta C_{2,n} \\ - m_1 \mathrm{div}(C_{1,n} \nabla \phi_n) - m_2 \mathrm{div}(C_{2,n} \nabla \phi_n) = f_{1,n} + f_{2,n},$$

$$\frac{\partial}{\partial t}[C_{1,n} + C_{2,n}] = d_2 \Delta (C_{1,n} + C_{2,n}) - (d_2 - d_1)\Delta C_{1,n} \\ + m_2 \mathrm{div}((C_{1,n} + C_{2,n})\nabla \phi_n) - (m_2 - m_1)\mathrm{div}(C_{1,n} \nabla \phi_n) + f_{1,n} + f_{2,n}.$$

Then use $\Psi\varphi(T_k(C_{1,n} + C_{2,n}) - T_K(C_{1,n} + C_{2,n})_\gamma)$ again as a test function and integrate on Q_T. Finally, we will use Lemma 2.12 to get the result. For simplicity, we denote

$$C_{r,n} = \sum_{1\leq j\leq r} c_{j,n}, \quad C_r = \sum_{1\leq j\leq r} c_j, \quad \text{for all } 2 \leq r \leq N_s.$$

Take $r = 2$; since

$$\frac{\partial}{\partial t} C_{2,n} = \rho_{1,n}^{(2)} + \rho_{2,n}^{(2)},$$

where

$$\begin{aligned}\rho^{(2)}_{1,n} &= d_2\Delta C_{2,n} - (d_2-d_1)\Delta c_{1,n} + m_2\mathrm{div}(C_{2,n}\nabla\phi_n)\\ &\qquad - (m_2-m_1)\mathrm{div}(c_{1,n}\nabla\phi_n) \in L^2(0,T,H^{-1}(\Omega)),\\ \rho^{(2)}_{2,n} &= f_{1,n}+f_{2,n} \in L^1(Q_T),\end{aligned}$$

we have, by (2.7),

$$\begin{aligned}\int_{Q_T}&\frac{\partial}{\partial t}(C_{2,n})\Psi\varphi(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &\geq w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + \int_\Omega \psi(0)\phi(T_k(C_2)-T_k(C_2)_\gamma)(0)\,dx\\ &\qquad - \int_\Omega (C_2)(0)\psi(0)\varphi(T_k(C_2)-T_k(C_2)_\gamma)(0),\end{aligned}$$

where

$$\phi(t) = \int_0^t \varphi(s)\,ds \quad \text{and} \quad G_k(s) = s - T_k(s).$$

We have

$$\begin{aligned}\int_{Q_T}&\frac{\partial}{\partial t}(C_{2,n})\Psi\varphi(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma) = \int_{Q_T}\rho^{(2)}_{1,n}\Psi\varphi(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &\quad - \int_{Q_T}\rho^{(2)}_{2,n}\Psi\varphi(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma) - \rho^{(2)}_{1,n}\\ &= -[d_2\Delta C_{2,n} - (d_2-d_1)\Delta c_{1,n} + m_2\mathrm{div}(C_{2,n}\nabla\phi_n)\\ &\quad - (m_2-m_1)\mathrm{div}(c_{1,n}\nabla\phi_n)] - \rho^{(2)}_{2,n} = -(f_{1,n}+f_{2,n}),\end{aligned}$$

$$\begin{aligned}-\int_{Q_T}&\rho^{(2)}_{1,n}\Psi\varphi(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &= d_2\int_{Q_T}\nabla C_{2,n}\Psi\varphi'(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &\quad - \int_{Q_T}(f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &\quad - (d_2-d_1)\\ &\quad \cdot\int_{Q_T}(\nabla c_{1,n}-\nabla c_1)\Psi\varphi'(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &\quad - (d_2-d_1)\\ &\quad \cdot\int_{Q_T}\nabla c_1)\Psi\varphi'(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\\ &\quad + m_2\int_{Q_T}C_{2,n}\nabla\phi_n\Psi\varphi'(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n})-T_k(C_{2,\ell})_\gamma)\end{aligned}$$

$$+ (m_2 - m_1)$$
$$\cdot \int_{Q_T} (c_{1,n} - c_1)\nabla\phi_n \Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)$$
$$+ (m_2 - m_1)$$
$$\cdot \int_{Q_T} c_1 \nabla\phi_n \Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma).$$

We denote

$$E_{r,n} = \sum_{1\le j\le r} c_{j,n}, \qquad E_r = \sum_{1\le j\le r} c_j, \quad \text{for all } 2 \le r \le N_s, \qquad E_2 = c_1 + c_2 = C_2,$$

$$I = d_2 \int_{Q_T} \nabla C_{2,n} \Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma),$$
$$\begin{aligned} I &= d_2 \int_{Q_T} \nabla T_k(C_{2,n}) \Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma) \\ &\quad + d_2 \int_{[E_n \ge n]} \nabla(C_{2,n}) \Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma) \\ &= I_1 + I_2. \end{aligned}$$

For I_2, we have

$$I_2 = - d_2 \int_{[E_n \ge n]} \nabla(C_{2,n}) \Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla T_k(C_{2,\ell})_\gamma \chi_{[E_n \ge k]},$$
$$\begin{aligned} I_2 &= w^{\gamma,n}\left(\frac{1}{n}\right) - d_2 \int_{Q_T} \nabla(C_{2,n}) \Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla T_k(C_2)_\gamma \chi_{[E_n \ge k]} \\ &= w^{\gamma,n}\left(\frac{1}{n}\right) \\ &\quad - d_2 \int_{Q_T} \nabla(C_{2,n}) \Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla T_k(C_2)_\gamma \chi_{[E_n \ge k]}\chi_{[E \ge k]} \\ &\quad - d_2 \int_{Q_T} \nabla(C_{2,n}) \Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla T_k(C_2)_\gamma \chi_{[E_n \ge k]}\chi_{[E < k]}, \end{aligned}$$
$$I_2 = w^{\gamma,n}\left(\frac{1}{n}\right) + I_{2,1} + I_{2,2}.$$

For $I_{2,1}$, we have by Cauchy–Schwartz inequality

$$|I_{2,1}| \le d_2 \|\nabla C_{2,n} \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\|_{L^2(Q_T)} \|\nabla T_k(C_2)_\gamma \chi_{[E_n \ge k]}\|_{L^2(Q_T)}.$$

$\varphi'(s) = \varphi'(|s|)$ and $|T_k(C_{2,n}) - T_k(C_2)_\gamma| \le 2k$ implies

$$\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \le \varphi'(2k).$$

We obtain

$$|I_{2,1}| \le d_2 C \|\nabla T_k(C_2)_\gamma \chi_{[E_n = C_2 \ge k]}\|_{L^2(Q_T)} = w\left(\frac{1}{\gamma}\right)$$

since

$$\begin{aligned} T_k(C_2)_\gamma \to T_k(C_2) &\quad \text{in } L^2(0,T;H_0^1(\Omega)),\\ \nabla T_k(C_2)_\gamma \chi_{[E_n=C_2\geq k]} = 0 &\quad \text{almost everywhere in } Q_T. \end{aligned}$$

Now, we study the term $I_{2,2}$:

$$\begin{aligned} I_{2,2} &= -\, d_2 \int_{Q_T} \nabla C_{2,n} \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla(T_k(C_2)_\gamma)\chi_{[E_n\geq k]}\chi_{[E_n=C_2\leq k]},\\ I_{2,2} &= w^\gamma\bigg(\frac{1}{n}\bigg), \end{aligned}$$

since $\chi_{[E_n\geq k]}\chi_{[E_n=C_2\leq k]} \to 0$ almost everywhere in Q_T. Thus

$$I_2 \geq w^{\gamma,n}\bigg(\frac{1}{\ell}\bigg) + w^\gamma\bigg(\frac{1}{n}\bigg) + w\bigg(\frac{1}{\gamma}\bigg).$$

We investigate

$$\begin{aligned} I_1 &= d_2 \int_{Q_T} \nabla T_k(C_{2,n})\Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma),\\ I_1 &= w^{\gamma,n}\bigg(\frac{1}{\ell}\bigg)\\ &\quad + d_2 \int_{Q_T} \nabla T_k(C_{2,n})\Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_2)_\gamma),\\ I_1 &= w^{\gamma,n}\bigg(\frac{1}{\ell}\bigg) + w^\gamma\bigg(\frac{1}{n}\bigg)\\ &\quad + d_2 \int_{Q_T} \nabla T_k(C_2)\Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_2)_\gamma)\\ &\quad + d_2 \int_{Q_T} \nabla(T_k(C_{2,n}) - T_k(C_2))\\ &\qquad\quad \cdot \Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_2)_\gamma)\\ &= w^{\gamma,n}\bigg(\frac{1}{\ell}\bigg) + w^\gamma\bigg(\frac{1}{n}\bigg) + w\bigg(\frac{1}{\gamma}\bigg)\\ &\quad + d_2 \int_{Q_T} \nabla(T_k(C_{2,n}) - T_k(C_2))\\ &\qquad\quad \cdot \Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_2)_\gamma),\\ I_1 &= w^{\gamma,n}\bigg(\frac{1}{\ell}\bigg) + w^\gamma\bigg(\frac{1}{n}\bigg) + w\bigg(\frac{1}{\gamma}\bigg)\\ &\quad + d_2 \int_{Q_T} |\nabla T_k(C_{2,n}) - \nabla T_k(C_2)|^2\Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\\ &\quad + d_2 \int_{Q_T} \nabla(T_k(C_{2,n}) - T_k(C_2))\\ &\qquad\quad \cdot \Psi\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_2)_\gamma \end{aligned}$$

$$= w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right)$$
$$+ d_2 \int_{Q_T} |\nabla T_k(C_{2,n}) - \nabla T_k(C_2)|^2 \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma),$$

then

$$I \geq w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right)$$
$$+ d_2 \int_{Q_T} |\nabla T_k(C_{2,n}) - \nabla T_k(C_2)|^2 \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma),$$

$$II = m_2 \int_{Q_T} C_{2,n} \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)$$
$$= w^{\gamma,n}\left(\frac{1}{\ell}\right)$$
$$+ m_2 \int_{Q_T} C_{2,n} \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_2)_\gamma)$$
$$= w^{\gamma,n}\left(\frac{1}{\ell}\right)$$
$$+ m_2 \int_{Q_T} (C_{2,n} - C_2) \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_2)_\gamma)$$
$$+ m_2 \int_{Q_T} C_2 \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_2)_\gamma).$$

$$II = w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right)$$
$$+ m_2 \int_{Q_T} C_2 \nabla \phi_n \Psi \varphi'(T_k(C_2) - T_k(C_2)_\gamma) \nabla (T_k(C_2) - T_k(C_2)_\gamma)$$
$$+ m_2 \int_{Q_T} (C_{2,n} - C_2) \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_2)_\gamma),$$

$$II = w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right)$$
$$+ m_2 \int_{Q_T} (C_{2,n} - C_2) \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_2)_\gamma)$$
$$+ m_2 \int_{Q_T} (C_{2,n} - C_2) \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \nabla (T_k(C_2) - T_k(C_2)_\gamma),$$

$$II = w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right) + (1) + (2)$$

or

$$C_{2,n} \to C_2 \quad \text{in } L^2(Q_T) \quad \text{and} \quad \int_{Q_T} |\nabla T_k(C_{2,n})|^2 \leq R_2,$$

and $\|\nabla\phi_n\|_{L^\infty}$ is bounded then (1) + (2) = $w^\gamma(1/n)$. So

$$II = w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right).$$

For J, we have

$$\begin{aligned} J &= -\int_{Q_T} (f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma), \\ J &= w^{\gamma,n}\left(\frac{1}{\ell}\right) - \int_{Q_T} (f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma) \\ &= w^{\gamma,n}\left(\frac{1}{\ell}\right) - \int_{E_{2,n}\geq k} (f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma) \\ &\quad - \int_{E_{2,n}\leq k} (f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma). \end{aligned}$$

Then

$$J \geq w^{\gamma,n}\left(\frac{1}{\ell}\right) - \int_{E_{2,n}\leq k} (f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)$$

since $\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma) \geq 0$ on $[E_2 \geq k]$ and $-(f_{1,n}+f_{2,n}) \geq 0$ by hypothesis $(\mathrm{H}_f)_2$.

On the other hand

$$\begin{aligned} &\left| \int_{E_{2,n}\leq k} (f_{1,n}+f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma) \right| \\ &\quad \leq h_1(k) \int_{E_{2,n}\leq k} F_1(t,x)\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\ &\quad\quad + h_1(k) \int_{E_{2,n}\leq k} |\nabla c_2|^{\alpha_2}\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\ &\quad\quad + h_1(k) \int_{E_{2,n}\leq k} |\nabla T_k(c_{1,n})|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\ &\quad\quad + h_2(k) \int_{E_{2,n}\leq k} F_2(t,x)\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\ &\quad\quad + h_2(k) \int_{E_{2,n}\leq k} |\nabla T_k(c_{2,n})|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\ &\quad = J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

We set

$$J_1 = h_1(k) \int_{E_{2,n}\leq k} F_1(t,x)\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| = w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right).$$

Similarly, for J_4,

$$J_4 = h_2(k) \int_{E_{2,n}\leq k} F_2(t,x)\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| = w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right),$$

$$J_2 = h_1(k)\int_{E_{2,n}\le k} |\nabla c_2|^{\alpha_2}\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| = w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right),$$

$$\begin{aligned}
J_3 &= h_1(k)\int_{E_{2,n}\le k} |\nabla T_k(c_{1,n})|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\
&= h_1(k)\int_{E_{2,n}\le k} |\nabla T_k(c_{1,n}) - \nabla T_k(c_1)|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\
&\quad + 2h_1(k)\int_{E_{2,n}\le k} \nabla T_k(c_{1,n})\nabla T_k(c_1)\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| \\
&\quad - h_1(k)\int_{E_{2,n}\le k} |\nabla T_k(c_1)|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)|, \\
J_3 &= w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right) \\
&\quad + h_1(k)\int_{E_{2,n}\le k} |\nabla T_k(c_{1,n}) - \nabla T_k(c_1)|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)|
\end{aligned}$$

and

$$J_5 = h_2(k)\int_{E_{2,n}\le k} |\nabla T_k(c_{2,n})|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)| = w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right)$$

since

$$\int_{E_{2,n}\le k} |\nabla T_k(c_{2,n})|^2 \le \liminf_{n\to+\infty}\left(\int_{E_{2,n}\le k} |\nabla T_k(c_{2,n})|^2\right) \le R_2.$$

Thus

$$\begin{aligned}
-\int_{E_{2,n}\le k} (f_{1,n} + f_{2,n})\Psi\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma) &\ge w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right) \\
&\quad - h_1(k)\int_{E_{2,n}\le k} |\nabla T_k(c_{1,n}) - \nabla T_k(c_1)|^2\Psi|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)|.
\end{aligned}$$

For λ and $\lambda\lambda$, we have

$$\begin{aligned}
\lambda &= -(d_2 - d_1) \\
&\quad \cdot \int_{Q_T} (\nabla c_{1,n} - \nabla c_1)\Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma), \\
\lambda &= w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right), \\
\lambda\lambda &= -(m_2 - m_1) \\
&\quad \cdot \int_{Q_T} (c_{1,n} - c_1)\nabla\phi_n\Psi\varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma)\nabla(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma), \\
\lambda\lambda &= w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^\gamma\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right).
\end{aligned}$$

We have β and $\beta\beta$

$$\beta = -(d_2 - d_1) \int_{Q_T} \nabla c_1 \Psi \varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma),$$

$$\beta = w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right),$$

$$\beta\beta = -(m_2 - m_1) \cdot \int_{Q_T} c_1 \nabla \phi_n \Psi \varphi'(T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma) \nabla (T_k(C_{2,n}) - T_k(C_{2,\ell})_\gamma),$$

$$\beta\beta = w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right).$$

Then

$$\begin{aligned} I + II + J + JJ + \lambda + \lambda\lambda + \beta + \beta\beta \geq {} & w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right) \\ & + d_2 \int_{Q_T} |\nabla (T_k(C_{2,n}) - T_k(C_2))|^2 \Psi \varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) \\ & - h_1(k) \int_{E_{2,n\leq k}} |\nabla (T_k(C_{1,n}) - T_k(C_1))|^2 \Psi |\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)|. \end{aligned}$$

We choose $\mu \geq (h_1(k)/(2d_2))^2$. Then we have $d_2\varphi'(s) - h_1(k)|\varphi(s)| > d_2/2$ and we conclude that

$$\begin{aligned} \int_{Q_T} & |\nabla (T_k(C_{2,n}) - T_k(C_2))|^2 \\ & \cdot \Psi[\varphi'(T_k(C_{2,n}) - T_k(C_2)_\gamma) - h_1(k)|\varphi(T_k(C_{2,n}) - T_k(C_2)_\gamma)|] \\ & \leq w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right). \end{aligned}$$

Then we have

$$\lim_{n\to+\infty} \int_{Q_T} |\nabla (T_k(C_{2,n}) - T_k(C_2))|^2 \chi_{[E_{2,n}\leq k]} = 0.$$

We get step by step by considering the equation satisfied by $C_{r,n} = \sum_{j=1}^{r} c_{j,n}$. Arguing in the same way as before, choosing $\mu \geq \max\{(h_1(k)/(2d_j))^2 : 1 \leq j \leq r\}$, we obtain

$$\begin{aligned} \int_{Q_T} & |\nabla (T_k(C_{r,n}) - T_k(C_r))|^2 \\ & \cdot \Psi[\varphi'(T_k(C_{r,n}) - T_k(C_r)_\gamma) - h_1(k)|\varphi(T_k(C_{r,n}) - T_k(C_r)_\gamma)|] \\ & \leq w^{\gamma,n}\left(\frac{1}{\ell}\right) + w^{\gamma}\left(\frac{1}{n}\right) + w\left(\frac{1}{\gamma}\right) \end{aligned}$$

which shows the desired result (b)

$$\lim_{n\to+\infty}\int_{Q_T}|\nabla(T_k(C_{r,n})-\nabla T_k(C_r))|^2\chi_{\left[E_{r,n}=\sum\limits_{j=1}^{r}c_{j,n}\le k\right]}=0. \qquad \square$$

PROOF OF LEMMA 2.10. Let A be a measurable subset Q_T. For all $k\ge 0$, we have

$$\begin{aligned}
&\int_A|f_{1,n}(t,x,C_n,\nabla C_n)|\,dx\\
&=\int_{A\cap[E_{N_s,n}\le k]}|f_{1,n}(t,x,C_n,\nabla C_n)|\,dx+\int_{A\cap[E_{N_s,n}>k]}|f_{1,n}(t,x,C_n,\nabla C_n)|\,dx\\
&\le\int_{A\cap[E_n\le k]}|f_{1,n}(t,x,C_n,\nabla C_n)|\,dx+\int_{A\cap[\theta_n>k]}|f_n(t,x,C_n,\nabla C_n)|\,dx
\end{aligned}$$

with

$$E_{N_s,n}=\sum_{1\le j\le N_s}C_{j,n}\quad\text{and}\quad\theta_n=\sum_{1\le k\le N_s}(N_s-k+1)C_{k,n}.$$

We obtain, for all $\varepsilon>0$, there exists k_0 such that, if $k\ge k_0$, then for all n

$$\frac{1}{k}\int_{A\cap[E_{N_s,n}>k]}k|f_{1,n}|\,dx\le\frac{1}{k}\int_{Q_T}E_{N_s,n}|f_{1,n}|\,dx\le\frac{1}{k}\int_{Q_T}\theta_n|f_{1,n}|\,dx.$$

We have

$$\int_{A\cap[E_{N_s,n}>k]}|f_{1,n}(t,x,C_n,\nabla C_n)|\,dx\le\frac{\varepsilon}{N_s+2},$$

$$\begin{aligned}
\int_A&|f_{1,n}(t,x,C_n,\nabla C_n)|\,dx\\
&\le\frac{\varepsilon}{N_s+2}+h_1(k)\Big[\int_A F_1(t,x)+\int_{A\cap[E_{N_s,n}\le k]}|\nabla C_{1,n}|^2\Big]\\
&\quad+h_1(k)\sum_{2\le j\le N_s}\Big(\int_{A\cap[E_{N_s,n}\le k]}|\nabla C_{j,n}|^{\alpha_j}\Big)\\
&\le\frac{\varepsilon}{2}+h_1(k)\Big[\int_A F_1(t,x)+\int_{A\cap[E_{N_s,n}\le k]}|\nabla T_k(C_{1,n})|^2\Big]\\
&\quad+h_1(k)\sum_{2\le j\le N_s}\Big(\int_{A\cap[E_{N_s,n}\le k]}|\nabla T_k(C_{j,n})|^{\alpha_j}\Big).
\end{aligned}$$

Using Hölder's inequality, for $1\le\alpha_j<2$, we obtain

$$\begin{aligned}
h_1(k)\int_{A\cap[E_{N_s,n}\le k]}|\nabla T_k(C_{j,n})|^{\alpha_j}&\le h_1(k)\Big(\int_A|\nabla T_k(C_{j,n})|^2\Big)^{\alpha_j/2}|A|^{(2-\alpha_j)/2}\\
&\le h_1(k)R_2^{\alpha_j/2}|A|^{(2-\alpha_j)/2}\le\frac{\varepsilon}{N_s+2}.
\end{aligned}$$

Whenever $|A| \le \rho_j$ with

$$\rho_j = \left(\frac{\varepsilon}{N_s+2}\, h_1^{-1}(k) R_2^{-\alpha_j/2}\right)^{2/(2-\alpha_j)}, \quad 2 \le j \le N_s$$

and

$$\int_{A\cap[E_n\le k]} |\nabla T_k(C_n)|^2 = \int_{A\cap[E_n\le k]} |[\nabla T_k(C_n) - \nabla T_k(C)] + \nabla T_k(C)|^2,$$
$$\int_{A\cap[E_n\le k]} |\nabla T_k(C_n)|^2 \le 2\int_{A\cap[E_n\le k]} |\nabla T_k(C_n) - \nabla T_k(C)|^2 + 2\int_A |\nabla T_k(C)|^2,$$

we have

$$\int_{Q_T} |\nabla T_k(C_n)|^2 \le C(k, \|C_0\|_{L^1(\Omega)}),$$

so

$$|\nabla T_k(C_n) - \nabla T_k(C)|^2 \chi_{[C_n\le k]} \to 0 \quad \text{in } L^1(\Omega) \text{ strongly}$$

and $|\nabla T_k(C_n) - \nabla T_k(C)|^2 \chi_{[C_n\le k]}$ is equi-integrable in $L^1(\Omega)$. Then there exists $\rho_{N_s+1} > 0$ such that, if $|A| \le \rho_{N_s+1}$, then

$$2h_1(k)\int_{A\cap[E_{N_s,n}\le k]} |\nabla T_k(C_n) - \nabla T_k(C)|^2 < \frac{\varepsilon}{N_s+2}.$$

On the other hand, $F_1, |\nabla T_k(C_1)|^2 \in L^1(\Omega)$, therefore there exists ρ_{N_s+2} such that if $|A| \le \rho_{N_s+2}$ we have

$$C_1\left(2\int_A |\nabla T_k(C_1)|^2 + \int_A F_1(t,x)\right) < \frac{\varepsilon}{N_s+2}.$$

Choose $\rho_0 = \inf\{\rho_j, 2 \le j \le N_s+2\}$, $|A| \le \rho_0$, we obtain

$$\int_A |f_{1,n}(t,x,C_n,\nabla C_n)|\,dx \le \varepsilon.$$

Similarly, we get for $2 \le i \le N_s$

$$\begin{aligned}\int_A |f_{i,n}| \le{}& \frac{\varepsilon}{N_s+2} + h_i(k)\bigg(\int_A F_i(t,x) \\ &+ \int_{A\cap[E_{N_s,n}\le k]} (6|\nabla C_1|^2 + 6|\nabla T_k(C_{1,n}) - \nabla T_k(C_1)|^2)\bigg) \\ &+ 8h_i(k)\sum_{2\le r\le N_s}\bigg(\int_{A\cap[E_{N_s,n}\le k]} \bigg|\nabla T_k\bigg(\sum_{1\le j\le r} C_j\bigg)\bigg|^2\bigg) \\ &+ 8h_i(k)\sum_{2\le r\le N_s}\bigg(\int_{A\cap[E_{N_s,n}\le k]} \bigg|\nabla T_k\bigg(\sum_{1\le j\le r} C_{j,n}\bigg) - \nabla T_k\bigg(\sum_{1\le j\le r} C_j\bigg)\bigg|^2\bigg).\end{aligned}$$

Arguing in the same way as before, we obtain the required result. □

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