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CONTRACTIBILITY OF MANIFOLDS BY MEANS OF STOCHASTIC FLOWS

Alexandra Antoniouk — Sergiy Maksymenko

Abstract. In the paper [Probab. Theory Relat. Fields, **100** (1994), 417–428] Xue-Mei Li has shown that the moment stability of an SDE is closely connected with the topology of the underlying manifold. In particular, she gave sufficient condition on SDE on a manifold M under which the fundamental group $\pi_1 M = 0$. We prove that under similar analytical conditions the manifold M is contractible, that is all homotopy groups $\pi_n M$, $n \geq 1$, vanish.

1. Introduction

The interplay between geometrical or topological structures of a manifold and the properties of differential operations on it forms a library of the most crucial results in analysis. For instance,

(1) if M is closed, then the number of (non-degenerate) critical points of index i of a Morse function $f\colon M \to \mathbb{R}$ bounds the rank of i-th homology group of M (Morse inequalities);
(2) de Rham cohomologies $H^*_{\mathrm{DR}}(M)$ of an orientable manifold M are isomorphic with its singular real cohomologies $H^*(M,\mathbb{R})$, (de Rham theory);

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(3) if M is Riemannian, then there is a lot of statements relating its Ricci and sectional curvatures with the topology of M and especially with the fundamental group $\pi_1 M$, (theorems by Cartan–Hadamard, Bonnet–Myers, Preissman, Byers, Bochner);
(4) for a vector field F on M having only isolated zeros the alternating sum of indexes of those zeros equals the Euler characteristic of M, (Poincaré–Hopf theorem);
(5) topological entropy of smooth dynamical systems on M can also be computed via Lyapunov exponents (Margulis–Ruelle Inequality, Pesin entropy formula).

The invention of the stochastic analysis since the milestone papers of Wiener and Ito gave rise a problem of finding stochastic counterparts of the above results. In 1962 Ito [9] (see also [10]) introduced a notion of a stochastic parallel transport which generalizes parallel transport in differential geometry. These ideas permitted further development of stochastic analysis on manifolds in the papers by Eells and Elworthy [6], Malliavin [18], Airault [1], Vauthier [20], Berthier and Gaveau [5], and many others. The principal problem which appears there is that the corresponding objects depend only *measurably* and not *continuously* on probabilistic parameters. This essentially prevents a usage of well developed homotopy invariants.

On the other hand, for a Riemannian manifold the Laplace operator Δ uniquely defines a Brownian motion on M. This allowed to prove analogues of results of type (3) in terms of stochastic differential equation (SDE) on M.

Another approach is based on extending results of type (5). Recall that a *maximal Lyapunov exponent* of a diffeomorphism $h\colon M \to M$ of a Riemannian manifold M at a point $x_0 \in M$ is defined by

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \|h'(x_i)\| \tag{1.1}$$

where $\|h'(x)\|$ is the norm of the tangent linear map $T_x h\colon T_x M \to T_{h(x)} M$, and $x_i = h^i(x_0)$ is an i-th iteration of x_0 under h. Thus if the limit (1.1) exists, then for large n we have that

$$e^{\lambda(x_0)} \approx \sqrt[n]{\prod_{i=0}^{n} \|h'(x_i)\|},$$

that is $e^{\lambda(x_0)}$ is an average value of the norm of the tangent map along the orbit of x_0.

In particular, if $\lambda(x_0) < 0$, then (saying non-strictly) "*in average, the orbit of x_0 should attract points that are sufficiently close to it*". Hence the points with negative Lyapunov exponents would detect attractors of dynamical systems.

Moreover, suppose there exists a diffeomorphism $h\colon M \to M$ isotopic to the identity (e.g. a diffeomorphism belonging to a flow) which has also negative Lyapunov exponents at some large subsets of M. Then one would expect that representatives of certain (co)homology classes of M can be deformed under iterations of h into subsets of small sizes, and therefore those classes could vanish. In other words, one would get triviality of some (co)homology or homotopy groups of M.

Stochastic analogues of Lyapunov exponents, the so called *p-moment exponents*, play a crucial role for investigation of stability of stochastic dynamical systems, see e.g. [11]–[14], [2]–[4], [7] and others.

Let ξ be a stochastic flow on M being a solution of SDE with smooth coefficients. Roughly speaking it is a family of differentiable flows depending on some parameter ω belonging to a probability space Ω. Then given a compact subset $K \subset M$ and $p > 0$ define the *p-th moment exponent* of ξ on K by

$$\mu_K(p) := \overline{\lim_{t\to\infty}} \sup_{x\in K} \frac{1}{t} \ln \mathbf{E}\|T_x\xi_t\|^p.$$

A stochastic flow ξ is called *p-th moment stable* whenever $\mu_x(p) < 0$ for each $x \in M$ and *strongly p-th moment stable* if $\mu_K(p) < 0$ for each compact subset $K \subset M$.

Then the strong p-th moment stability would imply that, in average, the flow decreases the sizes of compact sets. In particular, it was shown by Elworthy and Rosenberg [7] that for a compact manifold M

- $\mu_M(1) < 0$ implies triviality of the fundamental group $\pi_1 M$;
- $\mu_M(2) < 0$ implies triviality of the second homotopy group $\pi_2 M$;
- $\mu_M(q) < 0$ implies triviality of q-th homology group $H_q(M, \mathbb{Z}) = 0$;
- if $\mu_M([(n+1)/2]) < 0$, then M is a homotopy sphere.

For non-compact manifolds the situation is more complicated as a priori one can not expect uniform bounds for $\mu_K(p)$. That case was considered by Xue-Mei Li [17]. She studied moment stability of SDE of the form

$$dx_t = X(x_t) \circ dB_t + A(x_t)\,dt, \tag{1.2}$$

where B_t is an m-dimensional Brownian motion on $\mathcal{T}$, A is a vector field on M, and $X \in \mathrm{Hom}(\underline{\mathbb{R}}^n, TM)$ is a bundle homomorphism of class C^3 from trivial $\mathbb{R}^n$-bundle $\underline{\mathbb{R}}^n = \mathbb{R}^n \times M \to M$ over M to its tangent bundle $TM \to M$. Among other results she gave sufficient conditions for triviality of the fundamental group $\pi_1 M$ of the manifold M in terms of the coefficients of (1.2), see Theorem 1.1 below.

It is well known that under certain conditions SDE (1.2) generates a *stochastic flow*, that is a family of diffeomorphisms $\xi_{t,\omega}\colon M \to M$ depending on the time $t \in \mathbb{R}$ and a probabilistic parameter $\omega \in \Omega$. However, in general, ξ is not jointly continuous in $(x,t) \in M \times \mathbb{R}$ or even in $t \in \mathbb{R}$. A stochastic flow $\{\xi_{t,\omega}\}$ is called *strongly* 1*-complete* if for each smooth curve $\gamma\colon [0,1] \to M$ the map $\widehat{\gamma}\colon [0,1] \times \mathbb{R} \to M$, defined by $\widehat{\gamma}(s,t) = \xi_\omega(\gamma(s),t)$ is jointly continuous in t and x for almost all $\omega \in \Omega$. In particular, $\widehat{\gamma}_t$ is homotopic to $\widehat{\gamma}_0 = \gamma$.

The following statement is proved in [17, Theorem 4.1]. We present a slightly different formulation.

THEOREM 1.1 (c.f. [17, Theorem 4.1]). *Let M be a complete possibly non-compact connected Riemannian manifold and $h\colon M \to \mathbb{R}$ be a smooth function. Suppose there exists a strongly* 1*-complete recurrent h-Brownian system generating a stochastic flow ξ such that for each compact subset $K \subset M$*

$$\int_0^{+\infty} \sup_{x \in K} \mathbf{E}\, \|T_x \xi_{t,\omega}\|\, dt < \infty. \tag{1.3}$$

Then $\pi_1 M = 0$.

We refer the reader for precise definitions to the original paper, however let us briefly discuss the principal "topological" steps of the proof of Theorem 1.1. At first Xue-Mei Li deduces from inequality (1.3) (being a variant of a 1-st moment stability for non-compact manifolds) that every smooth loop σ on M is deformed by the flow to a loop of arbitrary small diameter. Then the recurrence assumption implies that σ will return into arbitrary compact set K at arbitrary large times with positive probability. For K being a geodesic ball it then follows that σ is null-homotopic.

In the present paper we extend Theorem 1.1 by clarifying topological assumptions needed for its proof. Moreover, we will show that if one can interchange "sup" and "$\mathbf{E}$" in (1.3) then *all* the homotopy groups $\pi_n M$ vanish, that is M is contractible, see Theorem 2.7 below.

2. Preliminaries

We start with a usual setting of the theory of SDE, see e.g. [16, §§1.2, 1.4, 4]. Given a map $f\colon A \times B \times C \to D$ of a product of sets we will often consider *restriction* maps obtained by fixing some coordinates, e.g. $f_a\colon \{a\} \times B \times C \to D$ defined by $f_a(b,c) = f(a,b,c)$, or $f_{a,b}\colon \{a\} \times \{b\} \times C \to D$, $f_{a,b}(c) = f(a,b,c)$, for $(a,b,c) \in A \times B \times C$. Thus we put the corresponding fixed coordinates as subindexes.

A *measurable space* $(\Omega, \mathcal{F})$ is a set Ω with a σ-algebra $\mathcal{F}$ of subsets.

Let $\mathbf{P}\colon \mathcal{F} \to [0,1]$ be a σ-additive measure, then the triple $(\Omega, \mathcal{F}, \mathbf{P})$ is called a *probability space*. If $(\Omega', \mathcal{F}')$ is another measurable space, then a map $f\colon \Omega \to \Omega'$ is called $\mathcal{F}/\mathcal{F}'$-measurable, if $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{F}'$.

Each topological space M can be regarded as a measurable space $(M, \mathcal{B}(M))$, where $\mathcal{B}(M)$ is the Borel σ-algebra of subsets of M.

Let N be another topological space and $(\Omega,\mathcal{F})$ be a measurable space. Then a *random N-valued field with parameter M* is a map $f\colon M\times\Omega\to N$ such that $f_x\colon\Omega\to N$ is $\mathcal{F}/\mathcal{B}(N)$-measurable for each $x\in M$.

A random field $f\colon M\times\Omega\to N$ is

- *measurable* if f is $(\mathcal{F}\otimes\mathcal{B}(M))/\mathcal{B}(N)$-measurable;
- *continuous* if $f_\omega : M\to N$ is continuous for almost all $\omega\in\Omega$;
- *bounded* if N is a metric space and $f_\omega(M)$ is bounded in N for almost all $\omega\in\Omega$.

Let $f'\colon M\times\Omega\to N$ be another random N-valued field with parameter M and $J_x=\{\omega\in\Omega\mid f_x(\omega)=f'_x(\omega)\}$. Then f' is a *modification* of f, whenever $\mathbf{P}(J_x)=1$ for all $x\in M$.

Notice that a priory for distinct $x\neq y\in M$ the sets J_x and J_y are distinct, though $\mathbf{P}(J_x\cap J_y)=1$. Even more, $\mathbf{P}\Big(\bigcap\limits_{x\in A}J_x\Big)=1$ for any countable set $A\subset M$. However, in general, one can not find a subset $J\in\Omega$ with $\mathbf{P}(J)=1$ and such that $f_x(\omega)=f'_x(\omega)$ for *all* $x\in M$ and $\omega\in J$.

Definition 2.1. Let M be a topological space, $\mathbb{R}_+=[0,+\infty)$, $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space, and

$$\xi\colon M\times\mathbb{R}_+\times\Omega\to M \tag{2.1}$$

be a map. Then ξ will be called a *stochastic deformation* whenever

(a) ξ is $\big(\mathcal{B}(M\times\mathbb{R}_+)\otimes\mathcal{F}\big)/\mathcal{B}(M)$-measurable;
(b) the map $\xi_\omega : M\times\mathbb{R}_+\to M$, $\xi_\omega(x,t)=\xi(x,t,\omega)$, is continuous for almost all $\omega\in\Omega$;
(c) $\xi(x,0,\omega)=x$ for all $x\in M$ and almost all $\omega\in\Omega$.

In other words, a *stochastic deformation* is a measurable continuous M-valued random field with parameter $M\times\mathbb{R}_+$ satisfying additional condition (c).

Remark 2.2. It is well known that for a large class of SDE on manifolds their solutions are stochastic deformations satisfying (semi-)group property, e.g. $\xi_{t+s,\omega}=\xi_{t,\omega}\circ\xi_{s,\omega}$ for $s,t\in\mathbb{R}_+$ and almost all $\omega\in\Omega$, and called *stochastic flows*. However, a priori not every stochastic flow is a solution of certain SDE. For details on this correspondence and definitions of distinct kinds of stochastic flows, see e.g. [16, Chapter 4], [15], [19].

Remark 2.3. It is often convenient to study random fields up to a modification, e.g. continuous or measurable. See e.g. [16, §1.2–1.4] for sufficient conditions of existence of such modifications. To simplify the exposition we assume in the definition of a stochastic deformation that it is *already measurable and continuous* and that ξ_ω is continuous for *all* $\omega\in\Omega$. However, in general, an M-valued random field with parameter $M\times\mathbb{R}_+$ does not necessarily admit a measurable and continuous modification.

Measures associated with a stochastic deformation. Let ξ be a stochastic deformation on a topological space M such that $\mathcal{B}(M)$ contains all one-point subsets. This holds e.g. when M is a T_1-space (i.e. every point is a closed subset) and, in particular, when M is a manifold. Then it follows measurability from the assumption (a) of Definition 2.1 on ξ, that for each $(x,t) \in M \times \mathbb{R}_+$ the map

$$\xi_{x,t}\colon \Omega \to M, \qquad \xi_{x,t}(\omega) = \xi(x,t,\omega).$$

is $\mathcal{F}/\mathcal{B}(M)$-measurable. Therefore one can define the following σ-additive probability measure $\mu_{x,t}$ on M by

$$\mu_{x,t}(K) := \mathbf{P}\big(\xi_{x,t}^{-1}(K)\big) = \mathbf{P}\{\omega \in \Omega \mid \xi_{t,\omega}(x) \in K\}, \quad K \in \mathcal{B}(M).$$

Suprema of continuous and measurable maps. We will use the following statements.

Lemma 2.4. *Let M be a topological space, $(\Omega, \mathcal{F})$ be a measurable space, and $f\colon M \times \Omega \to \mathbb{R}$ be a measurable bounded continuous random $\mathbb{R}$-valued field. Suppose also that M has a countable everywhere dense subset A such that every point of A belongs to $\mathcal{B}(M)$ as a one-point set. Then the function $g\colon \Omega \to \mathbb{R}$ defined by*

$$g(\omega) = \sup_{x \in M} f(x,\omega)$$

is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable.

Proof. By definition f is a $(\mathcal{B}(M) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable map such that for each $\omega \in \Omega$ the restriction $f_\omega\colon M \to \mathbb{R}$ is a bounded continuous map.

Notice that for every $a \in A$ the function $f_a\colon \Omega \to \mathbb{R}$, $f_a(\omega) = f(a,\omega)$, is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable. Indeed, we have that $\{a\} \times \Omega \in \mathcal{B}(M) \otimes \mathcal{F}$ since $\{a\} \in \mathcal{B}(M)$ for $a \in A$. Hence we should only check that the restriction $f|_{\{a\}\times\Omega}\colon \{a\} \times \Omega \to \mathbb{R}$ is $(\mathcal{B}(M) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable. But this follows from measurability of f and the identity

$$f_a^{-1}(Q) = \ f^{-1}(Q) \cap \{a\} \times \Omega \quad \text{for each } Q \in \mathcal{B}(\mathbb{R}).$$

Now, as $f_\omega\colon M \to \mathbb{R}$ is continuous and A is everywhere dense, we get that

$$g(\omega) = \sup_{x \in M = \overline{A}} f(x,\omega) = \sup_{a \in A} f(a,\omega) = \sup_{a \in A} f_a(\omega).$$

In other words, $g = \sup\limits_{a \in A} f_a$ is a supremum of countably many $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable functions. Hence g is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable as well. □

Lemma 2.5. *Let S, M be metric spaces and $f\colon S \times M \to \mathbb{R}$ be a continuous function. If S is compact, then the function $g\colon M \to \mathbb{R}$ defined by*

$$g(x) = \sup_{s \in S} f(s,x)$$

is continuous.

The proof is left to the reader.

Homotopies. Let S and M be two topological spaces, $f, g\colon S \to M$ be two continuous maps between them, and $I = [0, 1]$. These maps are called *homotopic* if there exists a (jointly) continuous map $H\colon S \times I \to M$ such that $H(0, x) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in S$. Any such map H is called a *homotopy* between f and g.

A map $f\colon S \to M$ homotopic to a constant map is also said to be *null homotopic*.

A *deformation* of M is a homotopy $H\colon M \times I \to M$ starting from id_M, i.e. $H_0 = \mathrm{id}_M$. Thus, roughly speaking, a *stochastic deformation* is a family of deformations "measurably depending" on some "probabilistic" parameter $\omega \in \Omega$.

A topological space M is *contractible* if the identity map $\mathrm{id}_M\colon M \to M$ is null-homotopic, i.e. homotopic to a constant map $*\colon M \to p \in M$ into some point $p \in M$. The corresponding homotopy between id_M and $*$, i.e. a "deformation of M into a point p", is called a *contraction of M*.

For instance, *any convex subset $M \subset \mathbb{R}^n$ is contractible*, and the contraction $H\colon M \times I \to M$ of M into a point $p \in M$ can be defined by the following formula: $H(x, t) = tp + (1-t)x$. On the other hand a compact manifold without boundary, e.g. the n-dimensional sphere S^n and the n-torus T^n, is never contractible.

Notice that *if M is contractible, then each continuous map $\sigma\colon S \to M$ is null homotopic*. Indeed, if $H\colon M \times I \to M$ is a contraction of M into some point $p \in M$, then the map $\Sigma\colon S \times I \to M$ defined by $\Sigma(x, t) = H(\sigma(x), t)$ is a homotopy between σ and a constant map into the point p.

The following statement is a particular case of the well-known Whitehead's theorem.

THEOREM 2.6 (J.H.C. Whitehead, e.g. [8, Theorem 4.5]). *A connected manifold M is contractible if and only if for each $n \geq 1$ each continuous map $\sigma\colon S^n \to M$ is null homotopic.*

Injectivity radius. Let M be a complete Riemannian manifold. Then it is also a complete metric space with the distance $\rho(x, y)$ between points $x, y \in M$ defined as the infimum of lengths of C^1-paths $\gamma\colon [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Also for each $x \in M$ and a unit tangent vector $v \in T_xM$ there exists a unique geodesic line $\gamma_{x,v}\colon \mathbb{R} \to M$ such that $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$. This allows to define the following *exponential* map $\exp_x\colon T_xM \to M$ by

$$\exp_x(w) = \gamma_{x,w/|w|}(|w|), \quad w \in T_xM.$$

It is well known and easy to show that $\exp_x$ is C^∞, $\exp_x(x) = 0$, and this map sends radial lines $\{tv\}_{t\in\mathbb{R}}$ onto geodesics passing through x. Moreover, let

$D_r(0) \subset T_xM$ be an open ball of radius r with center at the origin. Then there exists $r > 0$ such that $\exp_x$ diffeomorphically maps $D_r(0)$ onto some neighborhood $B_r(x)$ of x. Such a neighbourhood $B_r(x)$ is called a *geodesic ball* at x of radius r and the supremum of all such r for which $B_r(x)$ is defined is called the *injectivity* radius at x with respect to ρ and denoted by R_x. Thus

$$R_x = \sup_{r>0} \{r \mid \text{the restriction } \exp_x|_{D_r(0)} : D_r(0) \to M \text{ is an embedding}\}.$$

For a subset $K \subset M$ put

$$R_K = \inf_{x\in K} R_x. \tag{2.2}$$

If K is compact, then one easily checks that $R_K > 0$.

Main result. The following statement is an extension of [17, Theorem 4.1].

Theorem 2.7. *Let M be a smooth connected complete Riemannian manifold, $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and*

$$\xi \colon M \times \mathbb{R}_+ \times \Omega \to M$$

be a stochastic deformation having the following properties.

(a) *There exists $\Sigma \in \mathcal{F}$ such that $\mathbf{P}(\Sigma) = 1$, the map $\xi_{t,\omega}$ is C^1 for all $t \in \mathbb{R}_+$ and all $\omega \in \Sigma$, and the induced family of tangent maps*

$$\Xi \colon TM \times \mathbb{R}_+ \times \Sigma \to TM, \qquad \Xi(x, v, t, \omega) = T_x\xi_{t,\omega}(v),$$

is a stochastic deformation as well.

(b) *There exist a point $z \in M$ and a compact subset $K \subset M$ such that*

$$\varliminf_{t\to\infty} \mu_{z,t}(K) \equiv \varliminf_{t\to\infty} \mathbf{P}\{\omega \in \Omega \mid \xi_{t,\omega}(z) \in K\} > 0,$$

i.e. one can find $\varepsilon > 0$ and $A > 0$ satisfying $\mu_{z,t}(K) > \varepsilon$ for all $t > A$.

If for each compact subset $K \subset M$ we have that

$$\int_0^{+\infty} \sup_{x\in K} \mathbf{E}\, \|T_x\xi_{t,\omega}\| dt < \infty, \tag{2.3}$$

where the norm is taken with respect to the corresponding Riemannian metric, then $\pi_1 M = 0$. Moreover, if for each compact subset $K \subset M$ we have a stronger inequality

$$\int_0^{+\infty} \mathbf{E} \sup_{x\in K} \|T_x\xi_{t,\omega}\|\, dt < \infty, \tag{2.4}$$

then $\pi_n M = 0$ for all $n \geq 1$, that is M is contractible.

Remark 2.8. Condition (a) of Theorem 2.7 together with Lemmas 2.4, 2.5, and Fubini theorem guarantee that the integrals (2.3) and (2.4) are well defined.

The proof of Theorem 2.7 follows the lines of [17, Theorem 4.1], see Theorem 1.1 above. In fact the first statement about triviality of $\pi_1 M$ is proved in the same way as in [17, Theorem 4.1] but under weaker assumptions.

- First of all, ξ in Theorem 2.7 is just a stochastic deformation, not necessarily a stochastic flow, so the group property mentioned in Remark 2.2 is not needed.
- Furthermore, the recurrence of h-Brownian system in Theorem 1.1 implies that for *every* compact set $K \subset M$ and $z \in M$ the limit $\lim\limits_{t\to\infty} \mu_{z,t}(K)$ does not depend on z and equals to $\mu(K)$, a certain measure μ on M, see [17, §4]. At this point Theorem 2.7 requires just positivity of the lower limit $\varliminf\limits_{t\to\infty} \mu_{z,t}(K)$ for *some* compact set $K \subset M$ and a point $z \in M$. The latter assumption is essentially weaker.
- Also Theorem 1.1 requires strong 1-completeness implying that the deformation of each smooth loop along the flow is a homotopy. We require instead (just to simplify the exposition) measurability of the family of tangent maps Ξ and continuity of each Ξ_ω, see (a).
- Finally, conditions (1.3) and (2.3) coincide.

The second part of Theorem 2.7 about contractibility of M is new and holds when it is possible to interchange the order of "sup" and "$\mathbf{E}$" in (2.3) to get (2.4).

3. Proof of Theorem 2.7

Suppose that either

(a) $n = 1$ and condition (2.3) holds, or
(b) $n \geq 1$ and condition (2.4) is satisfied.

For the proof of Theorem 2.7 it suffices to show that every continuous map $\sigma\colon S^n \to M$ from n-dimensional sphere S^n into M is homotopic to a constant map. This will imply that $\pi_n M = 0$ for the corresponding values of n.

It is well known that σ is homotopic to a C^1-map, and therefore one can assume that σ *itself is of class* C^1.

Moreover, as M is connected, one can also assume that the point z from (b) of Theorem 2.7 belongs to $\sigma(S^n)$.

For each $\omega \in \Omega$ and $t \in \mathbb{R}_+$ define the map $\sigma_{t,\omega}\colon S^n \to M$ by

$$\sigma_{t,\omega} = \xi_{t,\omega} \circ \sigma\colon S^n \xrightarrow{\sigma} M \xrightarrow{\xi_{t,\omega}} M.$$

By assumption there exists $\Sigma \in \mathcal{F}$ such that $\mathbf{P}(\Sigma) = 1$ and $\xi_\omega\colon M \times \mathbb{R}_+ \to M$ is continuous for all $\omega \in \Sigma$. In other words, $\xi_{t,\omega}$ is homotopic to $\xi_{0,\omega} = \mathrm{id}_M$ and so $\sigma_{t,\omega}$ is homotopic to σ for all $\omega \in \Sigma$. Therefore it suffices to show that $\sigma_{t,\omega}$ is null homotopic for some $t \in \mathbb{R}_+$ and $\omega \in \Sigma$.

Define the function $f\colon S^n \times S^n \times \mathbb{R}_+ \times \Sigma \to \mathbb{R}$ by

$$f(x,y,t,\omega) = \rho\big(\sigma_{t,\omega}(x), \sigma_{t,\omega}(y)\big)$$

and let $\operatorname{diam}_\sigma\colon \mathbb{R}_+ \times \Sigma \to \mathbb{R}$ be another function given by

$$\operatorname{diam}_\sigma(t,\omega) = \sup_{(x,y)\in S^n\times S^n} f(x,y,t,\omega),$$

so $\operatorname{diam}_\sigma(t,\omega)$ is the diameter of the image $\sigma_{t,\omega}(S^n)$.

Notice that the restriction $f_\omega\colon S^n \times S^n \times \mathbb{R}_+ \to \mathbb{R}$ is continuous for each $\omega \in \Sigma$. Moreover, as $S^n \times S^n \times \mathbb{R}_+$ is a separable metric space, it follows from Lemma 2.4 that $\operatorname{diam}_\sigma$ is $\big(\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\big)/\mathcal{B}(\mathbb{R})$-measurable.

Lemma 3.1. *The following inequality holds: $\int_0^{+\infty} \mathbf{E}(\operatorname{diam}(\sigma_t))\,dt < \infty$. Hence there exists a sequence of numbers $\{t_j\} \subset \mathbb{R}_+$ converging to infinity and such that*

$$\lim_{j\to\infty} \mathbf{E}(\operatorname{diam}(\sigma_{t_j})) = 0. \tag{3.1}$$

Assuming that Lemma 3.1 is proved we will complete Theorem 2.7.

Let z and K be the same as in (b) of Theorem 2.7 and

$$L_t := \{\omega \in \Sigma \mid \xi_{t,\omega}(z) \in K\}, \quad t \in \mathbb{R}_+.$$

Then due to (b) there exist $\varepsilon > 0$ and $A > 0$ such that

$$\mu_{z,t}(K) = \mathbf{P}(L_t) > \varepsilon \quad \text{for all } t > A. \tag{3.2}$$

Let also $R_K > 0$ be the injectivity radius of K with respect to the metric ρ and $\{t_j\} \subset \mathbb{R}_+$ be a sequence satisfying (3.1). For each $t \in \mathbb{R}_+$ put

$$Q_t := \{\omega \in \Sigma \mid \operatorname{diam}(\sigma_t)(\omega) \geq R_K/2\}.$$

Then, due to (3.1), there exists $t_j > A$ such that $\mathbf{P}(Q_{t_j}) < \varepsilon/2$.

Now let $Z := L_{t_j} \setminus Q_{t_j} = \{\omega \in \Sigma \mid \operatorname{diam}(\sigma_{t_j})(\omega) < R_K/2 \text{ and } \xi_{t_j,\omega}(z) \in K\}$. Then

$$\mathbf{P}(Z) \geq \mathbf{P}(L_{t_j}) - \mathbf{P}(Q_{t_j}) - \mathbf{P}(N) = \varepsilon - \varepsilon/2 - 0 = \varepsilon/2 > 0,$$

whence $Z \neq \emptyset$. Since by assumption $z \in \sigma(S^n)$, it follows that for each $\omega \in Z$ the image of $\sigma_{t_j,\omega}(S^n) = \xi_{t_j,\omega} \circ \sigma(S^n)$ intersects K and is contained in the geodesic ball of radius smaller than R_K with center $\xi_{t_j,\omega}(z)$. Therefore the map $\sigma_{t_j,\omega}\colon S^n \to K \subset M$ is null homotopic, whence so is σ.

This proves Theorem 2.7 modulo Lemma 3.1.

Proof of Lemma 3.1. We should check that $\int_0^{+\infty} \mathbf{E}(\mathrm{diam}(\sigma_t))\,dt < \infty$.

(a) Suppose $n = 1$ and (2.3) holds. Then the proof is literally the same as in [17, Theorem 4.1].

By (a) of Definition 2.1 the family Ξ of tangent maps is a stochastic deformation. Hence there exists $\Sigma \in \mathcal{F}$ such that $\mathbf{P}(\Sigma) = 1$ and the restriction map $\Xi\colon TM \times \mathbb{R}_+ \to TM$ is continuous for all $\omega \in \Sigma$. Together with Fubini theorem this implies that the function

$$l\colon \mathbb{R}_+ \times \Sigma \to \mathbb{R}, \qquad l(t,\omega) = \int_0^{2\pi} \left| T_{\sigma(s)}\xi_{t,\omega}(\sigma'(s))\right| ds,$$

associating to (x,ω) the length of the C^1 loop $\sigma_{t,\omega}$, is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable. Then

$$\begin{aligned}\int_0^{+\infty} \mathbf{E}(\mathrm{diam}(\sigma_t))\,dt &\leq \int_0^{+\infty} \mathbf{E}(l(t,\omega))\,dt \\ &= \int_0^{+\infty} \mathbf{E}\int_0^{2\pi} \left|T_{\sigma(s)}\xi_{t,\omega}(\sigma'(s))\right| ds \\ &\leq 2\pi \sup_{s\in[0,2\pi]} |\sigma'(s)| \int_0^{+\infty} \sup_{x\in\sigma(S^1)} \mathbf{E}\,\|T_x\xi_{t,\omega}\| \overset{(2.3)}{<} \infty.\end{aligned}$$

(b) Assume now that $n \geq 1$ and (2.4) is satisfied. Notice that every great circle e in S^n of radius 1 can be regarded as a length preserving map $e\colon [0,2\pi] \to S^n$ and we will denote by $\dot{e} = \partial e/\partial s$ the unit tangent vector field along e.

This circle is uniquely determined by a 2-plane in $\mathbb{R}^{n+1}$ passing through the origin and so the space of all great circles in S^n can be identified with the Grassmannian manifold G_2^{n+1} of 2-planes in $\mathbb{R}^{n+1}$. Notice that G_2^{n+1} is comapct.

Similarly to the previous case there exists $\Sigma \in \mathcal{F}$ such that $\mathbf{P}(\Sigma) = 1$ and the function $l\colon G_2^{n+1} \times \mathbb{R}_+ \times \Sigma \to \mathbb{R}$ associating to (e,t,ω) the length of the C^1 loop $\sigma_{t,\omega}|_e\colon e \to M$ is $(\mathcal{B}(G_2^{n+1} \times \mathbb{R}_+) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable.

Notice that

$$\mathrm{diam}(\sigma_t)(\omega) \leq \frac{1}{2} \sup_{e\in G_2^{n+1}} l(e,t,\omega). \tag{3.3}$$

Let $p\colon TS^n \to S^n$ be the tangent bundle of S^n, $US^n \subset TS^n$ be the sphere-bundle consisting of all tangent vectors of length 1, and $\mathbf{L} = T\sigma(US^n) \subset TM$ the image of US^n in TM under the tangent map $T\sigma\colon TS^n \to TM$. Evidently, US^n and $\mathbf{L}$ are compact.

If $e\colon [0,2\pi] \to S^n$ is a great circle in S^n and $\dot{e} = \partial e/\partial s$, then

$$(e(s),\dot{e}(s)) \in US^n, \qquad T\sigma(e(s),\dot{e}(s)) = (\sigma\circ e(s), T_{e(s)}\sigma(\dot{e}(s))) \in \mathbf{L}.$$

Therefore

$$2\int_0^{+\infty} \mathbf{E}\,(\operatorname{diam}(\sigma_t))\,dt < \int_0^{+\infty} \mathbf{E} \sup_{e\in G_2^{n+1}} l(e,t,\omega)\,dt$$
$$= \int_0^{+\infty} \mathbf{E} \sup_{e\in G_2^{n+1}} \int_0^{2\pi} \big|T_{\sigma(e(s))}\xi_{t,\omega}\circ T_{e(s)}\sigma(\dot e(s))\big|\,ds\,dt$$
$$= 2\pi \sup_{(x,v)\in\mathbf{L}} |v| \int_0^{+\infty} \mathbf{E} \sup_{x\in\sigma(S^n)} \|T_x\xi_{t,\omega}\|\,dt \overset{(2.4)}{<} \infty. \qquad \square$$

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Alexandra Antoniouk and Sergiy Maksymenko
Institute of Mathematics of NAS of Ukraine
Tereshchenkivska st. 3
Kyiv, 01004, UKRAINE

E-mail address: antoniouk.a@gmail.com, maks@imath.kiev.ua