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A GRADIENT FLOW GENERATED BY A NONLOCAL MODEL OF A NEURAL FIELD IN AN UNBOUNDED DOMAIN

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Abstract. In this paper we consider the nonlocal evolution equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) = \int_{\mathbb{R}^N} J(x-y) f(u(y,t))\rho(y)\,dy + h(x).$$

We show that this equation defines a continuous flow in both the space $C_b(\mathbb{R}^N)$ of bounded continuous functions and the space $C_\rho(\mathbb{R}^N)$ of continuous functions u such that $u \cdot \rho$ is bounded, where ρ is a convenient "weight function". We show the existence of an absorbing ball for the flow in $C_b(\mathbb{R}^N)$ and the existence of a global compact attractor for the flow in $C_\rho(\mathbb{R}^N)$, under additional conditions on the nonlinearity. We then exhibit a continuous Lyapunov function which is well defined in the whole phase space and continuous in the $C_\rho(\mathbb{R}^N)$ topology, allowing the characterization of the attractor as the unstable set of the equilibrium point set. We also illustrate our result with a concrete example.

1. Introduction

We consider here the nonlocal evolution equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) = \int_{\mathbb{R}^N} J(x-y) f(u(y,t))\rho(y)\,dy + h(x), \tag{1.1}$$

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where f is a continuous real function, $J\colon \mathbb{R}^N \to \mathbb{R}$ is a nonnegative integrable function, $\rho\colon \mathbb{R}^N \to \mathbb{R}$ is a symmetric nonnegative bounded "weight" function with $\int_{\mathbb{R}^N} \rho(x)d(x) < \infty$ and h is a bounded continuous function. Additional hypotheses will be added when needed in the sequel. We can rewrite equation (1.1) as

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) = J{*_\rho}(f \circ u)(x,t) + h(x), \quad h \geq 0,$$

where the $*_\rho$ above denotes convolution product with respect to the measure $d\mu(y) = \rho(y)\,dy$, that is

$$J{*_\rho}(v)(x) := \int_{\mathbb{R}^N} J(x-y)v(y)\,d\mu(y) = \int_{\mathbb{R}^N} J(x-y)v(y)\rho(y)\,dy. \tag{1.2}$$

Equation (1.1) is a variation of the equation derived by Wilson and Cowan [24], to model neuronal activity. There are also other variations of this model in the literature (see, for example, [1], [3], [6], [10], [13]).

The function $u(x,t)$ denotes the mean membrane potential of a patch of tissue located at position $x \in \mathbb{R}^N$ at time $t \geq 0$. The connection function $J(x)$ determines the coupling between the elements at position x with the element at position y. The (usually nonnegative nondecreasing function) $f(u)$ gives the neural firing rate, or average rate at which spikes are generated, corresponding to an activity level u. The function h denotes an external stimulus applied to the entire neural field. Let us denote by $S(x,t) = f(u(x,t))$ the firing rate of a neuron at position x at time t. The neurons at a point x are said to be active if $S(x,t) > 0$ (see [1], [2], [22]).

There is already a vast literature on the analysis of similar neural field models, see [1]–[3], [5]–[9], [11]–[13], [16]–[18], [20], [22]. However, their asymptotic behavior have not been fully analyzed in the case of unbounded domains. In particular, the "Lyapunov functional" appearing in the literature is not well defined in the whole phase space (see, for example, [10, 13, 18]). One advantage of our model is that we will be able to define a continuous Lyapunov functional which is well defined in the whole phase space (see (4.1) in Section 4).

This paper is organized as follows. In Section 2, we consider the flow generated by (1.1) in the phase space of continuous bounded functions. In Subsection 2.1, we prove that the Cauchy problem for (1.1) is well posed in this phase space with globally defined solutions, and, in Subsection 2.2, we prove the existence of an absorbing set for the flow generated by (1.1). In Section 3, we consider the problem (1.1) in the phase space $C_\rho(\mathbb{R}^N) \equiv \{u : \mathbb{R}^N \to \mathbb{R}$ continuous with $\|u\|_\rho := \sup_{x\in\mathbb{R}^N} \{|u(x)|\rho(x)\} < \infty\}$, where ρ is a convenient "weight function". In this section, to obtain well-posedness, we impose more stringent conditions on the nonlinearity than in the previous section (see Subsection 3.1). On the other hand, we obtain stronger results, including existence of a compact global attractor for the corresponding flow. Our proof uses adaptations of the technique used in [5], replacing the compact embedding $H^1([-l,l]) \hookrightarrow L^2([-l,l])$ by the compact embedding $C^1(\mathbb{R}^N) \hookrightarrow C_\rho(\mathbb{R}^N)$ (see also [9, 10, 21] for related work). In Section 4, motivated by the energy functional from [2], [7], [10], [13], [18], [25], we exhibit a continuous Lyapunov functional for the flow generated by (1.1), well defined in the whole phase space $C_\rho(\mathbb{R}^N)$, and use it to prove that the flow is gradient in the sense of [14]. Finally, in Section 5, we present a concrete example illustrating our results.

2. The flow in the space $C_b(\mathbb{R}^N)$

In this section, we consider the problem (1.1) in the phase space

$$C_b(\mathbb{R}^N) \equiv \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ continuous with } \|u\|_\infty := \sup_{x\in\mathbb{R}^N} \{|u(x)|\} < \infty \right\}.$$

After establishing well-posedness, we prove that a ball of appropriate radius is an absorbing set for the corresponding flow.

2.1. Well-posedness. The following estimate will be useful in the sequel. The proof is straightforward and left to the reader.

LEMMA 2.1. *If $u \in C_b(\mathbb{R}^N)$ then $\|J*_\rho u\|_\infty \le \|J\|_{L^1(R^N)}\|\rho\|_\infty\|u\|_\infty$, where $J*_\rho u$ is given by* (1.2).

DEFINITION 2.2. If E and F are normed spaces, we say that a function $F\colon E \to F$ is *locally Lipschitz continuous* (*or simply locally Lipschitz*) if, for any $x_0 \in E$, there exists a constant C and a ball $B = \{x \in E : \|x - x_0\| < b\}$ such that, if x and y belong to B then $\|F(x) - F(y)\| \le C\|x - y\|$. We say that F is *Lipschitz continuous on bounded sets* if the ball B in the previous definition can be chosen as any bounded ball in E.

REMARK 2.3. The two notions in Definition 2.2 are equivalent if the normed space E is locally compact.

PROPOSITION 2.4. *If f is continuous and $h \in C_b(\mathbb{R}^N)$, then $F\colon C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$ given by*

$$F(u) = -u + J*_\rho(f \circ u) + h,$$

is well defined. If f is locally Lipschitz, then F is Lipschitz in bounded sets.

PROOF. The first assertion is immediate. Now, from the triangle inequality and Lemma 2.1, it follows that

$$\begin{aligned}\|F(u) - F(v)\|_\infty &\le \|v - u\|_\infty + \|J*_\rho(f\circ u) - J*_\rho(f \circ v)\|_\infty \\ &\le \|v - u\|_\infty + \|J\|_{L^1(R^N)}\|\rho\|_\infty\|(f\circ u) - (f\circ v)\|_\infty.\end{aligned}$$

If $\|u\|_\infty, \|v\|_\infty \leq R$ then

$$|(f \circ u)(x) - (f \circ v)(x)| \leq k_R |u(x) - v(x)|,$$

where k_R is a Lipschitz constant for f in the interval $[-R, R]$. It follows that

$$\|F(u) - F(v)\|_\infty \leq (1 + k_R \|J\|_{L^1(R^N)} \|\rho\|_\infty) \|u - v\|_\infty.$$

which concludes the proof. $\square$

Theorem 2.5. *If f is locally Lipschitz and satisfies the dissipative condition*

$$|f(x)| \leq \eta |x| + K, \quad \text{for some constants } \eta, K \text{ and any } x \in \mathbb{R}, \tag{2.1}$$

the Cauchy problem for (1.1) *is well posed in $C_b(\mathbb{R}^N)$ with globally defined solutions.*

Proof. It follows from Proposition 2.4 and well-known results (see [19] or [15], Theorems 3.3.3 and 3.3.4). $\square$

2.2. Existence of an absorbing set. In this section, we denote by $T(t)$ the flow generated by (1.1) in $C_b(\mathbb{R}^N)$. Under some additional hypotheses on the nonlinearity, we prove here the existence of an absorbing bounded ball $\mathcal{B} \subset C_b(\mathbb{R}^N)$ for $T(t)$. We recall that a set $\mathcal{B} \subset C_b(\mathbb{R}^N)$ is an *absorbing set* for the flow $T(t)$ if, for any bounded set $C \subset C_b(\mathbb{R}^N)$, there is a $t_1 = t_1(C) > 0$ such that $T(t)C \subset \mathcal{B}$ for any $t \geq t_1$ (see [23]).

Lemma 2.6. *Suppose that f is locally Lipschitz and satisfies the dissipative condition* (2.1)*, with $\eta \|J\|_{L^1(R^N)} \|\rho\|_\infty < 1$. Then, if $\eta \|J\|_{L^1(R^N)} \|\rho\|_\infty < \delta < 1$, the ball in $C_b(\mathbb{R}^N)$, centered at the origin with radius*

$$R = \frac{\|J\|_{L^1(R^N)} \|\rho\|_\infty K + \|h\|_\infty}{\delta - \|J\|_{L^1(R^N)} \|\rho\|_\infty \eta},$$

is an absorbing set for the flow $T(t)$.

Proof. Let $u(x,t)$ be the solution of (1.1)) with initial condition $u(\,\cdot\,,0) = u_0$. Then, by the variation of constants formula,

$$u(x,t) = e^{-t} u_0(x) + \int_0^t e^{s-t} [J *_\rho (f \circ u)(x,s) + h(x)] \, ds.$$

From (2.1), there exists a constant K such that $|f(x)| \leq \eta |x| + K$ for any $x \in \mathbb{R}$. Hence, using Lemma 2.1 and (2.1), we obtain

$$\begin{aligned}
|u(x,t)| &\leq e^{-t} |u_0(x)| + \int_0^t e^{s-t} [|J *_\rho (f \circ u)(x,s)| + |h(x)|] \, ds \\
&\leq e^{-t} \|u_0\|_\infty + \int_0^t e^{s-t} [\|J *_\rho (f \circ u)(\,\cdot\,,s)\|_\infty + \|h\|_\infty] \, ds \\
&\leq e^{-t} \|u_0\|_\infty + \int_0^t e^{s-t} [\|J\|_{L^1(R^N)} \|\rho\|_\infty \|(f \circ u)(\,\cdot\,,s)\|_\infty + \|h\|_\infty] \, ds
\end{aligned}$$

$$\leq e^{-t}\|u_0\|_\infty + \int_0^t e^{s-t}[\|J\|_{L^1(R^N)}\|\rho\|_\infty \eta \|u(\,\cdot\,,s)\|_\infty$$
$$+ \|J\|_{L^1(R^N)}\|\rho\|_\infty K + \|h\|_\infty]\, ds.$$

Suppose

$$\|u(\,\cdot\,,s)\|_\infty \geq \frac{1}{\delta - \|J\|_{L^1(R^N)}\|\rho\|_\infty \eta}\left(\|J\|_{L^1(R^N)}\|\rho\|_\infty K + \|h\|_\infty\right),$$

for $0 \leq t \leq T$. Then, for $t \in [0,T]$, we obtain

$$e^t|u(x,t)| \leq \|u_0\|_\infty + \delta \int_0^t e^s \|u(\,\cdot\,,s)\|_\infty\, ds \quad \text{for any } x \in \mathbb{R}^N.$$

Taking the supremum on the left hand side, it follows that

$$e^t\|u(x,\,\cdot\,)\|_\infty \leq \|u_0\|_\infty + \delta \int_0^t e^s \|u(\,\cdot\,,s)\|_\infty\, ds.$$

From Gronwall's inequality, it then follows that $e^t\|u(\,\cdot\,,t)\|_\infty \leq \|u_0\|_\infty e^{\delta t}$ and therefore

$$\|u(\,\cdot\,,t)\|_\infty \leq \|u_0\|_\infty e^{(\delta-1)t} \quad \text{for } t \in [0,T]. \tag{2.2}$$

It follows that there exists

$$T_0 \leq \frac{1}{(1-\delta)} \ln \frac{\|J\|_{L^1(R^N)}\|\rho\|_\infty K + \|h\|_\infty}{\|u_0\|_\infty(\delta - \|J\|_{L^1(R^N)}\|\rho\|_\infty \eta)}$$

such that

$$\|u(\,\cdot\,,T_0)\|_\infty \leq \frac{\|J\|_{L^1(R^N)}\|\rho\|_\infty K + \|h\|_\infty}{\delta - \|J\|_{L^1(R^N)}\|\rho\|_\infty \eta}.$$

Also, we must have

$$\|u(\,\cdot\,,t)\|_\infty \leq \frac{\|J\|_{L^1(R^N)}\|\rho\|_\infty K + \|h\|_\infty}{\delta - \|J\|_{L^1(R^N)}\|\rho\|_\infty \eta}$$

for any $t \geq T_0$, since $\|u(\,\cdot\,,t)\|_\infty$ decreases (exponentially) if the opposite inequality holds, by (2.2). □

Remark 2.7. From (2.2), it follows that the ball $B(0,R')$ is positively invariant under the flow $T(t)$ if $R' \geq R$.

3. The flow in the space $C_\rho(\mathbb{R}^N)$

In this section, we consider the problem (1.1) in the phase space

$$C_\rho(\mathbb{R}^N) \equiv \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ continuous with } \|u\|_\rho := \sup_{x\in\mathbb{R}^N} \{|u(x)|\rho(x)\} < \infty \right\}.$$

We will need to impose more stringent conditions on the nonlinearity than in the previous section, to obtain well-posedness. On the other hand, we will obtain stronger results, including existence of a compact global attractor for the corresponding flow.

3.1. Well-posedness. The following result is an analogue of Lemma 2.1. The proof is again straightforward and left to the reader.

Lemma 3.1. *If $u \in C_\rho(\mathbb{R}^N)$ then $\|J{*_\rho}u\|_\rho \leq \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\|u\|_\rho$.*

Proposition 3.2. *If f is globally Lipschitz and $h \in C_\rho(\mathbb{R}^N)$, then the map $F\colon C_\rho(\mathbb{R}^N) \to C_\rho(\mathbb{R}^N)$ given by*

$$F(u) = -u + J{*_\rho}(f \circ u) + h,$$

is well defined and globally Lipschitz.

Proof. Suppose $|f(x)-f(y)| \leq k|x-y|$ for any $x, y \in \mathbb{R}$. Then, in particular, $|f(x)| \leq k|x| + M$, where $M = f(0)$, for $x \in \mathbb{R}$. It follows that $\|f \circ u\|_\rho \leq k\|u\|_\rho + M\|\rho\|_\infty$. From Lemma 3.1, we then obtain

$$\begin{aligned}\|F(u)\|_\rho &\leq \|u\|_\rho + \|J{*_\rho}(f \circ u)\|_\rho + \|h\|_\rho \\ &\leq \|u\|_\rho + \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\|f \circ u\|_\rho + \|h\|_\rho \\ &\leq \|u\|_\rho + \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty(k\|u\|_\rho + M\|\rho\|_\infty) + \|h\|_\rho,\end{aligned}$$

so F is well defined. Furthermore,

$$\begin{aligned}\|F(u) - F(v)\|_\rho &\leq \|u - v\|_\rho + \|J{*_\rho}(f \circ u) - J{*_\rho}(f \circ v)\|_\rho \\ &\leq \|u - v\|_\rho + \|J\|_{L^1(R^N)}\|\rho\|_\infty\|(f \circ u) - (f \circ v)\|_\rho \\ &\leq \|u - v\|_\rho + \|J\|_{L^1(R^N)}\|\rho\|_\infty k\|u - v\|_\rho \\ &= (1 + k\|J\|_{L^1(R^N)}\|\rho\|_\infty)\|u - v\|_\rho\end{aligned}$$

Therefore F is globally Lipschitz in $C_\rho(\mathbb{R}^N)$. □

Theorem 3.3. *If f is globally Lipschitz, the Cauchy problem for* (1.1) *is well posed in $C_\rho(\mathbb{R}^N)$ with globally defined solutions.*

Proof. It follows from Proposition 3.2 and well-known results (see [4], [19], or [15], Theorems 3.3.3 and 3.3.4). □

3.2. Existence of an absorbing set. In this section, we denote by $T(t)$ the flow generated by (1.1) in $C_\rho(\mathbb{R}^N)$. Under some additional hypotheses on the nonlinearity, we prove the existence of a bounded ball $\mathcal{B} \subset C_\rho(\mathbb{R}^N)$ which is an absorbing set for $T(t)$.

Lemma 3.4. *Suppose that f is globally Lipschitz and satisfies the dissipative condition* (2.1)*, with $\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta < 1$. Then, if $\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta < \delta < 1$, the ball in $C_\rho(\mathbb{R}^N)$ centered at the origin with radius*

$$R = \frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\rho}{\delta - \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta}$$

is an absorbing set for the flow $T(t)$.

Proof. Let $u(x,t)$ be the solution of (1.1) with initial condition $u(\cdot\,,0) = u_0$. Then, by the variation of constants formula,

$$u(x,t) = e^{-t}u_0(x) + \int_0^t e^{s-t}[J*_\rho(f\circ u)(x,s) + h(x)]\,ds.$$

From (2.1) and Lemma 3.1, we obtain

$$\begin{aligned}|u(x,t)\rho(x)| &\le e^{-t}|u_0(x)\rho(x)| + \int_0^t e^{s-t}[|J*_\rho(f\circ u)(x,s)\rho(x)| + |h(x)\rho(x)|]\,ds\\
&\le e^{-t}\|u_0\|_\rho + \int_0^t e^{s-t}[\|J*_\rho(f\circ u)(\cdot\,,s)\|_\rho + \|h\|_\rho]\,ds\\
&\le e^{-t}\|u_0\|_\rho + \int_0^t e^{s-t}[\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\|(f\circ u)(\cdot\,,s)\|_\rho + \|h\|_\rho]\,ds\\
&\le e^{-t}\|u_0\|_\rho + \int_0^t e^{s-t}[\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta\|u(\cdot\,,s)\|_\rho\\
&\quad + \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\rho]\,ds.\end{aligned}$$

Suppose

$$\|u(\cdot\,,s)\|_\infty \ge \frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\rho}{\delta - \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta}$$

for $0 \le t \le T$. Then, for $t \in [0,T]$, we obtain

$$e^t|u(x,t)\rho(x)| \le \|u_0\|_\rho + \delta\int_0^t e^s\|u(\cdot\,,s)\|_\rho\,ds \quad \text{for any } x \in \mathbb{R}^N.$$

Taking the supremum on the left hand side, it follows that

$$e^t\|u(x,\cdot\,)\|_\rho \le \|u_0\|_\rho + \delta\int_0^t e^s\|u(\cdot,s)\|_\rho\,ds.$$

From Gronwall's inequality, $e^t\|u(\cdot\,,t)\|_\rho \le \|u_0\|_\rho e^{\delta t}$ and hence

$$\|u(\cdot\,,t)\|_\infty \le \|u_0\|_\infty e^{(\delta-1)t} \quad \text{for } t \in [0,T]. \tag{3.1}$$

Therefore, there exists

$$T_0 \le \frac{1}{1-\delta}\ln\big(\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\infty\|u_0\|_\infty(\delta - \|J\|_{L^1(R^N)}\|\rho\|_\infty\eta)\big)$$

such that

$$\|u(\cdot\,,T_0)\|_\rho \le \frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\rho}{\delta - \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta}.$$

Also, we must have

$$\|u(\cdot\,,T_0)\|_\rho \le \frac{\|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty K + \|h\|_\rho}{\delta - \|J\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty\eta}$$

for any $t \ge T_0$, since $\|u(\cdot\,,t)\|_\rho$ decreases (exponentially) if the opposite inequality holds by (3.1). □

Remark 3.5. From (3.1), it follows that the ball $B_\rho(0,R')$ of radius R' in $C_\rho(\mathbb{R}^N)$ is positively invariant under the flow $T(t)$ if $R' \ge R$.

3.3. Existence of a global attractor. We denote below by $C_b^1(\mathbb{R}^N)$ the subspace of functions in $C_b(\mathbb{R}^N)$ with bounded derivatives.

Lemma 3.6. *The inclusion map $i\colon C_b^1(\mathbb{R}^N) \to C_\rho(\mathbb{R}^N)$ is compact.*

Proof. Let C be a bounded set in $C_b^1(\mathbb{R}^N)$. For any $l > 0$, let $\varphi\colon \mathbb{R}^N \to [0,1]$ be a smooth function satisfying

$$\varphi(x) = \begin{cases} 0 & \text{if } \|x\| \geq l, \\ 1 & \text{if } \|x\| \leq l/2. \end{cases}$$

Let $C^0(B_l)$ denote the space of continuous functions defined in the ball of $\mathbb{R}^N$ with radius l and center at the origin, that vanish at the boundary. Consider the subset C_l of functions in $C^0(B_l)$ defined by $C_l := \{\varphi u_{|B_l}$ with $u \in C\}$. Then C_l is a bounded subset of $C_b^1(B_l)$ and, therefore, a precompact subset of $C^0(B_l)$, by the Arzelá–Ascoli theorem.

Let now E_1 be the subset of $C_\rho(\mathbb{R}^N)$ given by $G_l := \{E(u)$ with $u \in C_l\}$, where $E(u)$ is the extension by zero outside B_l. Since E is continuous as an operator from $C^0(B_l)$ into $C_\rho(\mathbb{R}^N)$, it follows that $\overline{C_l}$ is a compact subset of $C_\rho(\mathbb{R}^N)$. Let now $G_l^c := \{(1-\varphi)u$ with $u \in C\}$. Let R be such that $\|u\|_\infty \leq R$, for any $u \in C$. Then, for any $\varepsilon > 0$, we may find l such that $0 < \rho(x) < \varepsilon/R$, if $\|x\| \geq l/2$. Then, it follows that $\|u\|_\rho \leq \varepsilon$, for any $u \in G_l^c$, that is, G_l^c is contained in the ball of radius ε around the origin.

Since G_l is precompact, it can be covered by a finite number of balls of radius ε. Since any function u in C can be written as $u = u_1 + u_2$, with $u_1 = \varphi u \in G_l$ and $u_2 = (1-\varphi)u \in G_l^c$, it follows that C can be covered by a finite number of balls with radius 2ε, for any $\varepsilon > 0$. Thus C is precompact as a subset of $C_\rho(\mathbb{R}^N)$. □

Lemma 3.7. *In addition to the hypotheses of Lemma* 3.4*, suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and h has bounded derivative. Let C be a bounded set in $C_\rho(\mathbb{R}^N)$ Then, for any $\eta > 0$, there exists t_η such that $T(t_\eta)C$ has a finite covering by balls with radius smaller than η.*

Proof. Let $u(x,t)$ be the solution of (1.1) with initial condition $u_0 \in C$. We may suppose that C is contained in the ball B_R of radius R, centered at the origin. By the variation of constants formula,

$$T(t)u_0(x) = e^{-t}u_0(x) + \int_0^t e^{s-t}[J*_\rho(f \circ u)(x,s) + h(x)]\,ds.$$

Write $(T_1(t)u_0)(x) = e^{-t}u_0(x)$ and

$$(T_2(t)u_0)(x) = \int_0^t e^{-(t-s)}[J*_\rho(f \circ u)(x,s) + h(x)]\,ds.$$

Let $\eta > 0$ be given. Then there exists $t(\eta) > 0$, uniform for $u_0 \in C$, such that if $t \geq t(\eta)$ then $\|T_1(t)u_0\|_\rho \leq \eta/2$. In fact, $|(T_1(t)u_0)(x)|\rho(x) = e^{-t}|u_0(x)|\rho(x)$. Thus $\|T_1(t)u_0\|_\rho = e^{-t}\|u_0\|_\rho$. Hence, for $t > t_\eta = \ln(2R/\eta)$, $\|T_1(t)u_0\|_\rho \leq \eta/2$ for any $u_0 \in C$, that is, $T_1(t)C$ is contained in the ball of radius $\eta/2$ around the origin.

We now show that $T_2(t)C_\rho(\mathbb{R}^N)$ lies in a bounded ball of $C_b^1(\mathbb{R}^N)$. In fact, using Lemma 2.1 we have, for any $u_0 \in C_\rho(\mathbb{R}^N)$,

$$\begin{aligned}
\|T_2(t)u_0\|_\infty &\leq \int_0^t e^{s-t}[\|J*_\rho(f\circ u)(\,\cdot\,,s)\|_\infty + \|h\|_\infty]\,ds \\
&\leq \int_0^t e^{s-t}[\|J\|_{L^1(R^N)}\|\rho\|_\infty\|(f\circ u)(\,\cdot\,,s)\|_\infty + \|h\|_\infty]\,ds \\
&\leq (M\|J\|_{L^1(R^N)}\|\rho\|_\infty + \|h\|_\infty)\int_0^t e^{s-t}\,ds \\
&\leq M\|J\|_{L^1(R^N)}\|\rho\|_\infty + \|h\|_\infty,
\end{aligned}$$

where $M = \|f\|_\infty < \infty$, and

$$\begin{aligned}
\left\|\frac{\partial}{\partial x}T_2(t)u_0\right\|_\infty &\leq \int_0^t e^{s-t}[\|J'*_\rho(f\circ u)(\,\cdot\,,s)\|_\infty + \|h'\|_\infty]\,ds \\
&\leq \int_0^t e^{s-t}[\|J' * \rho\|_\infty\|(f\circ u)(\,\cdot\,,s)\|_\infty + \|h'\|_\infty]\,ds \\
&\leq (M\|J'\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty + \|h'\|_\infty)\int_0^t e^{s-t}\,ds \\
&\leq M\|J'\|_{L^1(\mathbb{R}^N)}\|\rho\|_\infty + \|h'\|_\infty
\end{aligned}$$

Then, for $t \geq 0$ and any $u_0 \in C_\rho(\mathbb{R}^N)$, $\left\|\frac{\partial}{\partial x}T_2(t)u_0\right\|_\rho$ is bounded by a constant independent of t and u. Therefore, by Lemma 3.6, it follows that $\{T_2(t)\}C_\rho(\mathbb{R}^N)$, is compact as a subset of $C_\rho(\mathbb{R}^N)$ and, therefore, it can be covered by a finite number of balls with radius $\eta/2$. Since $T(t)C = T_1(t)C + T_2(t)C$, we obtain that $T(t)C$ can be covered by a finite number of balls of radius η, as claimed. □

In what follows we denote by $\omega(B_\rho(0,R))$ the ω-limit set of the ball $B_\rho(0,R)$. Then, as a consequence from Lemma 3.7, we have the following result:

Theorem 3.8. *Assume the same hypotheses as in Lemma* 3.7. *Then* $\mathcal{A} = \omega(B_\rho(0,R))$, *is a global attractor for the flow* $T(t)$ *generated by* (1.1) *in* $C_\rho(\mathbb{R}^N)$, *which is contained in the ball* $B_\rho(0,R)$.

Proof. From Lemma 3.7, it follows that, for any $\eta > 0$, there exists $t_\eta > 0$ such that $T(t_\eta)B_\rho(0,R)$ can be covered by a finite number of ball of radius η. Since $B_\rho(0,R)$ is positively invariant (see Remark 3.5) we have, for any $t \geq t_\eta$,

$$T(t)B_\rho(0,R) = T(t_\eta)T(t-t_\eta)B_\rho(0,R) \subset T(t_\eta)B_\rho(0,R)$$

and thus

$$\bigcup_{t\geq t_\eta} T(t)B_\rho(0,R) \subset T(t_\eta)B_\rho(0,R)$$

can also be covered by a finite number of balls with radius η. Therefore

$$\mathcal{A} := \omega(B_\rho(0,R)) = \bigcap_{t_0\geq 0} \overline{\bigcup_{t\geq t_0} T(t)B_\rho(0,R)} = \bigcap_{t_0\geq 0} \overline{T(t)B_\rho(0,R)},$$

can be covered by a finite number of balls of arbitrarily small radius and is closed, so it is a compact set. From the positive invariance of $B\rho(0,R)$ (Remark 3.5), it is clear that $\mathcal{A} \subset B_\rho(0,R)$.

It remains to prove that $\mathcal{A}$ attracts bounded sets of $C_\rho(\mathbb{R}^N)$. It is enough to prove that it attracts the ball $B_\rho(0,R)$. Suppose, for contradiction, that there exist $\varepsilon > 0$ and sequences $t_n \to \infty$, $x_n \in B_\rho(0,R)$, with $d(T(t_n)(x_n),\mathcal{A}) > \varepsilon$. Now, the set $\{T(t_n)(x_n) : n \geq n_0\}$ is contained in $T(t_{n_0})B_\rho(0,R)$. Thus, for any $\eta > 0$, it can be covered by balls with radius η if n_0 is big enough. Since the remainder of the sequence is a finite set, the same happens with the whole sequence. It follows that the sequence $\{T(t_n)(x_n) : n \in \mathbb{N}\}$ is a precompact set and so, passing to a subsequence, it converges to a point $x_0 \in B_\rho(0,R)$. But then x_0 must belong to $\mathcal{A} = \omega(B_\rho(0,R))$ and we reach a contradiction. This concludes the proof. □

4. Existence of a Lyapunov functional

Energy-like Lyapunov functional for models of neural fields are well known in the literature (see for example, [2], [7]–[10], [13], [18], [25]). However, when dealing with unbounded domains, these functionals are frequently not well defined in the whole phase space, since they can assume the value ∞ at some points (see, for example, [10], [18]). In this section, under appropriate assumptions on f, we exhibit a continuous Lyapunov functional for the flow of (1.1), which is well defined in the whole phase space $C_\rho(\mathbb{R}^N)$, and use it to prove that this flow has the gradient property, in the sense of [14].

Suppose that f is strictly increasing. Motivated by the energy functionals appearing in [2], [13], [18], [25], we define the functional $F\colon C_\rho(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} F(u) = \int_{\mathbb{R}^N} \bigg[-\frac{1}{2} f(u(x)) \int_{\mathbb{R}^N} J(x-y)f(u(y))\rho(y)\,dy \\ + \int_0^{f(u(x))} f^{-1}(r)\,dr - hf(u(x))\bigg]\rho(x)\,dx. \end{aligned} \tag{4.1}$$

Equivalently, with $d\mu(x) = \rho(x)\,dx$, we can rewrite (4.1) as

$$F(u) = \int_{\mathbb{R}^N} \bigg[-\frac{1}{2} f(u(x)) \int_{\mathbb{R}^N} J(x-y) f(u(y))\,d\mu(y) \\ + \int_0^{f(u(x))} f^{-1}(r)\,dr - h f(u(x)) \bigg]\,d\mu(x).$$

We can then prove the following result:

Proposition 4.1. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and bounded. Then, if $\rho \in L^1(\mathbb{R}^N)$, the functional given in* (4.1) *satisfies $|F(u)| < \infty$ for all $u \in C_\rho(\mathbb{R}^N)$.*

Proof. We start by noting that $F(u) = F_1(u) + F_2(u) - F_3(u)$, where

$$F_1(u) = -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(u(x)) J(x-y) f(u(y)) \rho(y) \rho(x)\,dy\,dx,$$
$$F_2(u) = \int_{\mathbb{R}^N} \bigg[\int_0^{f(u(x))} f^{-1}(r)\,dr \bigg] \rho(x)\,dx,$$
$$F_3(u) = \int_{\mathbb{R}^N} h(x) f(u(x)) \rho(x)\,dx.$$

Let

$$G_1(x,y) := f(u(x)) J(x-y) f(u(y)) \rho(y) \rho(x) \tag{4.2}$$

denote the integrand of $F_1(u)$. Then, since $M = \|f \circ u\|_\infty < \infty$, we obtain

$$|G_1(x,y)| \leq M^2 J(x-y) \rho(y) \rho(x)$$

and, therefore,

$$\begin{aligned} |F_1(u)| &\leq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M^2 J(x-y) \rho(y) \rho(x)\,dy\,dx \\ &\leq \frac{1}{2}\, M^2 \|J\|_{L^1(R^N)} \|\rho\|_\infty \int_{\mathbb{R}^N} \rho(x)\,dx \\ &\leq \frac{1}{2}\, M^2 \|J\|_{L^1(R^N)} \|\rho\|_\infty \|\rho\|_{L^1(R^N)}, \end{aligned} \tag{4.3}$$

Let now

$$G_2(x) := \int_0^{f(u(x))} f^{-1}(r)\,dr\,\rho(x) \tag{4.4}$$

denote the integrand of $F_2(u)$. Then,

$$|G_2(x)| \leq \int_0^M |f^{-1}(r)|\,dr\,\rho(x)$$

and

$$|F_2(u)| \leq \int_{\mathbb{R}^N} \bigg[\int_0^M |f^{-1}(r)|\,dr \bigg] \rho(x)\,dx \leq \int_{\mathbb{R}^N} \mathcal{L} \rho(x)\,dx \leq \mathcal{L} \|\rho\|_{L^1(R^N)}, \tag{4.5}$$

where $\mathcal{L}$ is the integral of the continuous function f^{-1} in the (finite) interval $[0, M]$. Finally, let

$$G_3(x) := h(x)f(u(x))\rho(x) \tag{4.6}$$

denote the integrand of $F_3(u)$. Then $|G_3(x)| \le M\|h\|_\infty \rho(x)$ and

$$|F_3(u)| \le \int_{\mathbb{R}^N} M\|h\|_\infty \rho(x)\, dx \le M\|h\|_\infty \|\rho\|_{L^1(R^N)}. \tag{4.7}$$ □

Theorem 4.2. *Suppose f satisfies the same hypotheses as in Proposition* 4.1. *Then the functional given in* (4.1) *is continuous in the topology of $C_\rho(\mathbb{R}^N)$.*

Proof. Write $F(u) = F_1(u) + F_2(u) - F_3(u)$ as in the proof of Proposition 4.1. Let u_n be a sequence of functions converging to u in $C_\rho(\mathbb{R}^N)$. Let also $G_1(x, y)$, $G_2(x)$, $G_3(x)$ be as in (4.2), (4.4), (4.6) and $G_1^n(x, y)$, $G_2^n(x)$, $G_3^n(x)$ as in (4.2), (4.4), (4.6) with u replaced by u_n. Then

$$F_1(u_n) = -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_1^n(x, y)\, dy\, dx,$$
$$F_2(u_n) = \int_{\mathbb{R}^N} G_2^n(x)\, dx, \qquad F_3(u_n) = \int_{\mathbb{R}^N} G_3^n(x)\, dx.$$

By (4.2), (4.4), (4.6) and (4.3), (4.5), (4.7); the integrands $G_1^n(x, y)$, $G_2^n(x)$, $G_3^n(x)$ are all bounded by integrable functions independent of n. Also, from the pointwise convergence of u_n to u and the continuity of the functions f, ρ and h, it follows that $G_1^n(x, y) \to G_1(x, y)$, $G_2^n(x) \to G_2(x)$ and $G_3^n(x) \to G_3(x)$, for all $x, y \in \mathbb{R}^N$. Therefore $F(u_n) \to F(u)$ by the Lebesgue Dominated Convergence Theorem. This completes the proof. □

Theorem 4.3. *Suppose that f satisfies the hypotheses of Theorem* 3.3*, Proposition* 4.1*, and that $|f'(x)| \le c\rho^2(x)$ for all $x \in \mathbb{R}^N$ and some positive constant c. Let $u(\,\cdot\,, t)$ be a solution of* (1.1). *Then $F(u(\,\cdot\,, t))$ is differentiable with respect to t and*

$$\frac{dF}{dt} = -\int_{\mathbb{R}^N} [-u(x, t) + J*_\rho(f\circ u)(x, t) + h]^2 f'(u(x, t))\, d\mu(x) \le 0.$$

Proof. Let

$$\varphi(x, s) = -\frac{1}{2}f(u(x, s))\int_{\mathbb{R}^N} J(x - y)f(u(y, s))\rho(y)\, dy \\ + \int_0^{f(u(x,s))} f^{-1}(r)\, dr - h(x)f(u(x, s)).$$

Using the hypotheses on f and the fact that $|f'(x)| \le c\rho^2(x)$, it is easy to see that $\|\partial\varphi(\,\cdot\,, s)/\partial s\|_{L^1(\mathbb{R}^N, d\mu(x))} < \infty$, for all $s \in \mathbb{R}_+$. Hence, derivating under the integral sign, we obtain

$$\begin{aligned}\frac{d}{dt}F(u(\,\cdot\,,t)) = \int_{\mathbb{R}^N}\bigg[&-\frac{1}{2}\frac{\partial f(u(x,t))}{\partial t}\int_{\mathbb{R}^N} J(x-y)f(u(y,t))\,d\mu(y)\\ &-\frac{1}{2}f(u(x,t))\int_{\mathbb{R}^N} J(x-y)\frac{\partial f(u(y,t))}{\partial t}\,d\mu(y)\\ &+f^{-1}(f(u(x,t)))\frac{\partial f(u(x,t))}{\partial t} - h\frac{\partial f(u(x,t))}{\partial t}\bigg]\,d\mu(x)\\ = -\frac{1}{2}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)f(u(y,t))\frac{\partial f(u(x,t))}{\partial t}\,d\mu(y)\,d\mu(x)\\ &-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)f(u(x,t))\frac{\partial f(u(y,t))}{\partial t}\,d\mu(y)\,d\mu(x)\\ &+\int_{\mathbb{R}^N}[u(x,t)-h]\frac{\partial f(u(x,t))}{\partial t}\,d\mu(x).\end{aligned}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)f(u(y,t))\frac{\partial f(u(x,t))}{\partial t}\,d\mu(y)\,d\mu(x)\\ &\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)f(u(x,t))\frac{\partial f(u(y,t))}{\partial t}\,d\mu(y)\,d\mu(x),\end{aligned}$$

it follows that

$$\begin{aligned}\frac{d}{dt}F(u(\,\cdot\,,t)) &= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)f(u(y,t))\frac{\partial f(u(x,t))}{\partial t}\,d\mu(y)\,d\mu(x)\\ &\quad+\int_{\mathbb{R}^N}[u(x,t)-h]\frac{\partial f(u(x,t))}{\partial t}\,d\mu(x)\\ &= -\int_{\mathbb{R}^N}\bigg[-u(x,t)\\ &\qquad+\int_{\mathbb{R}^N} J(x-y)f(u(y,t))\,d\mu(y)+h\bigg]\frac{\partial f(u(x,t))}{\partial t}\,d\mu(x)\\ &= -\int_{\mathbb{R}^N}[-u(x,t)+J*_\rho(f\circ u)(x,t)+h]\frac{\partial f(u(x,t))}{\partial t}\,d\mu(x)\\ &= -\int_{\mathbb{R}^N}[-u(x,t)+J*_\rho(f\circ u)(x,t)+h]f'(u(x,t))\frac{\partial u(x,t)}{\partial t}\,d\mu(x)\\ &= -\int_{\mathbb{R}^N}[-u(x,t)+J*_\rho(f\circ u)(x,t)+h]^2 f'(u(x,t))\,d\mu(x).\end{aligned}$$

Using the fact that f is strictly increasing the result follows. □

REMARK 4.4. From Theorem 4.3 follows that, if $F(T(t)u_0) = F(u_0)$ for $t \in \mathbb{R}$, then u_0 is an equilibrium point for $T(t)$.

4.1. Gradient property. We recall that a semigroup, $T(t)$, is *gradient* if each bounded positive orbit is precompact and there exists a continuous Lyapunov functional for $T(t)$ (see [14]).

Proposition 4.5. *Assume the same hypotheses as in Theorems* 4.3 *and* 3.8. *Then the flow generated by equation* (1.1) *is gradient.*

Proof. The precompactness of the orbits follows from existence of the global attractor. From Proposition 4.1, Theorems 4.2, 4.3 and Remark 4.4 it follows that the functional given in (4.1) is a continuous Lyapunov functional. □

As a consequence of Proposition 4.5 we have the convergence of the solutions of (1.1) to the equilibrium point set of $T(t)$ (see [14, Lemma 3.8.2]).

Corollary 4.6. *For any $u \in C_\rho(\mathbb{R})$ the ω-limit set $\omega(u)$ of u under $T(t)$ belongs to E. Analogously, the α-limit set $\alpha(u)$ of u under $T(t)$ belongs to E.*

Also as a consequence of Proposition 4.5 we have that the global attractor given in Theorem 3.8 allows the following characterization (see [14, Theorem 3.8.5]).

Theorem 4.7. *Under the same hypotheses as in Theorem* 4.3*, the attractor $\mathcal{A}$ is the unstable set of the equilibrium point set of $T(t)$, that is, $\mathcal{A} = W^u(E)$, where $E = \{u \in B_\rho(0,R) : u(x) = J{*_\rho}(f \circ u)(x) + h\}$.*

Proof. Let $u \in \mathcal{A}$. Then, there exists a complete orbit through u which is contained in $\mathcal{A}$. Since $\mathcal{A}$ is compact, the α-limit set, $\alpha(u)$, of u under $T(t)$ is nonempty. By Corollary 4.6 it belongs to E and, therefore, $u \in W^u(E)$. Conversely, suppose that $u \in W^u(E)$ and let E^δ be the δ-neighbourhood of E. Then, for any $\delta > 0$, there exists $\overline{t}$ such that $T(-t)u \in E^\delta$, for any $t \geq \overline{t}$. Thus, $u \in T(t)(E^\delta)$, for any $t \geq \overline{t}$. It follows that u is arbitrarily close to $\mathcal{A}$, so it must belong to $\mathcal{A}$. □

5. A concrete example

To illustrate our results, we consider the one-dimensional case of (1.1) with $f(x) = \tanh(x)$,

$$J(x) = \begin{cases} e^{-1/(1-x^2)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

$\rho(x) = (1+x^2)^{-1}$, and h a real bounded function with bounded derivative, that is, we consider the equation

$$\frac{\partial u(t,x)}{\partial t} = -u(t,x) + \int_{\mathbb{R}} e^{-1/(1-(x-y)^2)} \tanh u(t,y)(1+y^2)^{-1}\, dy + h(x). \tag{5.1}$$

It is easy to see that the functions J and ρ satisfy all the hypotheses assumed in the introduction, that is, J and ρ are even nonnegative functions, J is integrable and ρ is integrable and bounded, with $\|J\|_{L^1(\mathbb{R})} \leq 2/e$ and $\|\rho\|_\infty = 1$. Also, the function f is bounded by 1 and globally Lipschitz with Lipschitz constant equal to 1. Therefore, it satisfies the condition (2.1) with $\eta = 0$ and $K = 1$. Thus, the hypotheses of Theorem 3.8 are all satisfied and the flow generated by (5.1) admits a global attractor contained in the ball of radius $R = 2/e + \|h\|_\rho$. We claim that the hypotheses of Theorem 4.3 are also satisfied. In fact, the function f is clearly continuous and strictly increasing. Also

$$\frac{f'(x)}{\rho^2(x)} = \frac{4(1+x^2)^2}{(e^x + e^{-x})^2} \to 0 \quad \text{as } |x| \to \infty,$$

so $f'(x) \leq c\rho^2(x)$ for some constant c. Therefore the hypotheses of Theorem 4.3 also hold, as claimed, and the results of Section 4.1 are valid for the flow generated by equation (5.1).

REMARK 5.1. For $f(x) = (1 + e^{-x})^{-1}$ and $J(x) = \rho(x) = (1 + x^2)^{-1}/\pi^2$ the hypotheses of our results are also easily verified.

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