Topological **M**ethods in **N**onlinear **A**nalysis
Volume 51, No. 1, 2018, 79–89
DOI: 10.12775/TMNA.2017.040

SCHRÖDINGER–POISSON SYSTEMS WITH RADIAL POTENTIALS AND DISCONTINUOUS QUASILINEAR NONLINEARITY

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Abstract. We consider the following Schrödinger–Poisson system:

$$\begin{cases} -\triangle u + V(|x|)u + \phi u = Q(|x|)f(u) & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

with more general radial potentials V, Q and discontinuous nonlinearity f. The Lagrange functional may be locally *Lipschitz*. Using nonsmooth critical point theorem, we obtain the multiplicity results of radial solutions, we also show concentration properties of the solutions. This is in contrast with some recent papers concerning similar problems by using the classical Sobolev embedding theorems.

1. Introduction and main results

In this paper we look for radial solutions of the following Schrödinger–Poisson system:

$$\begin{cases} -\triangle u + V(|x|)u + \phi u = Q(|x|)f(u) & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

Throughout the paper we assume V and Q are continuous, nonnegative functions in $(0, \infty)$. System (1.1) is also called the Schrödinger–Maxwell equation. Systems

2010 *Mathematics Subject Classification.* 35J50, 35J60.
Key words and phrases. Schrödinger–Poisson equation; radial potentials; weighted Sobolev embedding.
Supported by NSFC (11471187,11571197) and SNSFC (ZR2014AM034).

like (1.1) have been widely investigated due to its strong physical background. From a physical point of view, it describes systems of identical charged particles interacting with each other in the case that magnetic effects could be ignored, and its solution is a standing wave for such a system. The general nonlinear term f models the interaction between the particles. The first equation of (1.1) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term ϕu is nonlocal and concerns the interaction with the electric field. For more detailed mathematical and physical interpretation, we refer to [6] and references therein.

Recent studies of the system

$$\begin{cases} -\triangle u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

have focused on existence and nonexistence of solutions, multiplicity of solutions, ground states, radially and nonradial solutions, semiclassical limit and concentrations of solutions (see [3], [17], [25], [14], [13] and references therein). In [3], Azzollini and Pomponio proved the existence of ground state solutions of (1.2) when $3 < p < 6$ and V is a positive constant. The case nonconstant potential was also treated in [3] for $4 < p < 6$ and V is possibly unbounded below. L. Zhao and F. Zhao [25] proved the existence of ground state solutions for the system (1.2) and obtained at least a ground state solution when $f(x,u) = |u|^{p-1}u$ with $p \in (2,3]$. Using a Nehari-type manifold and gluing solution pieces together, Kim and Seok [14] proved the existence of radial sign-changing solutions with prescribed numbers of nodal domains for (1.2) in the case where $V(x) \equiv 1$, $f(u) = |u|^{p-2}u$, and $p \in (4,6)$. Ianni [13] obtained a similar result to [14] for $p \in [4,6)$, via a heat flow approach together with a limit procedure. There is an extensive literature concerning the existence of infinitely many large energy solutions (cf. Ambrosetti and Rabinowitz [3], Bartsch [4], Bartsch and Willem [5], etc).

In [16], the following Schrödinger–Poisson system has been studied:

$$\begin{cases} -\triangle u + V(x)u + \lambda\phi u = Q(x)f(u) & \text{in } \mathbb{R}^N, \\ -\triangle \phi = u^2, \quad \lim\limits_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{1.3}$$

where $N = 3,4,5$, $V,Q\colon \mathbb{R}^N \to \mathbb{R}$ are radial smooth. By using the classical Mountain Pass Theorem, Mercuri obtained the existence of positive solutions for (1.3) in the presence of external potentials V,Q. To be precise, V,Q in (1.3) are assumed to satisfy the same conditions as were introduced in [1] in the framework of nonlinear Schrödinger equations, i.e.

$$\frac{\overline{a}}{1+|x|^{\overline{\alpha}}} \le V(x) \le A \tag{1.4}$$

for some $\overline{\alpha} \in (0,2]$, $\overline{a}, A > 0$, and

$$0 < Q(x) \le \frac{\overline{b}}{1 + |x|^{\overline{\beta}}} \tag{1.5}$$

for some $\overline{\beta}, \overline{b} > 0$. The author proved that problem (1.3) has a positive solution for $\lambda \equiv 1$, $f(u) = u^p$, $1 < p < (N+2)/(N-2)$. In [26], under assumptions (1.4) and (1.5), Zhu studied (1.3) when $f(u)$ is asymptotically linear in u at ∞. Very recently, in [2], Azzollini, D'Avenia and Pomponio proved the existence of a nontrivial solution of (1.3) with V and Q being constants when the nonlinearity satisfies the general hypotheses introduced by Berestycki and Lions.

As mentioned above, most known results were obtained by using smooth critical point theory. However, when we study system (1.1) and focus on such discontinuous nonlinearity f, the corresponding Lagrange functional is locally Lipschitz continuous, it may not be differentiable, thus smooth critical point theory seems not to be suitable. In the past decades many efforts have been devoted to extending the theory of nonlinear partial differential equations (PDE) to PDE with discontinuous nonlinearities (DPDE). We refer the readers to the pioneering work [8] and the references therein for some historic developments and explanations in view of physical and mathematical aspects. The abstract methods for dealing with DPDE have been developed. In [8], Chang developed variational methods for nondifferentiable functional by using the generalized gradients for locally Lipschitz continuous functions on Banach space introduced by Clarke [10]. On the other hand, Schrödinger–Poisson equations with superlinear or sublinear nonlinearity have been well studied in the literature, but little has been done in the case of quasilinear nonlinearity. This is due to the fact that the quasilinear system is not easy to study.

In the present paper, we are interested in the case that f in system (1.1) is not only discontinuous but also quasilinear, and obtain multiplicity results of radial solutions, we also show concentration properties of the solutions. In contrast to previous works, we have much more difficulties to face. The main tool used here is nonsmooth critical point theory.

We make the following hypotheses on f, V and Q:

- (f_1) f is a measurable function.
- (f_2) There is $C_1 > 0$ such that $|f(u)| \le C_1$ for $u \in \mathbb{R}$.
- (f_3) There is $C_2 > 0$ such that $F(u) = \int_0^u f(t)\,dt \ge C_2|u|$ for $u \in \mathbb{R}$.
- (f_4) $f(-u) = -f(u)$, for $u \in \mathbb{R}$.
- (V) There exist numbers α_1 and α_2 such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^{\alpha_1}} > 0, \qquad \liminf_{r\to 0} \frac{V(r)}{r^{\alpha_2}} > 0.$$

(Q) There exist numbers β_1 and β_2 such that

$$\limsup_{r\to\infty}\frac{Q(r)}{r^{\beta_1}}<\infty,\qquad \limsup_{r\to0}\frac{Q(r)}{r^{\beta_2}}<\infty.$$

Before stating our main results, we give several notations. Denote the completion of $C_{0,r}^{\infty}(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{1,2}}=\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\bigg)^{1/2}$$

by $D_r^{1,2}(\mathbb{R}^3)$. Set

$$L^p(\mathbb{R}^3;Q)=\bigg\{u:\mathbb{R}^3\to\mathbb{R}\;\bigg|\;u \text{ is Lebesgue measurable, } \int_{\mathbb{R}^3}Q(|x|)|u|^p<\infty\bigg\},$$

$$E=W_r^{1,2}(\mathbb{R}^3;V)=\bigg\{u\in D_r^{1,2}(\mathbb{R}^3):\int_{\mathbb{R}^3}V(|x|)|u|^2<\infty\bigg\},$$

E is a Hilbert space (see [21]) equipped with the inner product and norm

$$(u,v)_E=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+V(|x|)uv)\,dx,\qquad \|u\|_E=(u,u)_E^{1/2}.$$

Now we are ready to give our main result.

THEOREM 1.1. *Assume* (V) *and* (Q) *with*

$$\beta_1<\max\bigg\{-\frac{5}{2},\frac{\alpha_1}{4}-2\bigg\},\qquad \beta_2>\min\bigg\{-\frac{5}{2},\frac{\alpha_2}{4}-2\bigg\}.$$

If f satisfies (f_1)–(f_4), *then* (1.1) *has infinitely many radial solutions u_k in $W_r^{1,2}(\mathbb{R}^3;V)$ with energy $I(u_k)=c_k<0$ satisfying $c_k\to0$ as $k\to\infty$.*

EXAMPLE 1.2. $f(x)=\operatorname{sgn}u(x)$ satisfies (f_1)–(f_4).

REMARK 1.3. It is easy to see that V,Q in Theorem 1.1 are more general than that in (1.4)–(1.5). Since there is no continuous function f that satisfies (f_3) and $f(0)=0$, (f_1) and (f_2) imply $F(u)=\int_0^u f(t)\,dt$ is locally Lipschitz continuous, and it may not be differentiable which implies that (1.1) considered here is completely different from those studied in the literature.

REMARK 1.4. The proof of the multiplicity result is different from that in the literature because the compactness of a (PS) sequence cannot be obtained by the usual methods. To overcome this difficulty, we adopt an idea from [19] to check that the functional

$$\psi(u):=\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+V(|x|)u^2)\,dx,\quad u\in E,$$

possesses the (S^+) property on E which is defined in (2.2).

The novelty of this paper is two-fold. One is that we consider the quasilinear Schrödinger–Poisson system (1.1) on the whole space $\mathbb{R}^N$ with discontinuous nonlinearity. Another one is that we obtain existence and concentration of infinitely many radial solutions via nonsmooth critical point theory. As showed in remarks, our results extend and improve some recent work.

This paper is organized as follows. In Sections 2 and 3, we state the variational setting and abstract critical point theorems for nondifferentiable functionals. In Section 4, we finish the proof of main results via abstract critical point theorems given in [15] for locally Lipschitz continuous functionals.

2. Preliminaries

Throughout this paper we denote by $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence. We use C_i to denote various constants independent of the functions in $W_r^{1,2}(\mathbb{R}^3;V)$.

LEMMA 2.1 ([18, Theorem 2.1]). *Assume* (V) *and* (Q) *with*

$$\beta_1 < \max\left\{-\frac{5}{2}, \frac{\alpha_1}{4} - 2\right\}, \qquad \beta_2 > \min\left\{-\frac{5}{2}, \frac{\alpha_2}{4} - 2\right\}.$$

Then the embedding $E \hookrightarrow L^1(\mathbb{R}^3;Q)$ is compact.

LEMMA 2.2 ([16, Lemma 2]). *Let $\gamma = 1 - \alpha/4$. The space E is compactly embedded in $L^q(\mathbb{R}^3)$ for any q such that $2 + \alpha/\gamma < q < 6$.*

See also [21] for a more general case of Lemma 2.2. Furthermore E is embedded in $L^q(\mathbb{R}^3)$ for any $q \in [2 + \alpha/\gamma, 6]$ (see Mercuri [16], Su, Wang and Willem [21]) and E is compactly embedded in $L^q_{\mathrm{loc}}(\mathbb{R}^3)$ for all $q < 6$ by the classical Rellich theorem. We should note that the related problem involving radial weighted embeddings for Schrödinger–Poisson systems has been studied by Bonheure and Mercuri in [7].

(1.1) is the Euler–Lagrange equations for the functional $J\colon E \times D^{1,2} \to R$ defined by

$$J(u,\phi) = \frac{1}{2}\,\|u\|_E^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_u(x)u^2\,dx - \int_{\mathbb{R}^3} Q(|x|)F(u)\,dx.$$

It follows from above-mentioned embedding results that J is well defined in $E \times D^{1,2}(\mathbb{R}^3)$ and its critical points are the solutions of (1.1). Since (f_1) and (f_2) imply that $F(u) = \int_0^u f(t)\,dt$ is locally Lipschitz continuous and it may not be differentiable, J is not of class C^1. Note that $J_0(u) := \int_{\mathbb{R}^3} Q(|x|)F(u)\,dx$ is uniformly Lipschitz continuous when (f_1), (f_2) are assumed. It is known that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [6], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

Now, we recall this method. For any $u \in E$, $0<\alpha<4/11$, $2+\alpha/\gamma<12/5<6$, the linear operator $T\colon D_r^{1,2} \to R$ defined as

$$T(v) = \int_{\mathbb{R}^3} u^2 v$$

is continuous in $D^{1,2}(\mathbb{R}^3)$. Then by the Riesz Representation Theorem, there exists a unique $\phi_u \in D_r^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v \, dx = \int_{\mathbb{R}^3} u^2 v \, dx.$$

Therefore, $-\triangle\phi = u^2$ in a weak sense. We can write an integral expression for ϕ_u in the form:

$$\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy, \tag{2.1}$$

for any $u \in E$ (for detail, see Section 2 of [9]). The functions ϕ_u possess the following properties:

Lemma 2.3 ([9, Lemma 2.2]). *For any $u \in E$, we have*:

(a) $\|\phi_u\|_{D^{1,2}} \le C_3 \|u\|_{L^{12/5}}$, *where $C_3 > 0$ does not depend on u. As a consequence there exists $C_4 > 0$ such that*

$$\int_{\mathbb{R}^3} \phi_u u^2 \, dx \le C_4 \|u\|_E^2;$$

(b) $\phi_u \ge 0$.

So, we can consider the functional $I\colon E \to R$ defined by $I(u) = J(u,\phi)$. After multiplying $-\triangle\phi = u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi|^2 \, dx = \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx.$$

Therefore, the reduced functional takes the form

$$I(u) = \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx - \int_{\mathbb{R}^3} Q(|x|) F(u) \, dx.$$

From Lemma 2.2, I is well defined with the derivative given by

$$(I'(u), v) = \int_{\mathbb{R}^3} \big(\nabla u \nabla v + V(|x|) uv + \phi_u(x) uv - Q(|x|) f(u) v\big) \, dx.$$

Set the functionals

$$I_1(u) = \frac{1}{2}\|u\|_E^2 = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(|x|) u^2) \, dx, \qquad I_2(u) = \int_{\mathbb{R}^3} Q(|x|) F(u) \, dx,$$

I_1 is of class $C^1(E,\mathbb{R})$ and for I_1' there holds the following (S^+) property on E (see [12], [20]) in the sense that for any sequence $\{u_n\} \subset E$:

(S^+) if $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty}(I_1'(u_n), u_n - u) \le 0$, then $u_n \to u$ in E.

It follows from (f_1) and (f_2) that I_2 is uniformly Lipschitz continuous on E which implies that $I(u)$ is a locally Lipschitz functional on E.

The following result has also been proved by Mercuri [16] and by Ruiz [17].

LEMMA 2.4. *Consider the operator* $\Phi\colon E \to D_r^{1,2}$, $\Phi[u] = \phi_u$, *that is, the solution in* $D^{1,2}$ *of the problem. Let* $\{u_n\}$ *be a sequence satisfying* $u_n \rightharpoonup u$ *in* E. *Then,* $\Phi[u_n] \to \Phi[u]$ *in* $D^{1,2}$ *and, as a consequence,*

$$\int \Phi[u_n]u_n^2 \to \int \Phi[u]u^2. \tag{2.2}$$

PROOF. Define the linear operators $T_n\colon D_r^{1,2} \to \mathbb{R}$,

$$T_n(v) = \int u_n^2 v, \qquad T(v) = \int u^2 v.$$

Recall that the inclusion $E \hookrightarrow L^q$ is compact for $2 < q < 6$. In particular, $u_n^2 \to u^2$ in the norm of $L^{6/5}$. Note that

$$|T_n(v) - T(v)| \le \bigg(\int |u_n^2 - u^2|\bigg)^{5/6}\bigg(\int v^6\bigg)^{1/6}.$$

This implies that T_n converges strongly (as a linear operator) to T. Hence, $\Phi[u_n] \to \Phi[u]$ in $D^{1,2}$. To conclude (2.2) it suffices to observe that $\Phi[u_n] \to \Phi[u]$ in L^6 and $u_n^2 \to u^2$ in the norm of $L^{6/5}$. □

3. A nonsmooth critical point theorem

In this section we establish the variational framework for locally *Lipschitz* functionals. Let X be a real Banach space and X^* be its dual space, $I\colon X \to R$ be a locally Lipschitz continuous functional. For each $v \in X$, the generalized directed derivative $I^0(u;v)$ of I at $u \in X$ in the direction v is defined as

$$I^0(u,v) = \lim_{h\to 0}\sup_{\lambda\downarrow 0}\frac{1}{\lambda}\,[I(u+h+\lambda v) - I(u+h)].$$

The generalized directed derivative $\varphi^0(u;\,\cdot\,)$ enjoys some basic properties: for each $u \in X$, the function $v \mapsto I^0(u;v)$ is continuous in v and satisfies $|I^0(u;v)| \le K\|v\|$, and furthermore, it is subadditive, positively homogenous, and then convex. We refer to [10] for these facts.

DEFINITION 3.1 ([10]). Let $I\colon X \to R$ be a locally Lipschitz continuous functional. The generalized gradient of I at $u \in X$ denoted by $\partial I(u)$, is defined to be the sub-differential of the convex function $I^0(u;v)$ at $v = 0$:

$$\partial I(u) := \{\omega \in X^* \mid \langle \omega, v\rangle \le I^0(u;v) \text{ for all } v \in X\}.$$

We refer the readers to [8] for some general information about the generalized gradient.

Definition 3.2 ([8]). Let $I\colon X\to R$ be a locally Lipschitz functional. We say that $u_0\in X$ is a critical point of I if $0\in\partial I(u_0)$, I satisfies the (PS) condition if any sequence $\{u_n\}\subset X$ along which $I(u_n)$ is bounded and

$$\lambda(u_n)=\min_{\omega\in\partial I(u_n)}\|\omega\|_{X^*}\to 0$$

possesses a convergent subsequence.

Theorem 3.3 ([15]). *Let X is a Banach space and $I\colon X\to R$ be a locally Lipschitz functional. Assume that I satisfies the* (PS) *condition and is even and bounded from below, and $\varphi(0)=0$. If for any $k\in\mathbb{N}$, there exists a k-dimensional subspace X_k and $\rho_k>0$ such that*

$$\sup_{X_k\cap S_{\rho_k}} I<0 \tag{3.1}$$

where $S_\rho=\{u\in X\mid \|u\|=\rho\}$, then I has a sequence of critical values $c_k<0$ satisfying $c_k\to 0$ as $k\to\infty$.

Proposition 3.4 ([8, Theorem 2.2]). *Let X and Y be two Banach spaces. Assume that X is reflexive, the embedding $X\hookrightarrow Y$ is continuous and X is dense in Y. Let $\widetilde{G}$ be a locally Lipschitz continuous functional in Y and $G=\widetilde{G}|_X$, then $\partial G(u)\subset\partial\widetilde{G}(u)$, for $u\in X$.*

4. Proof of Theorem 1.1

In this section, we will apply the above lemmas and Theorem 3.3 to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. It follows from the above lemmas that the functional

$$I(u)=\frac{1}{2}\|u\|_E^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\,dx-\int_{\mathbb{R}^3}Q(|x|)F(u)\,dx$$

is well defined.

(1) *I is coercive.* Denote by C_5 the constant of the embedding $E\hookrightarrow L^1(\mathbb{R}^N;Q)$. From (f_2), we have, for $u\in E$,

$$\begin{aligned}I(u)&=\frac{1}{2}\|u\|_E^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\,dx-\int_{\mathbb{R}^3}Q(|x|)F(u)\,dx\\&\geq\frac{a}{2}\|u\|_E^2-C_1\|u\|_{L^1(\mathbb{R}^3;Q)}\geq\frac{a}{2}\|u\|_E^2-C_1C_5\|u\|_E,\end{aligned}$$

thus,

$$I(u)\to+\infty\quad\text{as }\|u\|_E\to\infty. \tag{4.1}$$

Consequently, I is bounded from below.

(2) *I satisfies the* (PS) *condition.* Let

$$X=W_r^{1,2}(\mathbb{R}^3;V),\qquad Y=\overline{(W_r^{1,2}(\mathbb{R}^3;V),\|\cdot\|_{L^1(\mathbb{R}^3;Q)})}.$$

By Lemma 2.1, X and Y fit in with the conditions on spaces in Proposition 3.4 in the sense that X is reflexive, the embedding $X \hookrightarrow Y$ is continuous (and compact), and X is dense in Y. We use the notation X instead of $W_r^{1,2}(\mathbb{R}^3; V)$.

Let $\{u_n\} \subset X$ be such that $I(u_n)$ is bounded and

$$\lambda(u_n) = \min_{\omega \in \partial I(u_n) \subset X^*} \|\omega\|_{X^*} \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

By (4.1), $\{u_n\}$ is bounded in X. Since X is reflexive and the embedding $X \hookrightarrow Y$ is compact, one gets, for some $u \in X$,

$$u_n \rightharpoonup u \quad \text{in } X,\ n \to \infty, \tag{4.3}$$

$$u_n \to u \quad \text{in } Y \subset L^1(\mathbb{R}^N; Q),\ n \to \infty. \tag{4.4}$$

Using the properties of the generalized gradient (see [8]), there exists $u_n^* \in \partial I(u_n) \subset X^*$ for each $n \in N$ satisfying

$$\lambda(u_n) = \|u_n\|_{X^*}, \tag{4.5}$$

and there exists $v_n^* \in \partial I_2(u_n) \subset X^*$ satisfying

$$\langle u_n^*, v\rangle = \int_{\mathbb{R}^3} (\nabla u_n \nabla v + V(|x|) u_n v)\, dx - \langle v_n^*, v\rangle - \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n (u - u_n)\, dx, \tag{4.6}$$

for $v \in X$. Taking $v = u_n - u$ in (4.6), then

$$\begin{aligned} \langle I_1'(u_n), u_n - u\rangle &= \int_{\mathbb{R}^N} (\nabla u_n \nabla (u_n - u) + V(|x|) u_n (u_n - u))\, dx \\ &= \langle u_n^*, u_n - u\rangle + \langle v_n^*, u_n - u\rangle + \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n (u - u_n)\, dx. \end{aligned} \tag{4.7}$$

We conclude from (4.2), (4.3) and (4.5) that

$$\langle u_n^*, u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{4.8}$$

It follows from Proposition 3.4 that $v_n^* \in \partial I_2(u_n) \subset \partial \widetilde{I}_2(u_n) \subset Y^*$. Since ∂I_2 is uniformly Lipschitz continuous on Y, using the properties of the generalized gradient again one has that

$$\|v_n^*\|_{Y^*} \le \widehat{C}, \quad n \in N. \tag{4.9}$$

It follows from (4.4) and (4.9) that

$$|\langle v_n^*, u_n - u\rangle| \le \|v_n^*\|_{Y^*} \|u_n - u\|_Y \le \widehat{C} \|u_n - u\|_Y \to 0, \quad n \to \infty. \tag{4.10}$$

From Lemma 2.2 and $2 + \alpha/\gamma < 12/5 < 6$,

$$u_n \to u \quad \text{in } L^{12/5}(\mathbb{R}^3). \tag{4.11}$$

By the Hölder inequality, the Sobolev inequality and Lemma 2.4, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n)\, dx &\le \|\phi_{u_n}\|_6 \|u_n\|_{12/5} \|u - u_n\|_{12/5} \\ &\le C_6 \|\phi_{u_n}\|_{D^{1,2}} \|u_n\|_{12/5} \|u - u_n\|_{12/5} \le C_3 C_6 \|u_n\|_{12/5} \|u - u_n\|_{12/5} \end{aligned}$$

where $C_6 > 0$ is a constant. By (4.11), we have

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n)\, dx \to 0.$$

So we get the conclusion that $\langle I_1'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$. Since I_1' enjoys the (S^+) property, we have that $u_n \to u$ in X, $n \to \infty$. Then I satisfies the (PS) condition.

(3) *I verifies* (3.1). For any $k \in \mathbb{N}$, we take k independent smooth functions $\phi_i \in C_{0,r}^{\infty}(\mathbb{R}^N)$ for $i = 1, \ldots, k$, and define $X_k = \operatorname{span}\{\phi_1, \ldots, \phi_k\}$. Then $X_k \subset E \subset L^1(\mathbb{R}^N; Q)$ and $\dim X_k = k$. From (f_3) one has

$$I(u) = \frac{1}{2}\, \|u\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2\, dx - \int_{\mathbb{R}^3} Q(|x|) F(u)\, dx.$$

Since all norms on X_k are equivalent, one has, for $\rho_k > 0$ small enough,

$$\sup_{X_k \cap S_{\rho_k}} I(u) < 0,$$

that is, (3.1) holds. Since all conditions of Theorem 3.3 are checked, the proof of Theorem 1.1 is complete by applying Theorem 3.3. □

Acknowledgements. The authors thank the anonymous referee for carefully reading the manuscript and giving important comments which helped them to improve the work.

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Manuscript received December 12, 2016

accepted June 21, 2017

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