Topological **M**ethods in **N**onlinear **A**nalysis
Volume 50, No. 2, 2017, 623–642
DOI: 10.12775/TMNA.2017.033

CONCENTRATION OF GROUND STATE SOLUTIONS FOR FRACTIONAL HAMILTONIAN SYSTEMS

César Torres — Ziheng Zhang

Abstract. We are concerned with the existence of ground states solutions to the following fractional Hamiltonian systems:

$$(\mathrm{FHS})_\lambda \qquad \begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_{t}u(t)) - \lambda L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n), \end{cases}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $\lambda > 0$ is a parameter, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. Assuming that $L(t)$ is a positive semi-definite symmetric matrix for all $t \in \mathbb{R}$, that is, $L(t) \equiv 0$ is allowed to occur in some finite interval T of $\mathbb{R}$, $W(t,u)$ satisfies the Ambrosetti–Rabinowitz condition and some other reasonable hypotheses, we show that $(\mathrm{FHS})_\lambda$ has a ground sate solution which vanishes on $\mathbb{R}\setminus T$ as $\lambda \to \infty$, and converges to $u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n)$, where $u \in E^{\alpha}_0$ is a ground state solution of the Dirichlet BVP for fractional systems on the finite interval T. Recent results are generalized and significantly improved.

1. Introduction

Fractional differential equations both ordinary and partial ones are applied in mathematical modeling of processes in physics, mechanics, control theory, biochemistry, bioengineering and economics. Therefore, the theory of fractional

2010 *Mathematics Subject Classification.* Primary: 34C37; Secondary: 35A15, 35B38.

Key words and phrases. Fractional Hamiltonian systems; fractional Sobolev space; ground state solution; critical point theory; concentration phenomena.

The second author was supported by National natural Science Foundation of China (117771044).

differential equations is an area intensively developing during the last decades [1], [8]. The monographs [13], [16], [19] enclose a review of methods for solving fractional differential equations, which are an extension of procedures from differential equations theory.

Recently, also equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives are an interesting and new field in fractional differential equations theory. In this topic, many results are obtained in dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory (including Leray–Schauder nonlinear alternative) [2], topological degree theory (including co-incidence degree theory) [11] and comparison method (including upper and lower solutions and monotone iterative method) [32] and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions to a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books by to Mawhin and Willem [14], Rabinowitz [20], Schechter [23] and the references therein.

$(\mathrm{FHS})_\lambda$, if $\alpha = 1$ and $\lambda = 1$, reduces to the following second order Hamiltonian systems:

$$\ddot{u} - L(t)u + \nabla W(t,u) = 0. \tag{HS}$$

It is well known that the existence of homoclinic solutions to Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been recognized since Poincaré [18]. They may be "organizing centers" for the dynamics in their neighborhood. From their existence one may, under certain conditions, infer the existence of chaos nearby or the bifurcation behavior of periodic orbits. During the past two decades, with the works of [17] and [21] variational methods and critical point theory have been successfully applied to the study of the existence and multiplicity of homoclinic solutions to (HS).

Assuming that $L(t)$ and $W(t,u)$ are independent of t or periodic in t, many authors have studied the existence of homoclinic solutions to (HS), see for instance [3], [4], [21] and the references therein and some more general Hamiltonian systems are considered in the recent papers [9], [10]. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions to approximating problems. If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, the existence of homoclinic solutions to (HS) is quite different from the case of periodic systems, because of the lack of compactness of the Sobolev embedding, see [4], [17], [22] and the references therein.

Motivated by the above classical works, in [25] the author considered the following fractional Hamiltonian systems:

$$\text{(FHS)} \qquad \begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t,u(t)), \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n), \end{cases}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. Assuming that $L(t)$ and $W(t,u)$ satisfy the following hypotheses, the author showed that (FHS) possesses at least one nontrivial solution via the Mountain Pass Theorem:

(L) $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists $l \in C(\mathbb{R},(0,\infty))$ such that $l(t) \to \infty$ as $|t| \to \infty$ and

$$(L(t)u,u) \geq l(t)|u|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n. \tag{1.1}$$

(W_1) $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and there is a constant $\theta > 2$ such that

$$0 < \theta W(t,u) \leq (\nabla W(t,u),u) \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n \setminus \{0\}.$$

(W_2) $|\nabla W(t,u)| = o(|u|)$ as $|u| \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(W_3) There exists $\overline{W} \in C(\mathbb{R}^n,\mathbb{R})$ such that

$$|W(t,u)| + |\nabla W(t,u)| \leq |\overline{W}(u)| \quad \text{for every } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n.$$

(W_1) is the so-called Ambrosetti–Rabinowitz condition, which implies that, as $|u| \to \infty$, $W(t,u)$ has superquadratic growth. Inspired by this work, using the genus properties of critical point theory, in [33] the authors established some new criterion to guarantee the existence of infinitely many solutions to (FHS) for the case that $W(t,u)$ is subquadratic as $|u| \to \infty$, where the condition (L) is also needed to guarantee that the functional corresponding to (FHS) satisfies the (PS) condition (see [15] where a similar result was obtained). In addition, very recently, using the fountain theorem, in [31], the authors established the existence of infinitely many solutions to (FHS) for the case that $W(t,u)$ is superquadratic as $|u| \to \infty$ without the Ambrosetti–Rabinowitz condition. Moreover, recently in [26] the author firstly discussed the following perturbed fractional Hamiltonian systems:

$$\text{(PFHS)} \qquad \begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) - L(t)u(t) + \nabla W(t,u(t)) = f(t), \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n), \end{cases}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of

$W(t,u)$ at u, $f \in C(\mathbb{R},\mathbb{R}^n)$ and belongs to $L^2(\mathbb{R},\mathbb{R}^n)$. Under the conditions (L), (W_1)–(W_3) and assuming that the L^2-norm of f is sufficiently small, he showed that (PFHS) has at least two nontrivial solutions, this has been generalized in [31] where the condition (L) is also satisfied.

As is well known, the condition (L) is the so-called coercive condition and is very restrictive. In fact, for a simple choice like $L(t) = \tau \mathrm{Id}_n$, condition (1.1) is not satisfied, where $\tau > 0$ and Id_n is the $n \times n$ identity matrix. Motivated by this fact, in [34] the authors focused their attention on the case that $L(t)$ is bounded in the sense that

(L)$'$ $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there are constants $0 < \tau_1 < \tau_2 < \infty$ such that

$$\tau_1 |u|^2 \le (L(t)u,u) \le \tau_2 |u|^2 \quad \text{for all } (t,u) \in \mathbb{R}\times\mathbb{R}^n.$$

If the potential $W(t,u)$ is assumed to be subquadratic as $|u| \to +\infty$, then they also showed that (FHS) possesses infinitely many solutions. See [30] for a related result.

Here we must point out, to obtain the existence or multiplicity of solutions to (FHS) (or (PFHS)), that all the papers mentioned above need the assumption that the symmetric matrix $L(t)$ is positive definite, see (L) and (L)$'$. Inspired by [25], [26], [33], [34], in the present paper we deal with the following fractional Hamiltonian systems with a parameter:

$$(\mathrm{FHS})_\lambda \qquad \begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) - \lambda L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u \in H^\alpha(\mathbb{R},\mathbb{R}^n), \end{cases}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $\lambda > 0$ is a parameter, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. Unlike the papers on this problem, we require that $L(t)$ is a positive semi-definite symmetric matrix for all $t \in \mathbb{R}$, that is, $L(t) \equiv 0$ is allowed to occur in some finite interval T of $\mathbb{R}$. Explicitly,

$(\mathcal{L})_1$ $L \in C(\mathbb{R},\mathbb{R}^{n\times n})$ is a symmetric matrix for all $t \in \mathbb{R}$; there exists a non-negative continuous function $l\colon \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that

$$(L(t)u,u) \ge l(t)|u|^2,$$

and the set $\{l < c\} := \{t \in \mathbb{R} \mid l(t) < c\}$ is nonempty with $\mathrm{meas}\{l < c\} < 1/C_\infty^2$, where $\mathrm{meas}\{\,\cdot\,\}$ is the Lebesgue measure and C_∞ is the best Sobolev constant for the embedding of X^α into $L^\infty(\mathbb{R})$;

$(\mathcal{L})_2$ $J = \mathrm{int}(l^{-1}(0))$ is a nonempty finite interval and $\overline{J} = l^{-1}(0)$;

$(\mathcal{L})_3$ there exists an open interval $T \subset J$ such that $L(t) \equiv 0$ for all $t \in \overline{T}$.

In this case, we assume that $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ satisfy (W_1)–(W_3) and

(W_4) $s \mapsto \langle \nabla W(t, sq), q\rangle / s^{\theta-1}$ is strictly increasing for all $q \neq 0$ and $s > 0$, θ is given by (W_1).

REMARK 1.1. We note that, under assumption (W_1), there are constants $c_1 > 0$ and $c_2 > 0$ such that (see [25]):

(a) $W(t,u) \geq c_1 |u|^\theta$, $|u| \geq 1$,
(b) $W(t,u) \leq c_2 |u|^\theta$, $|u| \leq 1$.

Furthermore, by (a) we obtain that

$$\lim_{|u|\to\infty} \frac{W(t,u)}{|u|^2} = \infty \quad \text{uniformly in } t. \tag{1.2}$$

Since $W(t,q)$ can be replaced by $W(t,q) - W(t,0)$, we may also assume without loss of generality that $W(t,0) = 0$ for all t.

Now, we are in the position to state our main result.

THEOREM 1.2. *Suppose that* $(\mathcal{L})_1$–$(\mathcal{L})_3$*,* (W_1)–(W_4) *are satisfied, then there exists* $\Lambda_* > 0$ *such that for every* $\lambda > \Lambda_*$*,* $(\mathrm{FHS})_\lambda$ *has a ground state solution.*

REMARK 1.3. Note that in $(\mathcal{L})_1$–$(\mathcal{L})_3$, we assume that $L(t)$ is a positive semi-definite symmetric matrix for all $t \in \mathbb{R}$. Therefore, the hypotheses (L) and (L)$'$ on $L(t)$ are not satisfied. Thus the results in [25], [26], [33], [34] are generalized and improved significantly.

Moreover, as mentioned above, the coercive condition (L) is used to establish some compact embedding theorems to guarantee that the (PS) condition (or other weak compactness conditions) holds, which is the essential step to obtain the existence of homoclinic solutions to (FHS) via the Mountain Pass Theorem. In the present paper, we assume that $L(t)$ satisfies $(\mathcal{L})_1$–$(\mathcal{L})_3$ and could not obtain some compact embedding theorem. Therefore, the main difficulty is to adapt some new technique to overcome this issue and test the (PS) condition is verified, see Lemma 3.10.

Here we must mention the recent works [27], [35]. In fact, in [27], assuming that $L(t)$ satisfies $(\mathcal{L})_1$–$(\mathcal{L})_3$, the author showed that $(\mathrm{FHS})_\lambda$ has at least one nontrivial solution for the case that the potential $W(t,u)$ satisfies the following subquadratic assumptions as $|u| \to \infty$:

(W_5) there exist a constant $\gamma \in (1,2)$ and a positive function $b \in L^p(\mathbb{R})$ with $p \in (1, 2/(2-\gamma)]$ such that

$$|\nabla W(t,u)| \leq b(t)|u|^{\gamma-1} \quad \text{for all } (t, u \in \mathbb{R} \times \mathbb{R}^n);$$

(W_6) there exist two constants $\eta, \delta > 0$ such that

$$|W(t,u)| \geq \eta |u|^\gamma \quad \text{for all } x \in T \text{ and } u \in \mathbb{R} \text{ with } |u| \leq \delta.$$

Furthermore in [35], the authors have complemented the previous work by considering the superquadratic potential when $|u| \to \infty$. They obtained the same results as in [27].

For a technical reason, we assume that there exists $0 < L < +\infty$ such that $T = [0, L]$, where T is given by $(\mathcal{L})_3$. For the concentration of solutions we have the following result.

Theorem 1.4. *Let u_λ be a solution to problem* $(\mathrm{FHS})_\lambda$ *obtained in Theorem* 1.2*, then $u_\lambda \to \tilde{u}$ strongly in $H^\alpha(\mathbb{R})$ as $\lambda \to \infty$, where $\tilde{u}$ is a ground state solution to the equation*

$$\begin{gathered} {}_tD^\alpha_L {}_0D^\alpha_t u = \nabla W(t,u), \quad t \in (0,L), \\ u(0) = u(L) = 0. \end{gathered} \tag{1.3}$$

Remark 1.5. We recall that, Theorems 1.2 and 1.4 give a positive answer to the question formulated in [35]. For the proof of Theorems 1.2 and 1.4 we adapt some ideas from [5], [24], [35].

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we accomplish the proof of Theorem 1.2 and in Section 4 we present the proof of Theorem 1.4.

2. Preliminary results

In this section, for the reader's convenience, firstly we introduce some basic definitions of fractional calculus. The Liouville–Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$${}_{-\infty}I^\alpha_x u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-\xi)^{\alpha-1} u(\xi)\, d\xi$$

and

$${}_xI^\alpha_\infty u(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\xi - x)^{\alpha-1} u(\xi)\, d\xi.$$

The Liouville–Weyl fractional derivative of order $0 < \alpha < 1$ is defined as the left-inverse operator of the corresponding Liouville–Weyl fractional integral

$${}_{-\infty}D^\alpha_x u(x) = \frac{d}{dx}\, {}_{-\infty}I^{1-\alpha}_x u(x) \tag{2.1}$$

and

$${}_xD^\alpha_\infty u(x) = -\frac{d}{dx}\, {}_xI^{1-\alpha}_\infty u(x). \tag{2.2}$$

The definitions of (2.1) and (2.2) may be written in an alternative form as follows:

$${}_{-\infty}D^\alpha_x u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x) - u(x-\xi)}{\xi^{\alpha+1}}\, d\xi$$

and

$${}_xD^\alpha_\infty u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x) - u(x+\xi)}{\xi^{\alpha+1}}\, d\xi.$$

Moreover, recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-iwx} u(x)\, dx.$$

In order to establish the variational structure which enables us to reduce the existence of solutions to $(\mathrm{FHS})_\lambda$ to find critical points of the corresponding functional, it is necessary to construct appropriate function spaces. In what follows, we introduce some fractional spaces, for more details see [7]. To this end, denote by $L^p(\mathbb{R},\mathbb{R}^n)$ $(2 \leq p < \infty)$ the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|u\|_{L^p} = \biggl(\int_{\mathbb{R}} |u(t)|^p\, dt\biggr)^{1/p},$$

and $L^\infty(\mathbb{R},\mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}\{|u(t)| : t \in \mathbb{R}\}.$$

Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\alpha,p} = \biggl(\int_0^T |u(t)|^p\, dt + \int_0^T |{}_0D_t^\alpha u(t)|^p\, dt\biggr)^{1/p}, \quad \text{for all } u \in E_0^{\alpha,p}. \tag{2.3}$$

This space can be characterized by

$$E_0^{\alpha,p} = \{u \in L^p([0,T],\mathbb{R}^n) : {}_0D_t^\alpha u \in L^p([0,T],\mathbb{R}^n) \text{ and } u(0) = u(T) = 0\}.$$

Moreover, $(E_0^{\alpha,p}, \|\cdot\|_{\alpha,p})$ is a reflexive and separable Banach space. Considering the space $E_0^{\alpha,p}$, we have the following results.

PROPOSITION 2.1 ([12]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p}. \tag{2.4}$$

If $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0D_t^\alpha u\|_{L^p}. \tag{2.5}$$

Due to (2.4), we can consider in $E_0^{\alpha,p}$ the following norm:

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_{L^p}, \tag{2.6}$$

and (2.6) is equivalent to (2.3).

PROPOSITION 2.2 ([12]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and $\{u_k\} \rightharpoonup u$ in $E_0^{\alpha,p}$. Then $u_k \to u$ in $C[0,T]$, i.e.*

$$\|u_k - u\|_\infty \to 0, \quad \text{as } k \to \infty.$$

We denote by $E^\alpha = E_0^{\alpha,2}$, this is a Hilbert space with respect to the norm $\|u\|_\alpha = \|u\|_{\alpha,2}$ given by (2.6). For $\alpha > 0$, define the semi-norm

$$|u|_{I^\alpha_{-\infty}} = \|{}_{-\infty}D^\alpha_x u\|_{L^2}$$

and the norm

$$\|u\|_{I^\alpha_{-\infty}} = (\|u\|^2_{L^2} + |u|^2_{I^\alpha_{-\infty}})^{1/2} \tag{2.7}$$

and let

$$I^\alpha_{-\infty} = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_{-\infty}}},$$

where $C_0^\infty(\mathbb{R},\mathbb{R}^n)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^n$ with vanishing property at infinity.

Now we can define the fractional Sobolev space $H^\alpha(\mathbb{R},\mathbb{R}^n)$ in terms of the Fourier transform. Choose $0 < \alpha < 1$, define the semi-norm

$$|u|_\alpha = \||w|^\alpha \widehat{u}\|_{L^2}$$

and the norm

$$\|u\|_\alpha = (\|u\|^2_{L^2} + |u|^2_\alpha)^{1/2}$$

and let

$$H^\alpha = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_\alpha}. \tag{2.8}$$

Moreover, we note that a function $u \in L^2(\mathbb{R},\mathbb{R}^n)$ belongs to $I^\alpha_{-\infty}$ if and only if

$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R},\mathbb{R}^n).$$

Especially, we have

$$|u|_{I^\alpha_{-\infty}} = \||w|\widehat{u}\|_{L^2}.$$

Therefore, $I^\alpha_{-\infty}$ and H^α are equivalent with equivalent semi-norm and norm. Analogously to $I^\alpha_{-\infty}$, we introduce I^α_∞. Define the semi-norm

$$|u|_{I^\alpha_\infty} = \|{}_xD^\alpha_\infty u\|_{L^2}$$

and the norm

$$\|u\|_{I^\alpha_\infty} = (\|u\|^2_{L^2} + |u|^2_{I^\alpha_\infty})^{1/2} \tag{2.9}$$

and let

$$I^\alpha_\infty = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_\infty}}.$$

Then $I^\alpha_{-\infty}$ and I^α_∞ are equivalent with equivalent semi-norm and norm, see [7].

Let $C(\mathbb{R},\mathbb{R}^n)$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$. Then we obtain the following lemma.

Lemma 2.3 ([25, Theorem 2.1]). *If $\alpha > 1/2$, then $H^\alpha \subset C(\mathbb{R},\mathbb{R}^n)$ and there is a constant $C_\infty = C_{\alpha,\infty}$ such that*

$$\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)| \le C_\infty \|u\|_\alpha. \tag{2.10}$$

Remark 2.4. From Lemma 2.3, we know that if $u \in H^\alpha$ with $1/2 < \alpha < 1$, then $u \in L^p(\mathbb{R}, \mathbb{R}^n)$ for all $p \in [2, \infty)$, since

$$\int_{\mathbb{R}} |u(x)|^p \, dx \leq \|u\|_\infty^{p-2} \|u\|_{L^2}^2.$$

In what follows, we introduce the fractional space in which we will construct the variational framework for $(\mathrm{FHS})_\lambda$. Let

$$X^\alpha = \left\{ u \in H^\alpha : \int_{\mathbb{R}} \left[|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t)) \right] dt < \infty \right\},$$

then X^α is a reflexive and separable Hilbert space with the inner product

$$\langle u, v \rangle_{X^\alpha} = \int_{\mathbb{R}} \left[({}_{-\infty}D_t^\alpha u(t), {}_{-\infty} D_t^\alpha v(t)) + (L(t)u(t), v(t)) \right] dt$$

and the corresponding norm is $\|u\|_{X^\alpha}^2 = \langle u, u \rangle_{X^\alpha}$.

For $\lambda > 0$, we also need the following inner product:

$$\langle u, v \rangle_{X^{\alpha,\lambda}} = \int_{\mathbb{R}} \left[({}_{-\infty}D_t^\alpha u(t), {}_{-\infty} D_t^\alpha v(t)) + \lambda (L(t)u(t), v(t)) \right] dt$$

and the corresponding norm is $\|u\|_{X^{\alpha,\lambda}}^2 = \langle u, u \rangle_{X^{\alpha,\lambda}}$.

Lemma 2.5 ([35]). *Suppose $L(t)$ satisfies $(\mathcal{L})_1$ and $(\mathcal{L})_2$, then X^α is continuously embedded in H^α.*

Remark 2.6. Under the same conditions of Lemma 2.5, we also obtain

$$\int_{\mathbb{R}} |u(t)|^2 \, dt \leq \frac{C_\infty^2 \operatorname{meas}\{l < c\}}{1 - C_\infty^2 \operatorname{meas}\{l < c\}} \, \|u\|_{X^{\alpha,\lambda}} = \frac{1}{\Theta} \, \|u\|_{X^{\alpha,\lambda}}^2 \tag{2.11}$$

and

$$\|u\|_\alpha^2 \leq \left(1 + \frac{C_\infty^2 \operatorname{meas}\{l < c\}}{1 - C_\infty^2 \operatorname{meas}\{l < c\}} \right) \|u\|_{X^\alpha}^2 = \left(1 + \frac{1}{\Theta} \right) \|u\|_{X^{\alpha,\lambda}}^2 \tag{2.12}$$

for all $\lambda \geq 1/(cC_\infty^2 \operatorname{meas}\{l < c\})$. Furthermore, for every $p \in (2, \infty)$ and $\lambda \geq 1/(cC_\infty^2 \operatorname{meas}\{l < c\})$, we have

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leq \frac{1}{\Theta^{p/2} \left(\operatorname{meas}\{l < c\} \right)^{(p-2)/2}} \, \|u\|_{X^{\alpha,\lambda}}^p. \tag{2.13}$$

For more details see [35].

3. Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. For this purpose, we are going to establish the corresponding variational framework to obtain solutions to $(\mathrm{FHS})_\lambda$. Define the functional $I \colon \mathcal{B} = X^{\alpha,\lambda} \to \mathbb{R}$ by

$$\begin{aligned} I_\lambda(u) &= \int_{\mathbb{R}} \left[\frac{1}{2} \, |{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2} \, (\lambda L(t)u(t), u(t)) - W(t, u(t)) \right] dt \\ &= \frac{1}{2} \, \|u\|_{X^{\alpha,\lambda}}^2 - \int_{\mathbb{R}} W(t, u(t)) \, dt. \end{aligned} \tag{3.1}$$

Under the conditions of Theorem 1.2, as usual, we see that $I \in C^1(X^{\alpha,\lambda}, \mathbb{R})$, i.e. I is a continuously Fréchet-differentiable functional defined on $X^{\alpha,\lambda}$. Moreover, we have

$$(3.2)\quad I'_\lambda(u)v = \int_{\mathbb{R}} \big[({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (\lambda L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))\big]\, dt$$

for all u, $v \in X^\alpha$, which yields that

$$(3.3)\qquad I'_\lambda(u)u = \|u\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))\, dt.$$

Remark 3.1. We note that I_λ has the geometry property of the Mountain Pass Theorem. In fact, first we prove that there exist $\rho, \beta > 0$ such that $I_\lambda|_{\partial B_\rho} \geq \beta$. By Remark 2.6 and Lemma 2.3, we have

$$\|u\|^2_{L^2} \leq \frac{1}{\Theta}\|u\|^2_{X^{\alpha,\lambda}}, \quad \|u\|^2_\alpha \leq \left(1 + \frac{1}{\Theta}\right)\|u\|^2_{X^{\alpha,\lambda}} \quad \text{and} \quad \|u\|_\infty \leq C_\infty \|u\|_\alpha.$$

Therefore

$$(3.4)\qquad \|u\|_\infty \leq C_\infty \left(1 + \frac{1}{\Theta}\right)^{1/2} \|u\|_{X^{\alpha,\lambda}}.$$

Now choose $\varepsilon > 0$ sufficiently small such that $1/2 - \epsilon/\Theta > 0$. By (W_2), $|W(t, u)| = o(|u|^2)$ uniformly in t as $|u| \to 0$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|W(t, u(t))| \leq \varepsilon |u(t)|^2 \quad \text{whenever } |u(t)| < \delta.$$

Let $\rho = \delta/(C_\infty(1 + 1/\Theta)^{1/2})$ and $\|u\|_{X^{\alpha,\lambda}} \leq \rho$, then

$$|u(t)| \leq C_\infty \left(1 + \frac{1}{\Theta}\right)^{1/2} \|u\|_{X^{\alpha,\lambda}} \leq \delta.$$

Hence

$$(3.5)\qquad |W(t, u(t))| \leq \varepsilon |u(t)|^2, \quad \text{for all } t \in \mathbb{R}.$$

So, if $\|u\|_{X^{\alpha,\lambda}} = \rho$, then

$$(3.6)\qquad \begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} W(t, u(t))\, dt \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon}{\Theta}\right)\|u\|^2_{X^{\alpha,\lambda}} \geq \left(\frac{1}{2} - \frac{\varepsilon}{\Theta}\right)\rho^2 \equiv \beta > 0. \end{aligned}$$

Let $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ with $\|\varphi\|_{X^{\alpha,\lambda}} = 1$. It remains to prove that there exists $e \in X^{\alpha,\lambda}$ such that $\|e\|_{X^{\alpha,\lambda}} > \rho$ and $I_\lambda(e) \leq 0$, where ρ is defined above. Arguing by contradiction, we may assume that there exists $\{\sigma_k\} \subset \mathbb{R}$, $|\sigma_k| \to \infty$ such that $I_\lambda(\sigma_k\varphi) > 0$ for all k. Then, we have

$$(3.7)\qquad 0 < \frac{I_\lambda(\sigma_k\varphi)}{\sigma_k^2} = \frac{1}{2} - \int_{\mathbb{R}} \frac{W(t, \sigma_k\varphi)}{|\sigma_k\varphi|^2}|\varphi|^2\, dt.$$

Since $|\sigma_k\varphi(t)| \to \infty$ for t with $\varphi(t) \neq 0$, and since $\|\varphi\|_{X^{\alpha,\lambda}} = 1$, by (1.2) and Fatou's Lemma, we have that

$$\int_{\mathbb{R}} \frac{W(t,\sigma_k\varphi)}{|\sigma_k\varphi|^2} |\varphi|^2 \, dt \to \infty \quad \text{as } k \to \infty.$$

This contradicts (3.7). So we conclude taking $e = \sigma\varphi$ with σ large enough.

Now, let us introduce the Nehari's manifold defined by

$$\mathcal{N}_\lambda = \{u \in X^{\alpha,\lambda} \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\},$$

and we note that, for $u \in \mathcal{N}_\lambda$

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{2}\langle I'_\lambda(u), u\rangle = \int_{\mathbb{R}} \left(\frac{1}{2}(\nabla W(t,u(t)), u(t)) - W(t,u(t))\right) dt.$$

Define

$$c_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda(u).$$

In the following lemmas we assume that $(\mathcal{L}_1)$–$(\mathcal{L}_2)$, (W_1)–(W_4) hold and $\lambda > 0$.

LEMMA 3.2. *Let $S_\lambda = \{u \in X^{\alpha,\lambda} : \|u\|_{X^{\alpha,\lambda}} = 1\}$. For all $u \in S_\lambda$ there exists a unique $\sigma_u > 0$ such that $\sigma_u u \in \mathcal{N}_\lambda$. Furthermore*

$$I_\lambda(\sigma_u u) = \max_{\sigma \geq 0} I_\lambda(\sigma u).$$

PROOF. Let $u \in S_\lambda$ be fixed and define $h(\sigma) = I_\lambda(\sigma u)$ for $\sigma \geq 0$. Then

$$h(\sigma) = \frac{\sigma^2}{2}\|u\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} W(t,\sigma u(t))\, dt = \sigma^2\left(\frac{1}{2} - \int_{\mathbb{R}} \frac{W(t,\sigma u(t))}{\sigma^2}\, dt\right). \tag{3.8}$$

By (W_2), as $\sigma \to 0$

$$\int_{\mathbb{R}} \frac{W(t,\sigma u(t))}{\sigma^2}\, dt \to 0 \tag{3.9}$$

and by (1.2), as $\sigma \to \infty$,

$$\int_{\mathbb{R}} \frac{W(t,\sigma u(t))}{\sigma^2}\, dt \to \infty. \tag{3.10}$$

Consequently, by (W_4) and (3.8)–(3.10), there is a unique $\sigma_u = \sigma(u) > 0$ such that $h'(\sigma_u) = 0$ and

$$h(\sigma_u) = \max_{\sigma \geq 0} I_\lambda(\sigma u). \tag{3.11}$$

Furthermore $\sigma_u u \in \mathcal{N}_\lambda$. □

LEMMA 3.3. *The set $\mathcal{N}_\lambda$ is bounded away from 0. Furthermore, $\mathcal{N}_\lambda$ is closed in $X^{\alpha,\lambda}$.*

Proof. Following Remark 3.1, we can conclude that

$$I_\lambda(u) = \frac{1}{2}\|u\|^2_{X^{\alpha,\lambda}} + o(\|u\|^2_{X^{\alpha,\lambda}}) \quad \text{as } u \to 0. \tag{3.12}$$

Therefore there exists $\nu > 0$ such that $u \in \mathcal{N}_\lambda$ implies $\|u\|_{X^{\alpha,\lambda}} \geq \nu$. So, $\mathcal{N}_\lambda$ is bounded away from 0.

Now we prove that the set $\mathcal{N}_\lambda$ is closed in $X^{\alpha,\lambda}$. First, we note that I'_λ maps bounded sets in $X^{\alpha,\lambda}$ into bounded sets in $X^{\alpha,\lambda}$. In fact, let $\{u_k\}$ be a bounded sequence in $X^{\alpha,\lambda}$, then by (2.10) and (2.12), there exists $K_1 > 0$ such that $\|u_k\|_\infty \leq K_1$ for each $k \in \mathbb{N}$. From (W_2), there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$ and $|u| < \delta$, $|\nabla W(t,u)| \leq |u|$.

Now, let $M_1 = \max\{\overline{W}(u) : |u| \leq K_1\}$ and $K_2 = \max\{1, M_1/\delta\}$. If $|u_k(t)| < \delta$, then

$$|\nabla W(t, u_k(t))| \leq |u_k(t)|.$$

On the other hand, by (W_3), if $\delta \leq |u_k(t)| \leq K_1$, then

$$|\nabla W(t, u_k(t))| \leq \overline{W}(u_k(t)) \leq M_1 \leq \frac{M_1}{\delta}\,|u_k(t)|.$$

Therefore, for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$|\nabla W(t, u_k(t))| \leq K_2|u_k(t)|. \tag{3.13}$$

Next, by (3.13), the Hölder inequality and (2.11)

$$\bigg|\int_{\mathbb{R}^n} (\nabla W(t,u_k(t)), \varphi(t))\,dt\bigg| \leq K_2 \int_{\mathbb{R}} |u_k(t)||\varphi(t)|\,dt \leq \frac{K_2}{\Theta}\,\|u_k\|_{X^{\alpha,\lambda}}\|\varphi\|_{X^{\alpha,\lambda}}$$

for all $\varphi \in X^{\alpha,\lambda}$. So, for each $\varphi \in X^{\alpha,\lambda}$,

$$\begin{aligned} I'_\lambda(u_k)\varphi &= \langle u_k, \varphi\rangle_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} (\nabla W(t,u_k(t)), \varphi(t))\,dt \\ &\leq \|u_k\|^2_{X^{\alpha,\lambda}}\|\varphi\|^2_{X^{\alpha,\lambda}} + \frac{K_2}{\Theta}\,\|u_k\|_{X^{\alpha,\lambda}}\|\varphi\|_{X^{\alpha,\lambda}} \leq K_3. \end{aligned}$$

Now we are in position to prove that $\mathcal{N}_\lambda$ is closed in $X^{\alpha,\lambda}$. Let $u_k \in \mathcal{N}_\lambda$ be such that $u_k \to u$ in $X^{\alpha,\lambda}$. Since $I'_\lambda(u_k)$ is bounded, then we infer from

$$I'_\lambda(u_k)u_k - I'_\lambda(u)u = \langle I'_\lambda(u_k) - I'_\lambda(u), u\rangle - \langle I'_\lambda(u_k), u_k - u\rangle \to 0, \quad \text{as } k \to \infty,$$

that $I'_\lambda(u)u = 0$. Furthermore, since $\mathcal{N}_\lambda$ is bounded away from 0, we have

$$\|u\|_{X^{\alpha,\lambda}} = \lim_{k\to\infty} \|u_k\|_{X^{\alpha,\lambda}} \geq \nu > 0.$$

So $u \in \mathcal{N}_\lambda$. □

Lemma 3.4. *There exists $\kappa > 0$ such that $\sigma_u \geq \kappa$ for all $u \in S_\lambda$, and for each compact subset $\mathfrak{W} \subset S_\lambda$ there exists a constant $C_{\mathfrak{W}} > 0$ such that*

$$\sigma_u \leq C_{\mathfrak{W}} \quad \text{for all } u \in S_\lambda.$$

PROOF. For $u \in S_\lambda$, there exists $\sigma_u > 0$ such that $\sigma_u u \in \mathcal{N}_\lambda$. By Lemma 3.3, one sees that $\sigma_u \geq \nu > 0$. To prove that $\sigma_u \leq C_{\mathfrak{W}}$ for all $u \in \mathfrak{W} \subset S_\lambda$, we argue by contradiction. Suppose that there exist $u_k \in \mathfrak{W}$ such that $\sigma_k = \sigma_{u_k} \to \infty$. Since $\mathfrak{W}$ is compact, there exists $u \in \mathfrak{W}$ such that $u_k \to u$ in $X^{\alpha,\lambda}$ and $u_k(t) \to u(t)$ almost everywhere on $\mathbb{R}$. Therefore

$$\begin{aligned}\frac{I_\lambda(\sigma_k u_k)}{\sigma_k^2} &= \frac{1}{2}\|u_k\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} \frac{W(t,\sigma_k u_k(t))}{\sigma_k^2}\,dt \\ &= \frac{1}{2} - \int_{\mathbb{R}} \frac{W(t,\sigma_k u_k(t))}{|\sigma_k u_k(t)|^2}\,|u_k(t)|^2\,dt.\end{aligned} \tag{3.14}$$

Since $|\sigma_k u_k(t)| \to \infty$ if $u(t) \neq 0$, it follows from (1.2), (3.14) and Fatou's Lemma that $I_\lambda(\sigma_k u_k) \to -\infty$ as $k \to \infty$. □

LEMMA 3.5. *$c_\lambda \geq \rho > 0$, where $\rho > 0$ is independent of λ.*

PROOF. For $u \in \mathcal{N}_\lambda$, (W_1) and Lemma 3.3, we obtain:

$$\begin{aligned}I_\lambda(u) &= I_\lambda(u) - \frac{1}{\theta}\langle I'_\lambda(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2_{X^{\alpha,\lambda}} + \int_{\mathbb{R}} \left(\frac{1}{\theta}(\nabla W(t,u), u) - W(t,u)\right) dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2_{X^{\alpha,\lambda}} \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\nu := \rho > 0.\end{aligned}$$ □

REMARK 3.6. Following [29], by Lemma 3.2 we can get the following characterization:

$$c_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) = \inf_{u \in X^{\alpha,\lambda}\setminus\{0\}} \max_{s>0} I_\lambda(su) = \inf_{u \in S_\lambda\setminus\{0\}} \max_{s>0} I_\lambda(su).$$

On the other hand, choosing $\varphi_0 \in C_0^\infty(T)$, there exists a constant $\mathfrak{C}_0 > 0$ independent of λ, such that

$$c_\lambda = \inf_{u \in X^{\alpha,\lambda}\setminus\{0\}} \max_{s>0} I_\lambda(su) \leq \max_{s\geq 0} I_\lambda(s\varphi_0) \leq \mathfrak{C}_0. \tag{3.15}$$

Now, define the mapping $m_\lambda\colon S_\lambda \to \mathcal{N}_\lambda$ by setting $m_\lambda(u) := \sigma_u u$, where σ_u is as in Lemma 3.2 and S_λ is the unit sphere in $X^{\alpha,\lambda}$. Furthermore, by Lemma 3.2 and Proposition 3.1 of [24], m_λ is a homeomorphism between S_λ and $\mathcal{N}_\lambda$ and the inverse of m_λ is given by

$$m_\lambda^{-1}(u) = \frac{u}{\|u\|_{X^{\alpha,\lambda}}}. \tag{3.16}$$

Now we shall consider the functional $\Phi_\lambda\colon S_\lambda \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = I_\lambda(m_\lambda(u)).$$

As in [24], we have the following lemma.

Lemma 3.7.

(a) $\Phi_\lambda \in C^1(S_\lambda, \mathbb{R})$ *and*

$$\langle \Phi'_\lambda(u), v\rangle = \|m_\lambda(u)\|_{X^{\alpha,\lambda}} \langle I'_\lambda(m_\lambda(u)), v\rangle$$

for all $v \in \mathcal{T}_w(S_\lambda) = \{h \in X^{\alpha,\lambda} : \langle u, h\rangle_{X^{\alpha,\lambda}} = 0\}$.

(b) *If* $\{u_n\}$ *is a* (PS) *sequence for* Φ_λ *then* $\{m_\lambda(u_n)\}$ *is a* (PS) *sequence for* I_λ. *If* $\{u_n\} \subset \mathcal{N}_\lambda$ *is a bounded* (PS) *sequence for* I_λ, *then* $\{m_\lambda^{-1}(u_n)\}$ *is a* (PS) *sequence for* Φ_λ, *where* $m_\lambda^{-1}(u)$ *is given by* (3.16).

(c) $\inf_{S_\lambda} \Phi_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda$. *Furthermore, u is a critical point of* Φ_λ *if and only if* $m_\lambda(u)$ *is a nontrivial critical point of* I_λ. *Furthermore, the corresponding critical values of* Φ_λ *and* I_λ *coincide.*

Now, we investigate the minimizing sequence for I_λ.

Lemma 3.8. *Suppose that* $(\mathcal{L}_1)$–$(\mathcal{L}_2)$, (W_1)–(W_4) *hold and* $\lambda \geq 1$. *If* $\{u_n\} \subset \mathcal{N}_\lambda$ *be a minimizing sequence for* I_λ, *then* $\{u_n\}$ *is bounded in* $X^{\alpha,\lambda}$.

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda$ such that $I_\lambda(u_n) \to c_\lambda$, as $n \to \infty$. Then, by (W_1) and $\langle I'_\lambda(u_n), u_n\rangle = 0$, we obtain

$$\begin{aligned} c_\lambda + o(1) &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_{X^{\alpha,\lambda}} \\ &\quad + \int_{\mathbb{R}} \left(\frac{1}{\theta}(\nabla W(t, u_n(t)), u_n(t)) - W(t, u_n(t))\right) dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_{X^{\alpha,\lambda}}. \end{aligned} \tag{3.17}$$

Therefore, (3.17) implies that $\{u_n\}$ is bounded in $X^{\alpha,\lambda}$. □

Following Lemmas 2.1 and 3.3 in [29], we can show the following version of the Lions concentration compactness principle.

Lemma 3.9. *Let* $r > 0$ *and* $q \geq 2$. *Let* $\{u_n\} \in X^{\alpha,\lambda}$ *be bounded. If*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{(y-r,y+r)} |u_n(t)|^q \, dt = 0,$$

then $u_n \to 0$ *in* $L^p(\mathbb{R}, \mathbb{R}^n)$ *for any* $p > 2$.

Lemma 3.10. *Under the assumptions of Theorem 1.2, if* $\{u_n\} \subset \mathcal{N}_\lambda$ *is a sequence such that*

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0, \tag{3.18}$$

then there exists $\Lambda^* > 0$ *such that* $\{u_n\}$ *has a convergent subsequence in* $X^{\alpha,\lambda}$ *for all* $\lambda > \Lambda^*$.

PROOF. By (3.17) and (3.18) we deduce that $\{u_n\}$ is bounded in $X^{\alpha,\lambda}$. Since $X^{\alpha,\lambda}$ is a reflexive space, there is a subsequence still called $\{u_n\} \in X^{\alpha,\lambda}$ and $u \in X^{\alpha,\lambda}$ such that $u_n \rightharpoonup u$. Furthermore, by Remark 2.6 and the Sobolev Theorem

$$u_n \to u \quad \text{in } L^p_{\rm loc}(\mathbb{R}) \text{ for } p \in [2, \infty],$$

and, we have either $\{u_n\}$ is vanishing, namely

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{(t-r,t+r)} |u_n(s)|^2\, ds = 0 \tag{3.19}$$

or non-vanishing, namely, there exist $r, \beta > 0$ and a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{(t_n-r,t_n+r)} |u_n(s)|^2\, ds \geq \beta. \tag{3.20}$$

We claim that $u \neq 0$. By contradiction, we suppose that $u = 0$. If $\{u_n\}$ is vanishing, by Lemma 3.9, $u_n \to 0$ in $L^p(\mathbb{R}, \mathbb{R}^n)$ for $p > 2$. So, following the ideas of the proof of Lemma 3.3 , we deduce that

$$\int_{\mathbb{R}} (\nabla W(t, u_n(t)), u_n(t))\, dt \to 0. \tag{3.21}$$

Therefore, by (3.21) and $\langle I'_\lambda(u_n), u_n\rangle = 0$, we obtain that

$$\|u_n\|_{X^{\alpha,\lambda}} \to 0 \quad \text{as } n \to \infty.$$

This contradicts the conclusion of Lemma 3.3.

On the other hand, by (3.15) and (3.17) we have

$$\limsup_{n\to\infty} \|u_n\|^2_{X^{\alpha,\lambda}} \leq \frac{2\theta}{\theta-2}\, \mathfrak{C}_0. \tag{3.22}$$

Therefore, if $\{u_n\}$ is non-vanishing, then (3.20) implies that $|t_n| \to \infty$ as $n \to \infty$. Then $|(t_n - r, t_n + r) \cap \{t \in \mathbb{R} : l(t) < c\}| \to 0$ as $n \to \infty$. So, by the Hölder inequality, we obtain

$$\int_{(t_n-r,t_n+t)\cap\{l<c\}} u_n^2\, dt \to 0. \tag{3.23}$$

Combining (3.20), (3.22) and (3.23), one has

$$\begin{aligned}\frac{2\theta}{\theta-2}\, \mathfrak{C}_0 &\geq \limsup_{n\to\infty} \|u_n\|^2_{X^{\alpha,\lambda}} \geq \lambda c \limsup_{n\to\infty} \int_{(t_n-r,t_n+r)\cap\{l\geq c\}} u_n^2(t)\, dt \\ &= \lambda c \limsup_{n\to\infty} \bigg(\int_{(t_n-r,t_n+r)} u_n^2(t)\, dt \\ &\qquad - \int_{(t_n-r,t_n+r)\cap\{l<c\}} u_n^2(t)\, dt \bigg) \geq \lambda c\beta.\end{aligned} \tag{3.24}$$

Let

$$\Lambda_* = \max\left\{ \frac{1}{cC^2_\infty \text{meas}\{l < c\}}, \frac{2\theta\mathfrak{C}_0}{(\theta-2)c\beta} \right\},$$

then we obtain that $\lambda > \Lambda_* > 2\theta\, \mathfrak{C}_0/((\theta - 2)c\beta)$, which contradicts with (3.24).

To conclude, we need to prove that $u_k \to u$ in $X^{\alpha,\lambda}$. First, we note that the function $(\nabla W(t,su),su)/\theta - W(t,su)$ is non-decreasing for $s>0$. In fact, let $0<s_1<s_2$, then we have

$$\begin{aligned}
&(\nabla W(t,s_1u),s_1u) - \theta W(t,s_1u)\\
&= (\nabla W(t,s_1u),s_1u) + \theta W(t,s_2u) - \theta W(t,s_2u) - \theta W(t,s_1u)\\
&= (\nabla W(t,s_1u),s_1u) - \theta W(t,s_2u) + \theta \int_{s_1}^{s_2} (\nabla W(t,ru),u)\,dr\\
&\leq (\nabla W(t,s_1u),s_1u) - \theta W(t,s_2u) + \frac{(\nabla W(t,s_2u),u)}{s_2^{\theta-1}}\,(s_2^\theta - s_1^\theta)\\
&\leq (\nabla W(t,s_1u),s_1u) - \theta W(t,s_2u) + s_2(\nabla W(t,s_2u),u) - s_1(\nabla W(t,s_1u,u))\\
&= (\nabla W(t,s_2u),s_2u) - \theta W(t,s_2u).
\end{aligned}$$

Finally, since $u \neq 0$ and Lemma 3.2 there exists $\sigma \in (0,1]$ such that $\sigma u \in \mathcal{N}_\lambda$, then by Fatou's Lemma, it is easy to check that

$$\begin{aligned}
c_\lambda &\leq I_\lambda(\sigma u) = I_\lambda(\sigma u) - \frac{1}{\theta}\, I'_\lambda(\sigma u)\sigma u\\
&= \sigma^2\bigg(\frac12 - \frac1\theta\bigg)\|u\|^2_{X^{\alpha,\lambda}} + \int_{\mathbb{R}} \bigg(\frac1\theta\,(\nabla W(t,\sigma u(t)),\sigma u(t)) - W(t,\sigma u(t))\bigg)\,dt\\
&\leq \bigg(\frac12 - \frac1\theta\bigg)\|u\|^2_{X^{\alpha,\lambda}} + \int_{\mathbb{R}} \bigg(\frac1\theta\,(\nabla W(t,u(t)),u(t)) - W(t,u(t))\bigg)\,dt\\
&\leq \liminf_{n\to\infty}\bigg\{\bigg(\frac12 - \frac1\theta\bigg)\|u_n\|^2_{X^{\alpha,\lambda}} + \int_{\mathbb{R}} \bigg(\frac1\theta\,(\nabla W(t,u_n(t)),u_n(t)) - W(t,u_n(t))\bigg)\bigg\}\\
&\leq \limsup_{k\to\infty}\bigg\{\bigg(\frac12 - \frac1\theta\bigg)\|u_n\|^2_{X^{\alpha,\lambda}} + \int_{\mathbb{R}} \bigg(\frac1\theta\,(\nabla W(t,u_n(t)),u_n(t)) - W(t,u_n(t))\bigg)\bigg\}\\
&= \lim_{n\to\infty}\bigg\{I_\lambda(u_n) - \frac1\theta\, I'_\lambda(u_n)u_n\bigg\} = c_\lambda.
\end{aligned}$$

Hence, $\|u_n\|^2_{X^{\alpha,\lambda}} \to \|u\|^2_{X^{\alpha,\lambda}}$ in $\mathbb{R}$, from which it follows that $u_n \to u$ in $X^{\alpha,\lambda}$.□

4. Proof of Theorem 1.4

In the following, we study the concentration of solutions for problem $(\mathrm{FHS})_\lambda$ as $\lambda \to \infty$. Firstly, for technical reason we consider $T=[0,L]$ and the following fractional boundary value problem:

$$\begin{cases} {}_tD^\alpha_L {}_0D^\alpha_t u = \nabla W(t,u), & t \in (0,L),\\ u(0)=u(L)=0. \end{cases} \tag{4.1}$$

Associated to (4.1) we have the functional $I\colon E^\alpha_0 \to \mathbb{R}$ given by

$$I(u) := \frac12 \int_0^T |{}_0D^\alpha_t u(t)|^2\,dt - \int_0^T F(t,u(t))\,dt$$

and we have that $I \in C^1(E_0^\alpha, \mathbb{R})$ with

$$I'(u)v = \int_0^T \langle {}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t)\rangle \, dt - \int_0^T \langle \nabla W(t, u(t)), v(t)\rangle \, dt.$$

The Nehari manifold corresponding to I is defined by $\widetilde{\mathcal{N}} = \{u \in E_0^\alpha \setminus \{0\} : I'(u)u = 0\}$, and let $\widetilde{c} = \inf_{u \in \widetilde{\mathcal{N}}} I(u)$. Furthermore, we can show that

$$\widetilde{c} = \inf_{w \in E_0^\alpha} \max_{\sigma > 0} I(\sigma w) = \inf_{u \in \widetilde{S} \setminus \{0\}} \max_{\sigma > 0} I(\sigma u),$$

and if we follow the ideas of the proof of Theorem 1.2, we can get the following existence result:

THEOREM 4.1. *Suppose that W satisfies* (W_1)–(W_4) *with $t \in [0, L]$, then* (4.1) *has a ground state solution.*

Furthermore, under the assumptions $(\mathcal{L})_1$–$(\mathcal{L})_3$ and (W_1)–(W_2), we can get that $c_\lambda \le \widetilde{c}$ for $\lambda > 0$. In fact, by Theorem 4.1, let $\widetilde{u} \in E_0^\alpha$ be a ground state solution of (4.1), then $\widetilde{c} = I(\widetilde{u})$. Therefore,

$$c_\lambda \le \max_{\sigma > 0} I_\lambda(\sigma \widetilde{u}) = \max_{\sigma > 0} I(\sigma \widetilde{u}) = I(\widetilde{u}) = \widetilde{c} \quad \text{for all } \lambda > 0.$$

PROOF OF THEOREM 1.4. We follow the argument in [35]. For any sequence $\lambda_k \to \infty$, let $u_k = u_{\lambda_k}$ be the critical point of I_{λ_k}, namely $c_{\lambda_k} = I_{\lambda_k}(u_k)$ and $I'_{\lambda_k}(u_k) = 0$ and, by (W_1), we get

$$\begin{aligned} c_{\lambda_k} &= I_{\lambda_k}(u_k) = I_{\lambda_k}(u_k) - \frac{1}{\theta} I'_{\lambda_k}(u_k)u_k \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2_{X^{\alpha,\lambda_k}} + \int_{\mathbb{R}} \left[\frac{1}{\theta} (\nabla W(t, u_k(t)), u_k(t)) - W(t, u_k(t))\right] dt \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2_{X^{\alpha,\lambda_k}}. \end{aligned}$$

Therefore, by (3.15),

$$\sup_{k \ge 1} \|u_k\|^2_{X^{\alpha,\lambda_k}} \le \frac{2\theta}{\theta - 2} \mathfrak{C}_0, \tag{4.2}$$

where $\mathfrak{C}_0$ is independent of λ_k. Therefore, we may assume that $u_k \rightharpoonup \widetilde{u}$ weakly in X^{α,λ_k}. Moreover, by Fatou's Lemma, we have

$$\begin{aligned} \int_{\mathbb{R}} l(t)|\widetilde{u}(t)|^2 \, dt &\le \liminf_{k \to \infty} \int_{\mathbb{R}} l(t)|u_k(t)|^2 \, dt \\ &\le \liminf_{k \to \infty} \int_{\mathbb{R}} (L(t)u_k(t), u_k(t)) \, dt \le \liminf_{k \to \infty} \frac{\|u_k\|^2_{X^{\alpha,\lambda_k}}}{\lambda_k} = 0. \end{aligned}$$

Thus, $\widetilde{u} = 0$ almost everywhere in $\mathbb{R} \setminus J$. Now, for any $\varphi \in C_0^\infty(T, \mathbb{R}^n)$, since $I'_{\lambda_k}(u_k)\varphi = 0$, it is easy to see that

$$\int_0^L ({}_0D_t^\alpha \widetilde{u}(t), {}_0D_t^\alpha \varphi(t)) \, dt - \int_0^L (\nabla W(t, \widetilde{u}(t)), \varphi(t)) \, dt = 0,$$

that is, $\widetilde{u}$ is a solution to (4.1) by the density of $C_0^\infty(T,\mathbb{R}^n)$ in E^α.

Next, we show that $u_k \to \widetilde{u}$ strongly in $L^r(\mathbb{R})$ for $2 \le r < \infty$. Otherwise, by Lemma 3.9, there exist $\delta, R_0 > 0$ and $t_k \in \mathbb{R}$ such that

$$\int_{t_k-R_0}^{t_k+R_0} (u_k - \widetilde{u})^2\,dt \ge \delta.$$

Moreover, $t_n \to \infty$, hence $\operatorname{meas}\{(t_k - R_0, t_k + R_0) \cap \{l < c\}\} \to 0$. By the Hölder inequality, we have

$$\int_{(t_k-R_0,t_k+R_0)\cap\{l<c\}} |u_k - \widetilde{u}|^2\,dt \le \operatorname{meas}\{(t_k - R_0, t_k + R_0) \cap \{l < c\}\}\|u_k - \widetilde{u}\|_\infty \to 0.$$

Consequently,

$$\begin{aligned}
\|u_k\|^2_{X^{\alpha,\lambda_k}} &\ge \lambda_k c \int_{(t_k-R_0,t_k+R_0)\cap\{l\ge c\}} |u_k(t)|^2\,dt \\
&= \lambda_k c \int_{(t_k-R_0,t_k+R_0)\cap\{l\ge c\}} |u_k(t) - \widetilde{u}(t)|^2\,dt \\
&= \lambda_k c \bigg(\int_{(t_k-R_0,t_k+R_0)} |u_k(t) - \widetilde{u}(t)|^2\,dt \\
&\qquad - \int_{(t_k-R_0,t_k+R_0)\cap\{l<c\}} |u_k - \widetilde{u}|^2\,dt \bigg) + o(1) \to \infty,
\end{aligned}$$

which contradicts (4.2).

Now we show that $u_k \to \widetilde{u}$ in X^α. Since $I'_{\lambda_k}(u_k)u_k = I'_{\lambda_k}(u_k)\widetilde{u} = 0$, we have

$$\|u_k\|^2_{X^{\alpha,\lambda_k}} = \int_{\mathbb{R}} (\nabla W(t, u_k(t)), u_k(t))\,dt \tag{4.3}$$

and

$$\langle u_k, \widetilde{u}\rangle_{\lambda_k} = \int_{\mathbb{R}} (\nabla W(t, u_k(t)), \widetilde{u}(t))\,dt, \tag{4.4}$$

which implies that

$$\lim_{k\to\infty} \|u_k\|^2_{X^{\alpha,\lambda_k}} = \lim_{k\to\infty} \langle u_k, \widetilde{u}\rangle_{X^{\alpha,\lambda_k}} = \lim_{k\to\infty} \langle u_k, \widetilde{u}\rangle_{X^\alpha} = \|\widetilde{u}\|^2_{X^\alpha}.$$

Furthermore, by the weakly semi-continuity of norms, we obtain

$$\|\widetilde{u}\|^2_{X^\alpha} \le \liminf_{k\to\infty} \|u_k\|^2_{X^\alpha} \le \limsup_{k\to\infty} \|u_k\|^2_{X^\alpha} \le \lim_{k\to\infty} \|u_k\|^2_{X^{\alpha,\lambda_k}}.$$

So $u_k \to \widetilde{u}$ in X^α, and $u_k \to \widetilde{u}$ in $H^\alpha(\mathbb{R},\mathbb{R}^n)$ as $k \to \infty$. □

REFERENCES

[1] O. AGRAWAL, J. TENREIRO MACHADO AND J. SABATIER, *Introduction to Fractional Derivatives and Their Applications*, Nonlinear Dynamics **38** (2004), 1–2.

[2] Z.B. BAI AND H.S. LÜ, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495–505.

[3] V. COTI ZELATI AND P.H. RABINOWITZ, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc. **4** (1991), 693–727.

[4] Y.H. DING, *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Anal. **25** (1995), 1095–1113.

[5] M. DU, L. TIAN, J. WANG AND F. ZHANG, *Existence of ground state solutions for a superbiquadratic Kirchhoff-type equation with steep potential well*, Applicable Analysis (2015), DOI: 10.1080/00036811.2015.1022312.

[6] I. EKELAND, *Convexity Methods in Hamiltonian Mechanics*, Springer, Berlin, 1990.

[7] V. ERVIN AND J. ROOP, *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Methods Parial Differential Equations **22** (2006), 58–76.

[8] R. HILFER, *Applications of Fractional Calculus in Physics*, World Science, Singapore, 2000.

[9] M. IZYDOREK AND J. JANCZEWSKA, *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations **219** (2005), 375–389.

[10] M. IZYDOREK AND J. JANCZEWSKA, *Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential*, J. Math. Anal. Appl. **335** (2007), 1119–1127.

[11] W.H. JIANG, *The existence of solutions for boundary value problems of fractional differential equatios at resonance*, Nonlinear Anal. **74** (2011), 1987–1994.

[12] F. JIAO AND Y. ZHOU, *Existence results for fractional boundary value problem via critical point theory*, Internat. J. Bifur. Chaos appl. Sci. Engrg. **22** (2012), 1–17.

[13] A. KILBAS, H. SRIVASTAVA AND J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, North-Holland, Singapore, 2006.

[14] J. MAWHIN AND M. WILLEM, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[15] A. MENDEZ AND C. TORRES, *Multiplicity of solutions for fractional Hamiltonian systems with Liouville–Weyl fractional derivatives*, Fract. Calc. Appl. Anal. **18** (2015), 875–890.

[16] K. MILLER AND B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993.

[17] W. OMANA AND M. WILLEM, *Homoclinic orbits for a class of Hamiltonian systems*, Differential Integral Equations **5** (1992), 1115–1120.

[18] H. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, Gauthier–Villars, Paris, 1897–1899.

[19] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, New York, 1999.

[20] P. RABINOWITZ, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math., Vol. 65, American Mathematical Society, Provodence, 1986.

[21] P. RABINOWITZ, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A. **114** (1990), 33–38.

[22] P. RABINOWITZ AND K. TANAKA, *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z. **206** (1991), 473–499.

[23] M. SCHECHTER, *Linking Methods in Critical Point Theory*, Birkhäuser, Boston, 1999.

[24] A. Szulkin and T. Weth, *The method of Nehari manifold*, Handbook of Nonconvex Analysis and Applications (D. Gao and D. Motreanu, eds.) International Press, Boston, 2010, pp. 597–632.

[25] C. Torres, *Existence of solutions for a class of fractional Hamiltonian systems*, Electron. J. Differential Equations **2013** (2013), no. 259, 1–12.

[26] C. Torres, *Existence of solutions for perturbed fractional Hamiltonian systems*, J. Fract. Calc. Appl. **6** (2015), no. 1, 62–70.

[27] C. Torres, *Exstence and concentration of solution for a class of fractional Hamiltonian systems with subquadratic potential*, Proc. Math. Sci. (in press).

[28] C. Torres, *Mountain pass solution for a fractional boundary value problem*, J. Fract. Calc. Appl. **1** (2014), no. 1, 1–10.

[29] C. Torres, *Ground state solution for differential equations with left and right fractional derivatives*, Math. Meth. Appl. Sci. **38** (2015), 5063–5073.

[30] C. Torres, *Existence and symmetric result for Liouville–Weyl fractional nonlinear Schrödinger equation*, Commun. Nonlinear Sci. Numer Simul. **27** (2015), 314–327.

[31] J.F. Xu, D. O'Regan and K.Y. Zhang, *Multiple solutions for a class of fractional Hamiltonian systems*, Fract. Calc. Appl. Anal. **18** (2015), no. 1, 48–63.

[32] S.Q. Zhang, *Existence of a solution for the fractional differential equation with nonlinear boundary conditions*, Comput. Math. Appl. **61** (2011), 1202–1208.

[33] Z.H. Zhang and R. Yuan, *Variational approach to solutions for a class of fractional Hamiltonian systems*, Meth. Methods Appl. Sci. **37** (2014), no. 13, 1873–1883.

[34] Z.H. Zhang and R. Yuan, *Solutions for subquadratic fractional Hamiltonian systems without coercive conditions*, Meth. Methods Appl. Sci. **37** (2014), no. 18, 2934–2945.

[35] Z.H. Zhang and C. Torres, *Solutions for fractional Hamiltonian systems with a parameter*, J. Appl. Math. Comput. **54** (2017), 451–468.

Manuscript received November 25, 2016

accepted March 3, 2017

César Torres
Departamento de Matemáticas
Universidad Nacional de Trujillo
Av. Juan Pablo II
s/n. Trujillo, PERÚ

E-mail address: ctl_576@yahoo.es

Ziheng Zhang
Department of Mathematics
Tianjin Polytechnic University
Australian National University
Tianjin 300387, P.R. CHINA