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ON THE DYNAMICS OF A MODIFIED CAHN–HILLIARD EQUATION WITH BIOLOGICAL APPLICATIONS

Xiaopeng Zhao

Abstract. We study the global solvability and dynamical behaviour of the modified Cahn–Hilliard equation with biological applications in the Sobolev space $H^1(\mathbb{R}^N)$.

1. Introduction

The Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0,$$

is a classical higher-order nonlinear diffusion equation which arises in the study of phase separation on cooling binary solutions such as glasses, alloys and polymer mixtures (see [4], [24], [25]). When posed over a bounded domain, there exists a Lyapunov functional for the solutions of Cahn–Hilliard equation, therefore all solutions to the initial boundary value problem generically converge to a steady state solution asymptotically in time (see [13]). During the past years,

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the problems of stability and long time behavior of solutions to the Cahn–Hilliard equation have been studied by various authors (see e.g. Elliott and Zheng [15], Dłotko [10], Temam [28]). In addition, Debussche, Dettori [9], Cherfils, Miranville, Zelik [5] investigated the Cahn–Hilliard equation in phase separation with the thermodynamically relevant logarithmic potentials; Gilardi, Miranville and Schimperna [16], Colli, Gilardi and Sprekels [8], Wu and Zheng [30] considered the Cahn–Hilliard equation with dynamic boundary conditions.

Since Cahn–Hilliard equation is only a phenomenological model, various modifications of it have been proposed in order to capture the dynamical picture of the phase transition phenomena better. To name only a few, the Cahn–Hilliard equation with viscosity (see [11], [12]), convective Cahn–Hilliard equation (see [14], [32]), Cahn–Hilliard equation based on a microforce balance (see [17], [23]).

Recently, in [20], Khain and Sander proposed a generalized Cahn–Hilliard equation for biological applications:

$$\frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2}\left[\ln(1-q)\,\frac{\partial^2 u}{\partial x^2} + F'(u)\right] + \alpha u(u-1) = 0. \tag{1.1}$$

Equation (1.1) is modelling cells which move, proliferate and interact via adhesion in wound healing and tumor growth. Here, u is the local density of cells, q is the adhesion parameter, $\alpha > 0$ is the proliferation rate, F is the local free energy. Furthermore,

$$q = 1 - \exp\left(-\frac{J}{k_B T}\right),$$

where J corresponds to the interatomic interaction, k_B is the Boltzmann's constant and T is the absolute temperature, assumed constant. In addition, for simplicity, Cherfils, Miranville and Zelik [5] set all physical constants equal to 1 and solved the problem in high-dimensional spaces (in the two-dimensional space, the equation models, e.g. the clustering of malignant brain tumor cells, see [5], [20]), i.e. they studied asymptotic behavior the generalized Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(u) = 0 \tag{1.2}$$

endowed with Neumann boundary conditions, where $g(s) = \alpha s(s-1)$, $f(s) = s^3 - s$ and α is a positive constant. Furthermore, in [22], Miranville gave the generalized assumptions for the nonlinear terms of equation (1.2) and studied the asymptotic behaviour and finite-dimensional attractors of equation (1.2) endowed with the Dirichlet boundary condition.

Remark 1.1. In [7], Cohen and Murray introduced a generalized diffusion model for growth and dispersal in a population. Their model and the generalized Cahn–Hilliard equation with biological applications have similar form. There are some papers concerned with this diffusion model (see [21], [31]).

Dynamics properties of equation (1.2) with both Neumann and Dirichlet boundary conditions are studied in [5], [22]. The study of dynamics for equation (1.2) in unbounded domain is a rather natural extension. Note that the spectrum of the sole operator $(-\Delta)$ in $L^2(\mathbb{R})^N$ equals $[0,\infty)$ (see [12], [19]) and is purely absolutely continuous (see [12]), which means $(-\Delta)^{-1}$ exists but is unbounded with the domain dense in $L^2(\mathbb{R}^N)$. Hence, we modify the original generalized Cahn–Hilliard equation (1.2), for the needs of the Cauchy problem in $\mathbb{R}^N$, by adding the first term εI at the right hand side to deal with the invertible operator $(-\Delta+\varepsilon I)$ in $L^2(\mathbb{R}^N)$. For the explanation why the addition $(-\Delta+\varepsilon I)$ is needed in the whole space to stabilize the equation, compare Cholewa and Rodríguez–Bernal's paper [6].

In this article, we consider the global solvability and asymptotic behaviour of solutions to the Cauchy problem in $\mathbb{R}^N$ $(N \leq 3)$ for the modified Cahn–Hilliard equation with biological application

$$\begin{cases} \dfrac{\partial u}{\partial t} = (-\Delta+\varepsilon I)[(\Delta-\varepsilon I)u + f(u)] + g(u), \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where $\varepsilon > 0$, $f(s) = \gamma_2 s^3 + \gamma_1 s^2 - \gamma_0 s$, $g(s) = \alpha s(s-1)$, $\alpha, \gamma_2, \gamma_0 > 0$ and γ_1 are constants.

The main difficulties for treating problem (1.3) are caused by the unbounded domain and the nonlinear term $g(s) = \alpha s(s-1)$. Since we consider the Cauchy problem in unbounded domain, the compact embedding is always a serious problem (we are using the technique of tail estimate as in [12], [27], [29] to handle the problem). Since the term g is a polynomial of order 2 on $s \in \mathbb{R}$, it is difficult to deal with this term making a priori estimates for problem (1.3). We shall use the inverse operator $(-\Delta+\varepsilon I)^{-1}$ as in [6], [14], [22]) to handle the nonlinear term g.

This paper is organized as follows. In the next section, we give some known auxiliary results which will be used in this paper and prove global solvability of problem (1.3) in $H^1(\mathbb{R}^N)$. In Section 3, we give some a priori estimates. In Section 4, we prove the existence of $H^1(\mathbb{R}^N)$ global attractors for problem (1.3).

Throughout this paper, to simplify the notation in calculations, we agree hereafter that all the unspecified norms are taken over $L^2(\mathbb{R}^N)$, which means, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^N)}$. Furthermore, from now on we drop the dependence of the functional spaces on $\mathbb{R}^N$: therefore, L^2 means $L^2(\mathbb{R}^N)$, $W^{s,p}$ means $W^{s,p}(\mathbb{R}^N)$ and so on. In the following, the letter C will represent generic positive constants that may change from line to line even in the same inequality.

2. Global solvability in $H^1(\mathbb{R}^N)$

First of all, we recall some results used in the main part of this paper.

Lemma 2.1 (see [12]). *For $A = (-\Delta + \varepsilon I)^{-1}$ with $\varepsilon > 0$, there exists a positive constant C such that*

$$\|v\|_{H^{-1}} \leq C(\|Av\| + \|\nabla Av\|).$$

Lemma 2.2 (see [12]). *For each $r \in [2, 2N/(N-2))$, there exists a positive constant C such that*

$$\|v\|_{L^r} \leq C\|v\|_{H^s} \leq C\|v\|_{H^{-1}}^a \|v\|_{H^1}^{1-a}, \quad \text{for all } v \in H^1,$$

where $s = N/2 - N/r$ and $a = (s+1)/2 \in (0,1)$ depends on r.

Lemma 2.3 (see [26]). *$W^{k,p}(\mathbb{R}^N) \hookrightarrow W^{m,q}(\mathbb{R}^N)$ for $1 \leq p < q < \infty$ and $k - n/p \geq m - n/q$. It follows that*

$$\|f\|_{W^{m,q}} \leq C\|f\|_{W^{k,p}}.$$

Lemma 2.4. *For $A = (-\Delta + \varepsilon I)^{-1}$ with $\varepsilon > 0$, there exists a positive constant C such that*

$$\|Av\|^2 \leq \frac{1}{\varepsilon^2}\|v\|^2, \qquad \|\nabla Av\|^2 \leq \frac{1}{2\varepsilon}\|v\|^2.$$

Proof. To verify the estimate, observe that

$$\begin{aligned}
\|v\|^2 &= \|(-\Delta + \varepsilon I)Av\|^2 \\
&= \int |\Delta Av|^2\,dx - 2\varepsilon \int Av\Delta Av\,dx + \varepsilon^2 \int |Av|^2\,dx \\
&= \int |\Delta Av|^2\,dx + 2\varepsilon \int |\nabla Av|^2\,dx + \varepsilon^2 \int |Av|^2\,dx.
\end{aligned}$$

Hence

$$\int |Av|^2\,dx \leq \frac{1}{\varepsilon^2}\int |v|^2\,dx, \qquad \int |\nabla Av|^2\,dx \leq \frac{1}{2\varepsilon}\int v^2\,dx. \qquad \square$$

In the following, we consider problem (1.3) within the framework of the approach of [19] as an equation with a sectorial operator. We will use the lower index 0 in notation of the fractional order scale corresponding to realization of the sectorial operator $(-\Delta + \varepsilon I)^2$ in the space $X_0 = (H^2)^*$, with the domains $X_0^1 = H^2$ and $X_0^{1/2} = L^2$. We are interested in the $X_0^{3/4}$-solution, that is the case when the phase space equals to H^1. Problem (1.3) is then written abstractly as

$$\begin{cases} u_t = -(-\Delta_\varepsilon)^2 u - (-\Delta_\varepsilon) f(u) + g(u), \\ u(0) = u_0 \in X_0^{3/4}, \end{cases} \tag{2.1}$$

where $\Delta_\varepsilon = \Delta - \varepsilon I$.

To justify local solvability of that problem, we need to check that the nonlinear operator $\mathcal{F}(\phi) = -(-\Delta_\varepsilon)f(u) + g(u)$ is locally Lipschitz continuous as a map from $X_0^{3/4}$ to X_0. Take a bounded set $B \subset X_0^{3/4}$, let $\varphi, \psi \in B$. Note that $(-\Delta_\varepsilon)\colon X_0^{1/2} \to X_0$ is a linear isomorphism, we get

$$\begin{aligned}
&\|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\|_{X_0} \le \|(-\Delta_\varepsilon)(f(\varphi) - f(\psi))\|_{X_0} + \|g(\varphi) - g(\psi)\|_{X_0} \\
&\le c\big(\|f(\varphi) - f(\psi)\|_{X_0^{1/2}} + \|g(\varphi) - g(\psi)\|_{X_0^{1/2}}\big) \\
&\le c\big(\|(\varphi-\psi)(\varphi^2 + \psi^2 + \varphi\psi + \varphi + \psi + 1)\|_{X_0^{1/2}} + \|(\varphi - \psi)(\varphi + \psi + 1)\|_{X_0^{1/2}}\big) \\
&\le c\|(\varphi - \psi)(\varphi^2 + \psi^2 + 1)\|_{X_0^{1/2}} \\
&\le c\|\varphi - \psi\|_{L^6}(\|\varphi\|_{L^6}^2 + \|\psi\|_{L^6}^2) + \|\varphi - \psi\|_{L^2} \le C_B\|\varphi - \psi\|_{X_0^{3/4}}.
\end{aligned}$$

Therefore, the local solvability of problem (1.3) is obtained. To justify the global solvability of problem (1.3), we need to get some a priori estimates.

Having already local solvability, combining it with the a priori estimate (3.10) given in the next section, we obtain the global well-posedness of problem (1.3) in H^1. Furthermore, we can define the corresponding semigroup in H^1 through the solution to problem (1.3):

$$S(t)u_0 = u(t, u_0), \quad t \ge 0,$$

where $u(t, u_0)$ is a solution to problem (1.3) with initial data u_0.

3. Uniform a priori estimates

3.1. H^1-estimates. Let $A = (-\Delta + \varepsilon I)^{-1}$. Then, (1.3) is equivalent to

$$Au_t = (\Delta - \varepsilon I)u - (\gamma_2 u^3 + \gamma_1 u^2 - \gamma_0 u) + \alpha A[u^2 - u]. \tag{3.1}$$

In order to obtain the a priori estimate on H^1, we multiply (3.1) by u, then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\int |\nabla Au|^2\,dx + \frac{\varepsilon}{2}\frac{d}{dt}\int |Au|^2\,dx + \int |\nabla u|^2\,dx \\
&+ \varepsilon\int u^2\,dx + \int(\gamma_2 u^4 + \gamma_1 u^3 - \gamma_0 u^2)\,dx \\
&= \alpha\int A(u^2 - u)u\,dx = \alpha\int(-\Delta + \varepsilon I)AuA(u^2 - u)\,dx \\
&= -\alpha\int u^2 Au\,dx - \alpha\int |\nabla Au|^2\,dx - \varepsilon\alpha\int |Au|^2\,dx \\
&\le \frac{\gamma_2}{4}\int u^4\,dx + \frac{\alpha^2}{\gamma_2}\int |Au|^2\,dx - \alpha\int |\nabla Au|^2\,dx - \varepsilon\alpha\int |Au|^2\,dx.
\end{aligned}$$

On the other hand, we have

$$-\int(\gamma_1 u^3 - \gamma_0 u^2)\,dx \le \frac{\gamma_2}{4}\int u^4\,dx + \left(\frac{\gamma_1^2}{\gamma_2} + \gamma_0\right)\int u^2\,dx.$$

Combining the above two inequalities together gives

$$\begin{aligned}
(3.2)\quad & \frac{d}{dt}\left(\int |\nabla Au|^2\,dx + \varepsilon\int |Au|^2\,dx\right) + 2\int |\nabla u|^2\,dx \\
&\quad + 2\varepsilon\int u^2\,dx + \gamma_2\int u^4\,dx + 2\alpha\int |\nabla Au|^2\,dx \\
&\quad + \left(2\varepsilon\alpha - \frac{2\alpha^2}{\gamma_2}\right)\int |Au|^2\,dx \le 2\left(\frac{\gamma_1^2}{\gamma_2} + \gamma_0\right)\int u^2\,dx.
\end{aligned}$$

Also, multiplying (1.3) by u_t and integrating, we immediately obtain

$$\begin{aligned}
&\int |\nabla Au_t|^2\,dx + \varepsilon\int |Au_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx + \frac{\varepsilon}{2}\frac{d}{dt}\int u^2\,dx \\
&\quad + \frac{\gamma_2}{4}\frac{d}{dt}\int u^4\,dx - \frac{\gamma_0}{2}\frac{d}{dt}\int u^2\,dx + \frac{\gamma_1}{3}\frac{d}{dt}\int u^3\,dx \\
&= \alpha\int u_t A(u^2-u)\,dx = \alpha\int (u^2-u)Au_t\,dx \\
&\le \frac{\gamma_2}{2}\int u^4\,dx + \frac{\alpha^2}{2\gamma_2}\int |Au_t|^2\,dx - \alpha\int uAu_t\,dx \\
&\le \frac{\gamma_2}{2}\int u^4\,dx + \frac{\alpha^2}{2\gamma_2}\int |Au_t|^2\,dx - \alpha\int (\Delta-\varepsilon I)AuAu_t\,dx \\
&= \frac{\gamma_2}{2}\int u^4\,dx + \frac{\alpha^2}{2\gamma_2}\int |Au_t|^2\,dx + \frac{\alpha}{2}\frac{d}{dt}\int |\nabla Au|^2\,dx + \frac{\alpha\varepsilon}{2}\int |Au|^2\,dx,
\end{aligned}$$

that is

$$\begin{aligned}
(3.3)\quad & \frac{d}{dt}\left(\int |\nabla u|^2 dx + \varepsilon\int u^2\,dx + \frac{\gamma_2}{2}\int u^4\,dx - \gamma_0\int u^2\,dx\right. \\
&\quad \left. + \frac{2\gamma_1}{3}\int u^3\,dx - \alpha\int |\nabla Au|^2\,dx - \alpha\varepsilon\int |Au|^2\,dx\right) \\
&\quad + 2\int |\nabla Au_t|^2\,dx + 2\varepsilon\int |Au_t|^2\,dx - \gamma_2\int u^4\,dx - \frac{\alpha^2}{\gamma_2}\int |Au_t|^2\,dx \le 0.
\end{aligned}$$

We multiply (3.2) by α and add to (3.3),

$$\begin{aligned}
(3.4)\quad & \frac{d}{dt}\left(\int |\nabla u|^2\,dx + \varepsilon\int u^2\,dx + \frac{\gamma_2}{2}\int u^4\,dx + \frac{2}{3}\gamma_1\int u^3\,dx - \gamma_0\int u^2\,dx\right) \\
&\quad + 2\alpha\int |\nabla u|^2\,dx + 2\alpha\varepsilon\int u^2\,dx + \gamma_2(\alpha-1)\int u^4\,dx \\
&\quad + 2\alpha^2\int |\nabla Au|^2\,dx + 2\alpha^2\left(\varepsilon - \frac{\alpha^2}{\gamma_2}\right)\int |Au|^2 d\,x \\
&\quad + 2\int |\nabla Au_t|^2\,dx + \left(2\varepsilon - \frac{\alpha^2}{\gamma_2}\right)\int |Au_t|^2\,dx \le 2\alpha\left(\frac{\gamma_1^2}{\gamma_2} + \gamma_0\right)\int u^2\,dx,
\end{aligned}$$

where $\alpha > 1$ and γ_2 satisfies $\gamma_2\varepsilon - \alpha^2 > 0$. Furthermore, using Cauchy's inequality, we get

$$
\begin{aligned}
(3.5)\quad & \frac{d}{dt}\bigg(\int |\nabla u|^2\,dx + \varepsilon\int u^2\,dx + \frac{\gamma_2}{2}\int u^4\,dx + \frac{2}{3}\gamma_1\int u^3\,dx - \gamma_0\int u^2\,dx\bigg) \\
&\quad + 2\alpha\int |\nabla u|^2\,dx + 2\alpha\varepsilon\int u^2\,dx \\
&\quad + \gamma_2(\alpha-1)\int u^4\,dx + A\bigg[\int u^3\,dx - \gamma_0\int u^2\,dx\bigg] \\
&\le \bigg[2\alpha\bigg(\frac{\gamma_1^2}{\gamma_2} + \gamma_0\bigg) - A\gamma_0\bigg]\int u^2\,dx + A\int u^3\,dx \\
&\le \bigg[\frac{A^2}{2\gamma_2(\alpha-1)} - A\gamma_0 + 2\alpha\bigg(\frac{\gamma_1^2}{\gamma_2} + \gamma_0\bigg)\bigg]\int u^2\,dx + \frac{\gamma_2(\alpha-1)}{2}\int u^4\,dx.
\end{aligned}
$$

Applying Vieta's theorem, if γ_2 is sufficiently large, which satisfies

$$\gamma_0^2\gamma_2^2(\alpha-1) - 4\alpha(\gamma_1^2 + \gamma_0\gamma_2) \ge 0,$$

we can find

$$A = \gamma_2(\alpha-1)\bigg[\gamma_0 + \sqrt{\gamma_0 - \frac{4\alpha(\gamma_1^2/\gamma_2 + \gamma_0)}{\gamma_2(\alpha-1)}}\bigg],$$

such that

$$\frac{A^2}{2\gamma_2(\alpha-1)} + \alpha\bigg(\frac{\gamma_1^2}{\gamma_2} + 2\bigg) - A = 0.$$

Hence, (3.5) is equivalent to

$$
\begin{aligned}
(3.6)\quad & \frac{d}{dt}\bigg(\int |\nabla u|^2\,dx + \varepsilon\int u^2\,dx + \frac{\gamma_2}{2}\int u^4\,dx + \frac{2}{3}\gamma_1\int u^3\,dx - \gamma_0\int u^2\,dx\bigg) \\
&\quad + 2\alpha\int |\nabla u|^2\,dx + 2\alpha\varepsilon\int u^2\,dx + \frac{\gamma_2(\alpha-1)}{2}\int u^4\,dx \\
&\qquad + A\bigg[\int u^3\,dx - \gamma_0\int u^2\,dx\bigg] \le 0.
\end{aligned}
$$

There exists a positive constant M, which depends on $\varepsilon, \alpha, \gamma_0, \gamma_1$ and γ_2, such that

$$\frac{d}{dt}F(u) + MF(u) \le 0, \tag{3.7}$$

where

$$F(u) = \int |\nabla u|^2\,dx + \varepsilon\int u^2\,dx + \frac{\gamma_2}{2}\int u^4\,dx + \frac{2}{3}\gamma_1\int u^3\,dx - \gamma_0\int u^2\,dx. \tag{3.8}$$

Using Gronwall's inequality, we obtain

$$F(u) \le e^{-Mt}F(u_0). \tag{3.9}$$

Adding (3.8) and (3.9) together, we have

$$\begin{aligned}\int |\nabla u|^2\,dx + (\varepsilon-\gamma_0)\int u^2\,dx + \frac{\gamma_2}{2}\int u^4\,dx &\le e^{-Mt}F(u_0) - \frac{2}{3}\gamma_1\int u^3\,dx\\ &\le e^{-Mt}F(u_0) + \frac{\varepsilon-\gamma_0}{2}\int u^2\,dx + \frac{2}{9(\varepsilon-\gamma_0)}\int u^4\,dx.\end{aligned}$$

If $\gamma_0 < \varepsilon$ and $\gamma_2 \ge 4/(9(\varepsilon-\gamma_0))$, we obtain the H^1 and L^4 a priori estimates of the solution u of the form:

$$\|u(\,\cdot\,,t)\|_{H^1}^2 + \int u^4\,dx \le Ce^{-Mt}F(u_0). \tag{3.10}$$

We summarize the global H^1 estimate (3.10) in the following way:

Theorem 3.1. *Suppose that $\alpha > 1$, γ_2 is sufficiently large and γ_0 is sufficiently small. Then, for problem* (3.1) (*or problem* (1.3)), *there exists a positive constant r_1 such that for any bounded set $B \subset H^1$,*

$$\|S(t)B\|_{H^1} \le r_1, \quad \text{for all } t \ge T_{1B},$$

where $T_{1B} = T_1(B)$ depends only on $\|B\|_{H^1}$.

Remark 3.2. By (3.4) and Theorem 3.1, we have that for any set B bounded in H^1, there is T_{1B} (which only depends on $\|B\|_{H^1}$) such that the following estimate holds:

$$\begin{aligned}\int_t^{t+1}\bigg[2\alpha\int|\nabla u|^2\,dx + 2\alpha\varepsilon\int u^2\,dx + \gamma_2(\alpha-1)\int u^4\,dx + 2\alpha^2\int|\nabla Au|^2\,dx\\ + 2\alpha^2\Big(\varepsilon-\frac{\alpha^2}{\gamma_2}\Big)\int|Au|^2\,dx + 2\int|\nabla Au_t|^2\,dx + \Big(2\varepsilon-\frac{\alpha^2}{\gamma_2}\Big)\int|Au_t|^2\,dx\bigg]\\ \le Q(r_1,\gamma_2,\gamma_1,\gamma_0,\alpha),\end{aligned}$$

for all $t \ge T_{1B}$, where $Q(\,\cdot\,)$ is a continuous increasing function in each component.

3.2. H^2-estimates. To obtain the uniform estimate of the solution u in H^2, we set $v(t) = u_t(t)$ and differentiate (3.1) with respect to time t, then

$$Av_t = (\Delta-\varepsilon I)v - (3\gamma_2u^2 + 2\gamma_1u - \gamma_0)v + \alpha[A(u^2-u)]'_u v. \tag{3.11}$$

We multiply (3.11) by v and integrate over $\mathbb{R}^N$, use Lemma 2.4, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(|\nabla Av|^2 + \varepsilon|Av|^2)\,dx + \int|\nabla v|^2\,dx + \varepsilon\int v^2\,dx + 3\gamma_2\int v^2u^2\,dx\\ = -2\gamma_1\int uv^2\,dx + \gamma_0\int v^2\,dx + \alpha\int[A(u^2-u)]'_u v^2\,dx\\ = -2\gamma_1\int uv^2\,dx + \gamma_0\int v^2\,dx + \alpha\int A[(2u-1)v]v\,dx\\ = -2\gamma_1\int uv^2\,dx + \gamma_0\int v^2\,dx + \alpha\int(2u-1)vAv\,dx\end{aligned} \tag{3.12}$$

$$\leq 3\gamma_2 \int u^2 v^2\,dx + C_4 \int v^2\,dx + \int |Av|^2\,dx$$
$$\leq 3\gamma_2 \int u^2 v^2\,dx + C_5 \int v^2\,dx.$$

Using Lemma 2.2, for any $0 < \delta \ll 1$, we have

$$\int v^2\,dx \leq C\|v\|_{H^{-1}}\|v\|_{H^1} \leq \delta\|v\|_{H^1}^2 + C_\delta\|v\|_{H^{-1}}^2. \tag{3.13}$$

Taking δ small enough (depending on ε), combining (3.12) and (3.13) together, we obtain

$$\frac{d}{dt}\int (|\nabla Av|^2 + \varepsilon|Av|^2)\,dx \leq C'_\delta\|v\|_{H^{-1}}^2.$$

It then follows from Lemma 2.1 that

$$\frac{d}{dt}\int (|\nabla Av|^2 + \varepsilon|Av|^2)\,dx \leq C_6 \int (|\nabla Av|^2 + \varepsilon|Av|^2)\,dx.$$

Applying the uniform Gronwall lemma and Remark 3.2, for $t \geq T_{1B}$ and $u_0 \in B$, we derive that

$$\|\nabla Av(t)\|^2 + \varepsilon\|Av(t)\|^2 \leq C(r_1, \alpha, \varepsilon, \gamma_2, \gamma_1, \gamma_0). \tag{3.14}$$

On the basis of (3.1), we have

$$(\Delta - \varepsilon I)u = Au_t + (\gamma_2 u^3 + \gamma_1 u^2 - \gamma_0 u) - \alpha A[u^2 - u].$$

For the right hand side terms, we note that when $t \geq T_{1B}$, (3.14) implies

$$\|Au_t\| \leq C(r_1, \alpha, \varepsilon, \gamma_2, \gamma_1, \gamma_0).$$

In [2], the authors assume that:

$$f(x, s)s \leq C(x)s^2 + D(x)|s|, \quad \text{for all } s \in \mathbb{R},\ x \in \Omega, \tag{3.15}$$

for some suitable functions $C(x)$ and $D(x) \geq 0$ defined in Ω. It is easy to check that our assumption $f(u) = \gamma_2 u^2 + \gamma_1 u - \gamma_0 u$ is a typical case of (3.15). Similarly to [2], we have

$$\|\gamma_2 u^3 + \gamma_1 u^2 - u\| \leq Q(\|u\|_{H^1}) \leq C(r_1, \alpha, \varepsilon, \gamma_2, \gamma_1, \gamma_0).$$

By Lemma 2.3 and (3.10), we have $H^1 \hookrightarrow L^4$ for $N \leq 3$. By Lemma 2.4, we have

$$\|A[u^2 - u]\| \leq \frac{1}{\varepsilon}\,\|u^2 - u\| \leq \frac{2}{\varepsilon}\,(\|u\|_{L^4}^2 + \|u\|^2) \leq C(r_1, \alpha, \varepsilon, \gamma_2, \gamma_1, \gamma_0).$$

Hence, for all $t \geq T_{1B}$, $u_0 \in B$, we get

$$\|u(t)\|_{H^2} \leq C(r_1, \alpha, \varepsilon, \gamma_2, \gamma_1, \gamma_0).$$

That is, we obtain a uniform estimate of the solution u in H^2:

THEOREM 3.3. *Let $\alpha > 1$, γ_2 be sufficiently large and γ_0 be sufficiently small. Then for problem* (3.1) (*or problem* (1.3)), *there is a positive constant r_2 such that for any bounded set $B \subset H^1$,*

$$\|S(t)B\|_{H^2} \leq r_2, \quad \text{for all } t \geq T_{2B},$$

where $T_{2B} = T_2(B)$ depends only on $\|B\|_{H^1}$.

4. Existence of global attractors in $H^1(\mathbb{R}^N)$

In this section, on the basis of the a priori estimates obtained in the above section, we will show that the semigroup $\{S(t)\}_{t\geq 0}$ has a global H^1-attractor.

As shown in [12], [2], [29], in order to obtain the necessary (H^1, H^1)-asymptotic compactness, thanks to Theorem 3.3, we only need to prove the following tail estimate:

LEMMA 4.1. *Under the assumption of Theorem* 3.3, *for any $\eta > 0$ and any bounded set $B \subset H^1$, there exist $h = h(\eta, \|B\|_{H^1})$ and $T = t(\eta, \|B\|_{H^1})$ such that*

$$\int_{\mathcal{O}_h} (|S(t)u_0|^2 + |\nabla S(t)u_0|^2)\,dx \leq \eta, \quad \text{for all } t \geq T,\ u_0 \in B,$$

where $\mathcal{O}_h = \{x \in \mathbb{R}^N : |x| \geq h\}$.

PROOF. Similarly to [12], we choose a smooth function $\theta = \theta(s) \in [0,1]$ for any $s \in \mathbb{R}^+$, and

$$\theta(s) = 0, \quad \text{for all } s \in [0,1], \quad \text{and} \quad \theta(s) = 1, \quad \text{for all } s \geq 2.$$

Then there exists a constant C such that $|\theta(s)| + |\theta'(s)| + |\theta''(s)| \leq C$ for any $s \in \mathbb{R}^+$.

Multiplying (3.1) by $\lambda(\theta^2(|x|^2/h^2)u)$, integrating in $\mathbb{R}^N$, we get

$$\begin{aligned}
\int \lambda\bigg(\theta^2\bigg(\frac{|x|^2}{h^2}\bigg)u\bigg)Au_t\,dx &= -\lambda\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)|\nabla u|^2\,dx - \lambda\varepsilon\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)u^2\,dx \\
&- \lambda\int u\nabla u\nabla\theta^2\bigg(\frac{|x|^2}{h^2}\bigg)dx - \gamma_2\lambda\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)u^4\,dx - \gamma_1\lambda\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)u^3\,dx \\
&+ \gamma_0\lambda\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)u^2\,dx + \alpha\lambda\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)uA[u^2 - u]\,dx.
\end{aligned}$$

Multiplying (3.1) by $(\theta^2(|x|^2/h^2)u_t)$, integrating in $\mathbb{R}^N$, we obtain

$$\begin{aligned}
\varepsilon\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)|Au_t|^2\,dx &+ \int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)|\nabla Au_t|^2\,dx \\
&+ \int \nabla\theta^2\bigg(\frac{|x|^2}{h^2}\bigg)\nabla Au_t\cdot Au_t\,dx \\
&= -\frac{1}{2}\frac{d}{dt}\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)|\nabla u|^2\,dx - \frac{\varepsilon}{2}\frac{d}{dt}\int \theta^2\bigg(\frac{|x|^2}{h^2}\bigg)u^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\gamma_2}{4}\frac{d}{dt}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^4\,dx - \frac{\gamma_1}{3}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^3\,dx \\
&+ \frac{\gamma_0}{2}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^2\,dx + \alpha\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u_t A[u^2-u]\,dx.
\end{aligned}
$$

Summing up, we have

$$
\frac{d}{dt}E_u(t) + G_u(t) \le L_u(t), \tag{4.1}
$$

where λ is a positive constant and

$$
\begin{aligned}
E_u(t) = {}& \frac{1}{2}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla u|^2\,dx + \frac{\varepsilon-\gamma_0}{2}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^2\,dx \\
&+ \frac{\gamma_2}{4}\frac{d}{dt}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^4\,dx + \frac{\gamma_1}{3}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^3\,dx,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
G_u(t) = {}& \lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla u|^2\,dx + \lambda(\varepsilon-\gamma_0)\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^2\,dx \\
&+ \gamma_2\lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^4\,dx + \gamma_1\lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^3\,dx \\
&+ \varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx + \int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|\nabla Au_t|^2\,dx,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
L_u(t) = {}& -\lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)uAu_t\,dx - \lambda\int u\nabla u\nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\,dx \\
&+ \alpha\lambda\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)uA[u^2-u]\,dx - \int \nabla\theta^2\Big(\frac{|x|^2}{h^2}\Big)\nabla Au_t\cdot Au_t\,dx \\
&+ \alpha\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u_tA[u^2-u]\,dx = I_1+I_2+I_3+I_4+I_5.
\end{aligned} \tag{4.4}
$$

Note that

$$
\begin{aligned}
I_1 &\le \varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx + \frac{\lambda^2}{4\varepsilon}\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)u^2\,dx \\
&\le \varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx + \frac{\lambda^2}{4\varepsilon}\int_{h\le|x|\le\sqrt{2}h} u^2\,dx \\
&\le \varepsilon\int \theta^2\Big(\frac{|x|^2}{h^2}\Big)|Au_t|^2\,dx + \frac{\lambda^2}{4\varepsilon}\|u\|^2, \\
I_2 &\le \frac{C'}{h}\|u\|\|\nabla u\|, \\
I_3 &\le \alpha\lambda\int_{h\le|x|\le\sqrt{2}h}|uA(u^2-u)|\,dx \\
&\le \frac{\alpha\lambda}{\varepsilon^2}\|u\|\|u^2-u\| \le C'(\|u\|^2+\|u\|_{L^4}^4), \\
I_4 &\le \frac{C'}{h}\|Au_t\|\|\nabla Au_t\|.
\end{aligned}
$$

On the other hand, we have

$$\begin{aligned}
&\int \left[\theta^2\left(\frac{|x|^2}{h^2}\right)A(u^2-u)\right]^2 dx + \int \left[\nabla\left(\theta^2\left(\frac{|x|^2}{h^2}\right)A(u^2-u)\right)\right]^2 dx \\
&\quad \le C_\theta \int_{h\le|x|\le\sqrt{2}h} |A(u^2-u)|^2\,dx + \int \left|\frac{4x}{h^2}\,\theta\left(\frac{|x|^2}{h^2}\right)\theta'\left(\frac{|x|^2}{h^2}\right)A(u^2-u)\right|^2 dx \\
&\qquad + \int \left|\theta^2\left(\frac{|x|^2}{h^2}\right)\nabla A(u^2-u)\right|^2 dx \\
&\quad \le C_\theta \int |A(u^2-u)|^2\,dx + C'_\theta \int |\nabla A(u^2-u)|^2\,dx \le {C''}_\theta \|u^2-u\|^2,
\end{aligned}$$

where the constant C_θ depends only on the cutoff function θ. Then,

$$\begin{aligned}
I_5 &\le \|u_t\|_{H^{-1}} \left\|\theta^2\left(\frac{|x|^2}{h^2}\right)A(u^2-u)\right\|_{H^1} \\
&= \|u_t\|_{H^{-1}} \left\{ \int \left[\theta^2\left(\frac{|x|^2}{h^2}\right)A(u^2-u)\right]^2 dx \right. \\
&\qquad \left. + \int \left[\nabla\left(\theta^2\left(\frac{|x|^2}{h^2}\right)A(u^2-u)\right)\right]^2 dx \right\}^{1/2} \\
&\le C_7\|u_t\|_{H^{-1}}\|u^2-u\| \le C_8\|u_t\|_{H^{-1}}(\|u\|_{L^4}^2 + \|u\|).
\end{aligned}$$

By (3.10), we have $\|u\|_{L^4} \le C\|u\|_{H^1} \le Cr_1$, for all $t \ge T_{1B}$. Summing up, we deduce that

$$\begin{aligned}
(4.5)\qquad L_u(t) &\le \frac{C'}{h}\,(\|u\|\|\nabla u\| + \|Au_t\|\|\nabla Au_t\|) + C_8\|u_t\|_{H^{-1}}^2 \\
&\quad + C_9(\|u\|^2 + \|u\|_{L^4}^4) + \varepsilon \int \theta^2\left(\frac{|x|^2}{h^2}\right)|Au_t|^2\,dx \\
&\le \frac{C'}{h}(\|Au_t\|^2 + \|\nabla Au_t\|^2 + r_1^2 + r_2^2) + \varepsilon \int \theta^2\left(\frac{|x|^2}{h^2}\right)|Au_t|^2\,dx \\
&\quad + C_9(r_1^2 + r_1^4) + C_{10}(\|Au_t\|^2 + \|\nabla Au_t\|^2),
\end{aligned}$$

for all $t \ge T_{1B} + T_{2B}$. Combining (4.1), (4.2) and (4.5) together gives

$$\begin{aligned}
(4.6)\qquad &\frac{d}{dt}\,E_u(t) + 2\lambda E_u(t) + \frac{\gamma_2\lambda}{2}\int \theta^2\left(\frac{|x|^2}{h^2}\right)u^4\,dx + \frac{\gamma_1\lambda}{3}\int \theta^2\left(\frac{|x|^2}{h^2}\right)u^3\,dx \\
&\quad \le \left(\frac{C'}{h} + C_{10}\right)(\|Au_t\|^2 + \|\nabla Au_t\|^2) + \frac{C'}{h}\,(r_1^2 + r_2^2) + C_9(r_1^2 + r_1^4).
\end{aligned}$$

Note that

$$\begin{aligned}
-\frac{\gamma_1\lambda}{3}\int \theta^2\left(\frac{|x|^2}{h^2}\right)u^3\,dx &\le \frac{\gamma_2\lambda}{4}\int \theta^2\left(\frac{|x|^2}{h^2}\right)u^4\,dx + \frac{\gamma_1^2\lambda}{9\gamma_2}\int \theta^2\left(\frac{|x|^2}{h^2}\right)u^2\,dx \\
&\le \frac{\gamma_2\lambda}{4}\int \theta^2\left(\frac{|x|^2}{h^2}\right)u^4\,dx + C_\theta r_1^2.
\end{aligned}$$

Hence, (4.6) implies

$$\frac{d}{dt} E_u(t) + 2\lambda E_u(t) \leq \left(\frac{C'}{h} + C_{10}\right)(\|Au_t\|^2 + \|\nabla Au_t\|^2) + \frac{C'}{h}(r_1^2 + r_2^2) + C_{11}(r_1^2 + r_1^4). \tag{4.7}$$

On the other hand, (3.14) implies that

$$\int_t^{t+1} (\|Au_t\|^2 + \|\nabla Au_t\|^2)\, ds \leq C(r_1, r_2, \alpha, \gamma_2, \gamma_1), \tag{4.8}$$

for all $t \geq t_1 = T_{1B} + T_{2B}$. Adding (4.7) and (4.8) together, using the uniform Gronwall lemma (see[28]), we obtain

$$\begin{aligned} E_u(t) \leq{}& e^{-2\lambda(t-t_1)} E_u(t_1) \\ &+ \frac{e^{2\lambda}}{1-e^{-2\lambda}} \left[\left(\frac{C'}{h} + C_{10}\right) C(r_1, r_2, \alpha, \gamma_2, \gamma_1) \right. \\ &\left. + \frac{C'}{h}(r_1^2 + r_2^2) + C_{11}(r_1^2 + r_1^4)\right] \\ \leq{}& e^{-2\lambda(t-t_1)} E_u(t_1) + \frac{e^{2\lambda}}{1-e^{-2\lambda}} C(r_1, r_2, \alpha, \gamma_2, \gamma_1), \end{aligned} \tag{4.9}$$

for all $t \geq t_1 = T_{1B} + T_{2B}$, for all $u_0 \in B$. Then, combining with (4.2), we find that

$$\int_{\mathcal{O}_h} (|S(t)u_0|^2 + |\nabla S(t)u_0|^2)\, dx \leq \eta$$

as t, h are taken large enough. □

LEMMA 4.2 ((H^1, H^1)-asymptotic compactness). *Under the assumptions of Theorem* 3.3*, the semigroup* $\{S(t)\}_{t\geq 0}$ *is* (H^1, H^1)*-asymptotically compact.*

PROOF. Based on Lemma 4.1, Theorem 3.3 and the compact embedding

$$H^2(\mathbb{R}^N/\mathcal{O}_h) \hookrightarrow H^1(\mathbb{R}^N/\mathcal{O}_h),$$

we complete the proof. □

Furthermore, on the basis of our local existence theorem presented in Section 2, we obtain the result on (H^1, H^1)-continuity of $\{S(t)\}_{t\geq 0}$.

LEMMA 4.3 ((H^1, H^1)-continuity). *Under the assumptions of Theorem* 3.3*, the semigroup* $\{S(t)\}_{t\geq 0}\colon H^1 \to H^1$ *is continuous.*

Therefore, we give the main result of this section, which is a direct consequence of Lemmas 4.2 and 4.3 (see [12], [3], [18]):

THEOREM 4.4. *Let* $\alpha > 1$ *and* γ_2 *satisfy* $\gamma_2\varepsilon > \alpha^2$*, then the semigroup* $\{S(t)\}_{t\geq 0}$ *of problem* (1.3) *has an* H^1*-global attractor* $\mathcal{A}$*, which is compact in* H^1*, invariant under* $\{S(t)\}_{t\geq 0}$ *and attracts every* H^1*-bounded set with respect to the* H^1*-norm.*

Remark 4.5. We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow. It appears as a suitable object in view of the study of the asymptotic behavior of the system (1.3). Our theorem might be useful in the study of cell reproduction, cell movement, cell proliferation and tumor growth.

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Xiaopeng Zhao
Department of Mathematics
Southeast University
Nanjing 210096, P.R. CHINA
and
School of Science
Jiangnan University
Wuxi 214122, P.R. CHINA

E-mail address: zhaoxiaopeng@jiangnan.edu.cn