

EXISTENCE OF SOLUTIONS FOR NONLINEAR p -LAPLACIAN DIFFERENCE EQUATIONS

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ABSTRACT. The aim of this paper is the study of existence of solutions for nonlinear $2n^{\text{th}}$ -order difference equations involving p -Laplacian. In the first part, the existence of a nontrivial homoclinic solution for a discrete p -Laplacian problem is proved. The proof is based on the mountain-pass theorem of Brezis and Nirenberg. Then, we study the existence of multiple solutions for a discrete p -Laplacian boundary value problem. In this case the proof is based on the three critical points theorem of Averna and Bonanno.

1. Introduction

Consider the fourth-order p -Laplacian difference equation

$$(1.1) \quad \Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) - a\Delta(\varphi_{p_1}(\Delta u(k-1))) \\ + V(k)\varphi_q(u(k)) = \lambda f(k, u(k))$$

where $p > 1$, $\varphi_p(t) = |t|^{p-2}t$, $V: \mathbb{Z} \rightarrow \mathbb{R}$ is a T -periodic positive function for T a fixed integer and $f: \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function with growth conditions.

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The above equation is a discretization of a fourth order p -Laplacian equation studied by authors in [18], where the existence of solution for a periodic problem involving p -Laplacian differential equation is considered. The partial cases where $p_1 = p_2 = 2$ are known as stationary extended Fisher–Kolmogorov equation (see Peletier and Troy [17], [19] and references therein).

The theory of nonlinear difference equations is widely used in the study of discrete models in different fields of science. Recently, the problems for difference equations are treated by topological and variational methods. Topological methods for higher order difference equations using Green’s functions and fixed point theorems are used in [2], [3]. The variational methods coupled with critical point theory have been extensively applied to the solvability of problems for difference equations during the last decade. We refer the reader to [1], [12], [20] and references therein. A survey on applications of critical point theory to existence results for difference equations is given in [7]. Periodic and homoclinic orbits for $2n^{\text{th}}$ order difference equations are studied in [8] using linking theorem and in [9] by mountain-pass and symmetric mountain-pass theorems.

This paper is divided in two parts. The first part is based on the mountain-pass theorem of Brezis and Nirenberg [5]. Following the steps of [6], we obtain the existence of a nontrivial homoclinic solution of equation (1.1), i.e. a nonzero solution u , such that $\lim_{|k| \rightarrow +\infty} |u(k)| = 0$.

In the second part, we obtain the existence of at least three solutions for the difference equation with $p_1 = p_2 = q$ and the Dirichlet boundary conditions, by generalizing a result given in [10] to the problem

$$\begin{aligned} -\Delta(\varphi_p(\Delta u(k-1))) &= \lambda f(k, u(k)), \quad k \in [1, T], \\ u(0) &= u(T+1) = 0. \end{aligned}$$

Such a result is obtained by applying [10, Theorem 2.1], which is a modification of the theorem of Averna and Bonanno (see [4]), to our boundary value problem.

In both cases, we show how our result should be modified for higher order problems.

The study of p -Laplacian difference equations has been developed in the literature. In addition to the previously mentioned [6], [10], we refer to [13], where the following problem is studied:

$$\begin{aligned} \Delta(\varphi_p(\Delta u(k-1))) + a(t) f(k, u(k)) &= 0, \quad k \in [1, T+1], \\ \Delta u(0) &= u(T+2) = 0, \end{aligned}$$

where $a(t)$ is a positive function. Moreover, in [21], the existence of three positive solutions of this problem is studied.

Recently, in [11], it was proved the existence of at least three solutions of the problem

$$\begin{aligned} \Delta^2(\varphi_p(\Delta^2 u(k-2))) + \alpha\varphi_p(u(k)) &= \lambda f(k, u(k)), \quad k \in [1, T], \\ u(0) = \Delta u(-1) = \Delta^2 u(T) &= 0, \\ \Delta(\varphi_p(\Delta^2 u(T-1))) &= \mu g(u(T+1)), \end{aligned}$$

where α, λ and μ are real parameters, f and g are continuous.

Moreover, we refer to [14], [16] and [20], where the existence of homoclinic solution for different discrete second order problems is studied. Finally, in [15] there is studied the existence of periodic solutions for a higher order difference equation involving the p -Laplacian. In addition to its particular interest separately, we study these two problems together to point out the differences on the generalization of a result for a second order problem to higher order. In the first case, we will see that the hypotheses on the function f are not modified in order to obtain the results for higher order. However, for the boundary value problems, the hypotheses should be modified to increase the order of the problem and also the used arguments suffer more modifications than in the homoclinic case.

This paper is structured in two parts: Sections 2 and 3. We introduce the considered problems in the beginning of these parts. Then, we construct the related variational formulation. After some preliminaries, the existence solutions results are proved. Finally, we give examples to the results obtained. In Section 4, we show, as a remark, how the results will look like for $n \geq 3$. Finally, in Section 5, we comment on the obtained results.

2. Homoclinic solutions

This section is focused on the study of the existence of homoclinic solutions for the following problem:

$$\begin{aligned} \Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) - a\Delta(\varphi_{p_1}(\Delta u(k-1))) \\ + V(k)\varphi_q(u(k)) &= \lambda f(k, u(k)), \\ \lim_{|k| \rightarrow +\infty} |u(k)| &= 0, \end{aligned} \tag{2.1}$$

where $a > 0$ is fixed, $p_i \geq q > 1$ for $i = 1, 2$ and

$$\Delta u(k) = u(k+1) - u(k), \tag{2.2}$$

$$\Delta^i u(k) = \Delta^{i-1} u(k+1) - \Delta^{i-1} u(k), \quad \text{if } i \geq 2, \tag{2.3}$$

are the forward difference operators.

We suppose that V is a T -periodic positive function for a fixed integer T and denote

$$0 < V_0 = \min\{V(0), \dots, V(T-1)\} \quad \text{and} \quad V_1 = \max\{V(0), \dots, V(T-1)\}.$$

Let us define the potential function $F: \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$F(k, t) = \int_0^t f(k, s) ds.$$

Moreover, let $f: \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ verify the following assumptions:

(F₁) For each $k \in \mathbb{Z}$, $f(k, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Moreover, for each $t \in \mathbb{R}$, $f(\cdot, t): \mathbb{Z} \rightarrow \mathbb{R}$ is a T -periodic function.

(F₂) The potential function F satisfies the Ambrosetti–Rabinowitz type condition:

There exists $\mu \in \mathbb{R}$, such that $\mu > p_i$, for $i = 1, 2$, and

$$\mu F(k, t) \leq t f(k, t), \quad \text{for all } k \in \mathbb{Z}, t \neq 0.$$

(F₃) There exists $s > 0$ such that $F(k, t) > 0$, for all $k \in \mathbb{Z}$, for all $t \geq s > 0$.

(F₄) Uniformly, for $k \in \mathbb{Z}$, $f(k, t) = o(|t|^{q-1})$ as $|t| \rightarrow 0$.

Define $\phi_p(t) = |t|^p/p$. It is trivial that $\phi'_p(t) = \varphi_p(t)$ for every $p > 1$. Let

$$\ell^q = \left\{ (u(k))_{k \in \mathbb{Z}} \mid \sum_{k \in \mathbb{Z}} |u(k)|^q < \infty \right\}$$

be the considered Banach space with the norm $|u|_q = \sum_{k \in \mathbb{Z}} |u(k)|^q$ and $J: \ell^q \rightarrow \mathbb{R}$ be the functional

$$J(u) = \Phi(u) - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)),$$

where

$$\Phi(u) = \sum_{k \in \mathbb{Z}} (\phi_{p_2}(\Delta^2 u(k-2)) + a \phi_{p_1}(\Delta u(k-1)) + V(k) \phi_q(u(k))).$$

We have the following result:

LEMMA 2.1. *Let the function f satisfy assumptions (F₁) and (F₄). Then the functional $J: \ell^q \rightarrow \mathbb{R}$ is well defined and C^1 . Moreover, its critical points are solutions of problem (2.1).*

PROOF. Let us see first that the functional J is well defined. In order to do that, we use the following elementary inequality:

$$(2.4) \quad (x + y)^p \leq 2^{p-1}(x^p + y^p),$$

which is fulfilled for every non-negative x, y and $p > 1$.

Applying (2.4) for $p_2 > 1$ twice, we have:

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \phi_{p_2}(\Delta^2(u(k-2))) \\ & \leq \frac{1}{p_2} \sum_{k \in \mathbb{Z}} 4^{p_2-1} (|u(k)|^{p_2} + 2|u(k-1)|^{p_2} + |u(k-2)|^{p_2}) = \frac{4^{p_2}}{p_2} \sum_{k \in \mathbb{Z}} |u(k)|^{p_2}. \end{aligned}$$

Now, since $p_2 \geq q$, it is well known that $\ell^q \subset \ell^{p_2}$ and for $u \in \ell^q$, we conclude that

$$\sum_{k \in \mathbb{Z}} \phi_{p_2}(\Delta^2(u(k-2))) \leq \frac{4^{p_2}}{p_2} \sum_{k \in \mathbb{Z}} |u(k)|^{p_2} < +\infty.$$

Analogously, let us apply inequality (2.4) for p_1 . We have, taking into account that $u \in \ell^q \subset \ell^{p_1}$:

$$\sum_{k \in \mathbb{Z}} |\Delta u(k-1)|^{p_1} \leq \frac{2^{p_1}}{p_1} \sum_{k \in \mathbb{Z}} |u(k)|^{p_1} < +\infty.$$

Moreover,

$$\sum_{k \in \mathbb{Z}} V(k) |u(k)|^q \leq V_1 \sum_{k \in \mathbb{Z}} |u(k)|^q < +\infty.$$

Finally, for all $\delta \in (0, 1)$, there exists $N > 0$ sufficiently large such that $|u(k)|^q < \delta < 1$ if $|k| > N$. Moreover, under assumption (F_4) , we have

$$\exists \delta \in (0, 1) \text{ such that } F(k, u(k)) < |u(k)|^q < \delta < 1, |k| > N.$$

Thus, $\sum_{k \in \mathbb{Z}} F(k, u(k)) < +\infty$ and J is a well-defined functional in ℓ^q . For all $v \in \ell^q$, we have:

$$\begin{aligned} \langle J'(u), v \rangle &= \sum_{k \in \mathbb{Z}} (\varphi_{p_2}(\Delta^2 u(k-2)) \Delta^2 v(k-2) + a \varphi_{p_2}(\Delta u(k-1)) \Delta v(k-1)) \\ &\quad + \sum_{k \in \mathbb{Z}} V(k) \varphi_q(u(k)) v(k) - \lambda \sum_{k \in \mathbb{Z}} f(k, u(k)) v(k). \end{aligned}$$

By direct calculations, we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \varphi_{p_2}(\Delta^2 u(k-2)) \Delta^2 v(k-2) &= \sum_{k \in \mathbb{Z}} \Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) v(k), \\ \sum_{k \in \mathbb{Z}} \varphi_{p_2}(\Delta u(k-1)) \Delta v(k-1) &= - \sum_{k \in \mathbb{Z}} \Delta(\varphi_{p_2}(\Delta u(k-1))) v(k). \end{aligned}$$

Hence, for all $v \in \ell^q$,

$$\begin{aligned} \langle J'(u), v \rangle &= \sum_{k \in \mathbb{Z}} (\Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) - a \Delta(\varphi_{p_2}(\Delta u(k-1)))) v(k) \\ &\quad + \sum_{k \in \mathbb{Z}} V(k) \varphi_q(u(k)) v(k) - \lambda \sum_{k \in \mathbb{Z}} f(k, u(k)) v(k). \end{aligned}$$

So, we can obtain the partial derivatives as follows:

$$\begin{aligned} \frac{\partial J(u)}{\partial u(k)} &= \Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) - a \Delta(\varphi_{p_2}(\Delta u(k-1))) \\ &\quad + V(k) \varphi_q(u(k)) - \lambda f(k, u(k)), \end{aligned}$$

which are continuous functions. Following the arguments of Iannizotto and Tersian [14], Propositions 5–7, one can prove that the functional J is continuously differentiable and its critical points of J are the solutions of (2.1). \square

Now, let us recall the mountain-pass theorem of Brezis and Nirenberg [5], which we use to obtain the homoclinic solutions of (2.1) and (4.1).

Let X be a Banach space with norm $\|\cdot\|$ and $I: X \rightarrow \mathbb{R}$ be a C^1 -functional. We say that I satisfies the $(PS)_c$ condition if every sequence $(x_k) \subset X$ such that

$$(2.5) \quad I(x_k) \rightarrow c, \quad I'(x_k) \rightarrow 0,$$

has a convergent subsequence in X . Let us denote by a $(PS)_c$ -sequence every sequence $(x_k) \subset X$ that verifies (2.5).

THEOREM 2.2 (Mountain-pass theorem, Brezis and Nirenberg [5]). *Let X be a Banach space with norm $\|\cdot\|$, $I \in C^1(X, \mathbb{R})$ and suppose that there exist $r > 0$, $\alpha > 0$ and $e \in X$ such that $\|e\| > r$ and*

- (a) $I(x) \geq \alpha$ if $\|x\| = r$,
- (b) $I(e) < 0$.

Let $c = \inf_{\gamma \in \Gamma} \left\{ \max_{t \in [0,1]} I(\gamma(t)) \right\} \geq \alpha$, where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

Then, there exists a $(PS)_c$ -sequence for I . Moreover, if I satisfies the $(PS)_c$ condition, then c is a critical value of I , that is, there exists $u_0 \in X$ such that $I(u_0) = c$ and $I'(u_0) = 0$.

Let us consider the following norm in ℓ^q :

$$\|u\|_q := \left(\frac{1}{q} \sum_{k \in \mathbb{Z}} V(k) |u(k)|^q \right)^{1/q}.$$

From the assumption on V , we have that it is an equivalent norm to $|\cdot|_q$, since we have:

$$\frac{V_0}{q} |u|_q^q \leq \|u\|_q^q \leq \frac{V_1}{q} |u|_q^q.$$

Now, we have the following result, which can be proved in the same way as [6, Lemma 2.3]:

LEMMA 2.3. *Suppose that assumptions (F_1) – (F_4) are verified. Then, there exist $\rho > 0$, $\alpha > 0$ and $e \in \ell^q$ such that $\|e\| > \rho$ and*

- (a) $J(u) \geq \alpha$ if $\|u\| = \rho$.
- (b) $J(e) < 0$.

We obtain the following result.

LEMMA 2.4. *Assume that (F_1) – (F_4) are verified. Then, there exists $c > 0$ and an ℓ^q -bounded $(PS)_c$ sequence for J .*

PROOF. From Lemma 2.3 and Theorem 2.2, we can ensure that there exists a $(PS)_c$ -sequence for J , $(u_m) \subset \ell^q$, i.e. (2.5) is verified for $I = J$, where c has been introduced in Theorem 2.2.

Now, we have to prove that the sequence (u_m) is bounded in ℓ^q . We have,

$$\begin{aligned} \langle J'(u_m), u_m \rangle &= \sum_{k \in \mathbb{Z}} (|\Delta^2 u_m(k-2)|^{p_2} + a|\Delta u_m(k-1)|^{p_1} + V(k)|u_m(k)|^q) \\ &\quad - \sum_{k \in \mathbb{Z}} \lambda f(k, u_m(k)) u_m(k). \end{aligned}$$

Now, using (F_2) and taking into account that $\mu > p_i \geq q > 1$ for $i = 1, 2$, using the same arguments as in [6, Lemma 2.4] we conclude that

$$\mu J(u_m) - \langle J'(u_m), u_m \rangle \geq (\mu - q) \|u_m\|_q^q.$$

Thus, (u_m) is a bounded sequence in ℓ^q . \square

Now, we can prove the main result of this part:

THEOREM 2.5. *Suppose that $a > 0$, the function $V: \mathbb{Z} \rightarrow \mathbb{R}$ is positive and T -periodic and assumptions (F_1) – (F_4) are fulfilled. Then, for $\lambda > 0$, problem (2.1) has a non-trivial homoclinic solution $u \in \ell^q$, which is a critical point of the functional $J: \ell^q \rightarrow \mathbb{R}$.*

PROOF. The proof is analogous to the proof of [6, Theorem 1.1]. We only have to take into account that $q/p_i \leq 1$ for $i = 1, 2$, thus all the arguments represented there remain valid for such a problem. \square

EXAMPLE 2.6. Let $r > p_i \geq q > 1$ for $i = 1, 2$ and $b: \mathbb{Z} \rightarrow \mathbb{R}$ a positive T -periodic function. Consider $f(k, t) = b(k) \varphi_r(t)$. Let us verify that such f satisfies (F_1) – (F_4) .

(F_1) Obviously f is continuous as a function of t and T -periodic as a function of k ,

(F_2) $F(k, t) = b(k) \Phi_r(t)$. There exists $r \in \mathbb{R}$, such that $r > p_i \geq q > 1$, for $i = 1, 2$, such that

$$rF(k, t) = b(k) |t|^r = tb(k) t |t|^{r-2} = tf(k, t), \quad \text{for all } k \in \mathbb{Z}, t \neq 0.$$

(F_3) $F(k, t) > 0$, for all $k \in \mathbb{Z}$, for all $t > 0$.

(F_4) Since $r > q$, we have:

$$\lim_{|t| \rightarrow 0} \frac{f(k, t)}{|t|^{q-1}} = \lim_{|t| \rightarrow 0} b(k) t |t|^{r-q-1} = 0.$$

Then, for $V, b: \mathbb{Z} \rightarrow \mathbb{R}$ two positive T -periodic functions the problem

$$\begin{aligned} \Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) - a\Delta(\varphi_{p_1}(\Delta u(k-1))) + V(k) \varphi_q(u(k)) &= \lambda b(k) \varphi_r(u(k)), \\ \lim_{|k| \rightarrow +\infty} |u(k)| &= 0, \end{aligned}$$

where $r > p_i \geq q > 1$ for $i = 1, 2$, has a non-trivial homoclinic solution for every $\lambda > 0$.

3. Boundary value problems

In this part we study the existence of multiple solutions for the following boundary value problem:

$$(3.1) \quad \Delta^2(\varphi_q(\Delta^2 u(k-2))) - a\Delta(\varphi_q(\Delta u(k-1))) \\ + V(k)\varphi_q(u(k)) - \lambda f(k, u(k)) = 0, \quad k \in [1, T],$$

$$(3.2) \quad u(0) = u(T+1) = \Delta u(-1) = \Delta u(T) = 0,$$

where $a > 0$, T is a fixed positive integer, $[1, T] = \{1, \dots, T\}$, the difference operators have been introduced in (2.2)–(2.3) and φ_q has been defined in the previous section for $1 < q < +\infty$.

Moreover, we consider $V: [1, T] \rightarrow \mathbb{R}$ as a positive function. We can consider it as a restriction to the discrete interval $[1, T]$ of the T -periodic function V introduced in the first part of the paper. We also denote $V_0 = \min\{V(1), \dots, V(T)\}$ and $V_1 = \max\{V(1), \dots, V(T)\}$. Finally, let $f: [1, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.

As in the first part, we obtain the existence result by means of variational methods. In order to obtain our variational approach, we consider the following T -dimensional Banach space:

$$(3.3) \quad X = \{u: [-1, T+2] \rightarrow \mathbb{R} \mid u(0) = u(T+1) = \Delta u(-1) = \Delta u(T+1) = 0\},$$

coupled with the following norm:

$$\|u\|_X = \left(\sum_{k=1}^{T+2} |\Delta^2(k-2)|^q + a \sum_{k=1}^{T+1} |\Delta u(k-1)|^q + \sum_{k=1}^T V(k) |u(k)|^q \right)^{1/q}.$$

We have the following result, in terms of the norm $\|\cdot\|_X$.

LEMMA 3.1. *For every $u \in X$, the following inequality holds:*

$$\max_{k \in [1, T]} |u(k)| \leq \rho \|u\|_X,$$

where

$$(3.4) \quad \rho = \frac{(T+1)(T+2)^{(q-1)/q}}{(4q + 2^q a(T+1)(T+2)^{q-1} + V_0(T+1)^q(T+2)^{q-1})^{1/q}}.$$

PROOF. First, we have:

$$(3.5) \quad \max_{k \in [1, T]} |u(k)|^q \leq \sum_{k=1}^T |u(k)|^q \leq \frac{1}{V_0} \sum_{k=1}^T V(k) |u(k)|^q.$$

Secondly, taking into account the boundary conditions (3.2), for every $u \in X$ and all $j = 1, \dots, T$, we have

$$\begin{aligned}
 (3.6) \quad \sum_{k=1}^{T+1} |\Delta u(k-1)| &= \sum_{k=1}^j |u(k) - u(k-1)| + \sum_{k=j+1}^{T+1} |u(k-1) - u(k)| \\
 &\geq \sum_{k=1}^j (|u(k)| - |u(k-1)|) + \sum_{k=j+1}^{T+1} (|u(k-1)| - |u(k)|) \\
 &= 2|u(j)| - |u(0)| - |u(T+1)| = 2|u(j)|.
 \end{aligned}$$

Analogously, for every $u \in X$ and all $j = 1, \dots, T$, we obtain

$$(3.7) \quad \sum_{k=1}^{T+2} |\Delta^2 u(k-2)| \geq 2|\Delta u(j-1)|.$$

Now, combining (3.6) with the discrete Hölder inequality, we have

$$\begin{aligned}
 (3.8) \quad \max_{k \in [1, T]} |u(k)| &\leq \frac{1}{2} \sum_{k=1}^{T+1} |\Delta u(k-1)| \\
 &\leq \frac{1}{2} (T+1)^{(q-1)/q} \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^q \right)^{1/q}.
 \end{aligned}$$

Using (3.6) and (3.7), we have:

$$\begin{aligned}
 (3.9) \quad \max_{k \in [1, T]} |u(k)| &\leq \frac{1}{2} \sum_{k=1}^{T+1} |\Delta u(k-1)| \leq \frac{1}{2} (T+1) \max_{k \in [1, T+1]} |\Delta u(k-1)|^q \\
 &\leq \frac{T+1}{4} \sum_{k=1}^{T+2} |\Delta^2 u(k-2)| \\
 &\leq \frac{(T+1)(T+2)^{(q-1)/q}}{4} \left(\sum_{k=1}^{T+2} |\Delta^2 u(k-2)|^q \right)^{1/q}.
 \end{aligned}$$

Thus, from (3.5), (3.8) and (3.9), we obtain

$$\left(\frac{4^q}{(T+1)^q (T+2)^{q-1}} + \frac{a2^q}{(T+1)^{q-1}} + V_0 \right) \max_{k \in [1, T]} |u(k)|^q \leq \|u\|_X^q,$$

and the result is proved, taking into account that

$$\frac{4^q}{(T+1)^q (T+2)^{q-1}} + \frac{a2^q}{(T+1)^{q-1}} + V_0 = \frac{1}{\rho^q}. \quad \square$$

Now, let us consider $J_1: X \rightarrow \mathbb{R}$, the functional

$$J_1(u) = \Phi_1(u) - \lambda \sum_{k=1}^T F(k, u(k)), \quad \text{where } F(k, t) = \int_0^t f(k, s) ds$$

for every $(k, t) \in [1, T] \times \mathbb{R}$ and

$$\Phi_1(u) = \sum_{k=1}^{T+2} \phi_q(\Delta^2 u(k-2)) + a \sum_{k=1}^{T+1} \phi_q(\Delta u(k-1)) + \sum_{k=1}^T V(k) \phi_q(u(k)).$$

REMARK 3.2. Observe that $\Phi_1(u) = \|u\|_X^q/q$, hence trivially Φ_1 is coercive, that is, $\lim_{\|u\|_X \rightarrow +\infty} \Phi_1(u) = +\infty$.

We have the analogous result to Lemma 2.1 for this case.

LEMMA 3.3. *The functional $J: X \rightarrow \mathbb{R}$ is C^1 -differentiable and its critical points are solutions of (3.1).*

Let us denote $\Psi_1(u) = -\sum_{k=1}^T F(k, u(k))$. Let E be a finite dimensional Banach space and consider $J: E \rightarrow \mathbb{R}$ the functional $J(u) = \Phi(u) + \lambda\Psi(u)$, where $\Phi, \Psi: X \rightarrow \mathbb{R}$ are of class $C^1(E)$ and Φ is coercive.

REMARK 3.4. Realize that $J_1(u) = \Phi_1(u) + \lambda\Psi_1(u)$ satisfies this condition.

For every $r > \inf_E \Phi$, let us define:

$$\begin{aligned} \psi_1(r) &:= \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\Psi(u) - \inf_{\Phi^{-1}((-\infty, r])} \Psi}{r - \Phi(u)}, \\ \psi_2(r) &:= \inf_{u \in \Phi^{-1}((-\infty, r))} \sup_{v \in \Phi^{-1}([r, +\infty))} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)}, \end{aligned}$$

Now, we are under conditions of applying [10, Theorem 2.1] to our problem.

THEOREM 3.5 ([10, Theorem 2.1]). *Assume that:*

- (a) *there exists $r > \inf_E \Phi$ such that $\psi_1(r) < \psi_2(r)$,*
- (b) *for each $\lambda \in (1/\psi_2(r), 1/\psi_1(r))$ we have $\lim_{\|u\|_E \rightarrow +\infty} J(u) = +\infty$.*

Then, for each $\lambda \in (1/\psi_2(r), 1/\psi_1(r))$, J has at least three critical points.

Let us define, for $c, d > 0$,

$$\begin{aligned} \Theta(c) &:= \frac{1}{c^q} \sum_{k=1}^T \sup_{|s| \leq c} F(k, s) > 0, \\ \Lambda(d) &:= \frac{1}{d^q} \sum_{k=1}^T \left(F(k, d) - \sup_{|s| \leq c} F(k, s) \right) > 0. \end{aligned}$$

Now, we can state the main result of this section:

THEOREM 3.6. *Assume that there exist four positive constants b, c, d , and p , such that $c < d$ and $p < q$ verifying:*

$$(d_1) \quad \Theta(c) < \frac{\Lambda(d)}{(4 + 2a + TV_1) \rho^q}, \text{ where } \rho \text{ has been introduced in (3.4).}$$

(d₂) $F(k, t) \leq b(1 + |t|^p)$ for all $(k, t) \in [1, T] \times \mathbb{R}$.

Then, for every $\lambda \in ((4 + 2a + TV_1)/(q\Lambda(d)), 1/(q\rho^q\Theta(c)))$, problem (3.1)–(3.2) admits at least three solutions which are critical points of J_1 .

PROOF. We just need to find $r > \inf_X \Phi$ such that the hypotheses of Theorem 3.5 are verified. Let

$$r = \frac{c^q}{q\rho^q}.$$

Taking into account the relationship between Φ_1 and the norm $\|\cdot\|_X$, we have:

$$\begin{aligned} \psi_1(r) &= \inf_{\|u\|_X < (qr)^{1/q}} \frac{\Psi_1(u) - \inf_{\|u\|_X \leq (qr)^{1/q}} \Psi_1(u)}{r - \Phi_1(u)} \\ &\leq \frac{1}{r} \inf_{\|u\|_X \leq (qr)^{1/q}} \Psi(u) = \frac{1}{r} \sup_{\|u\|_X \leq (qr)^{1/q}} \sum_{k=1}^T F(k, u(k)). \end{aligned}$$

Now, from Lemma 3.1, if $\|u\|_X \leq (qr)^{1/q}$, then for all $k \in [1, T]$:

$$|u(k)| \leq \rho(qr)^{1/q} = c,$$

thus

$$\psi_1(r) \leq \frac{1}{r} \sum_{k=1}^T \sup_{|u(k)| \leq c} F(k, u(k)) = q\rho^q\Theta(c).$$

Now, let us see that $c^q < \rho^q(4 + 2a + V_0)d^p$. In order to do that, let us choose $k^* \in [1, T]$, such that $V(k^*) = V_0$, and consider:

$$v_c(k) := \begin{cases} c & \text{if } k = k^*, \\ 0 & \text{if } k \neq k^*, \end{cases}$$

from Lemma 3.1, since $c < d$, we have:

$$c^q \leq \rho^q \|v_c\|_X^q = \rho^q(4 + 2a + V_0)c^q < \rho^q(4 + 2a + V_0)d^q.$$

Now, consider $v_d \in X$, such that:

$$v_d(k) := \begin{cases} d & \text{if } k \in [1, T], \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\|v_d\|_X^q = \left(4 + 2a + \sum_{k=1}^T V(k)\right) d^q \geq (4 + 2a + V_0)d^q > \left(\frac{c}{\rho}\right)^q = qr.$$

Hence,

$$\psi_2(r) = \inf_{\|u\|_X < (qr)^{1/q}} \sup_{\|v\|_X \geq (qr)^{1/q}} \frac{\Psi_1(u) - \Psi_1(v)}{\Phi_1(v) - \Phi_1(u)} \geq \inf_{\|u\|_X < (qr)^{1/q}} \frac{\Psi_1(u) - \Psi_1(v_d)}{\Phi_1(v_d) - \Phi_1(u)}$$

$$\begin{aligned}
&= q \inf_{\|u\|_X < (qr)^{1/q}} \frac{\sum_{k=1}^T F(k, d) - \sum_{k=1}^T F(k, u(k))}{\left(4 + 2a + \sum_{k=1}^T V(k)\right) d^p - \|u\|_X^q} \\
&\geq q \frac{\sum_{k=1}^T F(k, d) - \sum_{k=1}^T \sup_{|s| \leq c} F(k, u(k))}{\left(4 + 2a + \sum_{k=1}^T V(k)\right) d^p}.
\end{aligned}$$

Now, taking into account that $\sum_{k=1}^T V(k) \leq TV_1$, we have

$$\psi_2(r) \geq q \frac{\Lambda(d)}{4 + 2a + TV_1}.$$

Thus, from condition (d₁), we conclude $\psi_2(r) > q\rho^q\Theta(c) \geq \psi_1(r)$. By other hand, from condition (d₂), we have:

$$J_1(u) = \frac{\|u\|_X^q}{q} - \lambda \sum_{k=1}^T F(k, u(k)) \geq \frac{\|u\|_X^q}{q} - \lambda \sum_{k=1}^T (1 + |u(k)|^s).$$

Using again Lemma 3.1, we conclude

$$J_1(u) \geq \frac{\|u\|_X^q}{q} - \lambda Tb - \lambda T\rho^p \|u\|_X^p,$$

which, since $p < q$, ensures that $\lim_{\|u\|_X \rightarrow +\infty} J_1(u) = +\infty$. Therefore, by applying Theorem 3.5 the result is proved. \square

EXAMPLE 3.7. Let $T = 8$ and $V(k) = 6(k+6)^2$ for each $k \in [1, T]$. Then, in this case $V_0 = 294$ and $V_1 = 1176$. Moreover, consider $f(k, t) = kg(t)$, where

$$g(t) = \begin{cases} e^t & \text{if } t \leq 14, \\ e^{14} & \text{if } t > 14, \end{cases}$$

then, $F(k, t) = k^2G(t)$, where

$$G(t) = \begin{cases} e^t & \text{if } t \leq 14 \\ e^{14}(t-13) & \text{if } t > 14. \end{cases}$$

So, we can see that $F(k, t) \leq 8e^{14}(1 + |t|^p)$ for all $p > 1$.

Let us choose $c = 3$, $d = 14$ and $q = 3$. We have:

$$(3.10) \quad \Theta(3) := \frac{1}{3^3} \sum_{k=1}^8 \sup_{|s| \leq 3} F(k, s) = \frac{4}{3} e^3,$$

$$(3.11) \quad \Lambda(14) := \frac{1}{14^3} \sum_{k=1}^8 \left(F(k, 14) - \sup_{|s| \leq 3} F(k, s) \right) = \frac{9}{686} (e^{14} - e^3).$$

Now, consider the problem:

$$(3.12) \quad \Delta^2(\varphi_3(\Delta^2 u(k-2))) - 10\Delta(\varphi_3(\Delta u(k-1))) \\ + 6(k+6)^2 \varphi_3(u(k)) - \lambda k g(u(k)) = 0, \quad k \in [1, 8],$$

coupled with the boundary conditions (3.2). In this case,

$$\rho^3 = \frac{18225}{5376166} \cong 0.0038996 < \frac{1}{294} \cong 0.0034.$$

Moreover, we have

$$\frac{\Lambda(14)}{(4 + 20 + 8 \cdot 1176) \rho^3} = \frac{2688083}{655123140} (e^{14} - e^3) \cong 493.44 > \frac{4}{3} e^3 \cong 26.78.$$

Then, we can apply Theorem 3.6 to conclude that for each

$$\lambda \in \left(\frac{718928}{3(e^{14} - e^3)}, \frac{2688083}{36450 e^3} \right) = (0.2, 3.67),$$

problem (3.12) has at least three solutions.

4. Higher order problems

In this section, we want to show how our previously obtained results could be generalized for $n \geq 3$. First, we consider the homoclinic problem:

$$(4.1) \quad \Delta^n(\varphi_{p_n}(\Delta^n u(k-2))) + \sum_{i=1}^{n-1} (-1)^i a_i \Delta^{n-i}(\varphi_{p_{n-i}}(\Delta^{n-i} u(k-1))) \\ + (-1)^n V(k) \varphi_q(u(k)) + (-1)^{n+1} \lambda f(k, u(k)) = 0, \\ \lim_{|k| \rightarrow +\infty} |u(k)| = 0,$$

where V, f, φ_p and Δ^j have been previously introduced, $\mu > p_i \geq q > 1$ and $a_i \geq 0$ for all $j \in \{1, \dots, n\}$, with $a_n = 1$. In this case, with small modifications on the arguments, we obtain the generalization of Theorem 2.5.

THEOREM 4.1. *Suppose that $a_i > 0$ for $i = 1, \dots, n-1$, $V: \mathbb{Z} \rightarrow \mathbb{R}$ is a positive and T -periodic function and assumptions (F₁)–(F₄) are fulfilled. Then, for $\lambda > 0$, problem (4.1) has a non-trivial homoclinic solution $u \in \ell^q$, which is a critical point of the functional $J: \ell^q \rightarrow \mathbb{R}$.*

Clearly, the functions introduced in Example 2.6 are still valid for this generalized case with $r > p_i$ for $i = 1, \dots, n$.

Now, we consider the boundary generalized boundary value problem. Let $a_i > 0$ for $i = 1, \dots, n-1$. We will show an existence result for the $2n^{\text{th}}$ -order problem:

$$(4.2) \quad \Delta^n(\varphi_q(\Delta^n u(k-2))) + \sum_{i=1}^{n-1} (-1)^i a_i \Delta^{n-i}(\varphi_q(\Delta^{n-i} u(k-1))) \\ + (-1)^n V(k) \varphi_q(u(k)) + (-1)^{n+1} \lambda f(k, u(k)) = 0,$$

$$(4.3) \quad u(0) = \Delta u(-1) = \Delta^2 u(-2) = \dots = \Delta^{n-1} u(1-n) = 0,$$

$$(4.4) \quad u(T+1) = \Delta u(T+1) = \Delta^2 u(T+1) = \dots = \Delta^{n-1} u(T+1) = 0.$$

Such a result is obtained with slight modifications on the arguments used for fourth order. Let ρ given by the expression:

$$(4.5) \quad \frac{\prod_{j=1}^n (T+j)}{\left(2^{qn}(T+n) + \sum_{i=1}^{n-1} a_i 2^{q(n-i)}(T+n-i) \prod_{j=n-i+1}^n (T+j)^q + V_0 \prod_{j=1}^n (T+j)^q \right)^{1/q}}.$$

REMARK 4.2. Realize that the constant defined in (3.4) is a particular case of the previous one for $n = 2$. Its general expression seems complicated. However, it can be calculated for each particular problem. Moreover, from its expression it can be seen that it is clearly bounded as follows:

$$\rho \leq \frac{\prod_{j=1}^n (T+j)}{\left(V_0 \prod_{j=1}^n (T+j)^q \right)^{1/q}} = \left(\frac{1}{V_0} \right)^{1/q}.$$

We can state the equivalent to Theorem 3.6 for problem (4.2)–(4.4) as follows:

THEOREM 4.3. *Assume that there exist four positive constants b, c, d , and p , such that $c < d$ and $p < q$ verifying:*

$$(d_1) \quad \Theta(c) < \Lambda(d) / \left(2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + TV_1 \right) \rho^q, \text{ where } \rho \text{ has been introduced in (4.5).}$$

$$(d_2) \quad F(k, t) \leq b(1 + |t|^p) \text{ for all } (k, t) \in [1, T] \times \mathbb{R}.$$

Then, for every

$$\lambda \in \left(\frac{1}{q\Lambda(d)} \left(2^n + \sum_{i=1}^{n-1} 2^{n-i} a_i + TV_1 \right), \frac{1}{q\rho^q\Theta(c)} \right),$$

problem (4.2)–(4.4) admits at least three solutions.

Now, let us consider a higher order example, for instance let us choose $n = 4$.

EXAMPLE 4.4. Let us choose $T, V(k)$ and $f(k, t)$ as in Example 3.7. For the choice $c = 3, d = 14$ and $q = 3$, (3.10)–(3.11) are still true. Now, consider the problem:

$$(4.6) \quad \begin{aligned} &\Delta^4(\varphi_3(\Delta^4 u(k-4))) - \Delta^3(\varphi_3(\Delta^3 u(k-3))) \\ &\quad + 2\Delta^2(\varphi_3(\Delta^2 u(k-2))) - 3\Delta(\varphi_3(\Delta u(k-1))) \\ &\quad + 6(k+6)^2 \varphi_3(u(k)) - \lambda kg(u(k)) = 0, \end{aligned}$$

for $k \in [1, 8]$, coupled with the boundary conditions (4.3)–(4.4) for $n = 4$. Observe that $a_i = i$ for $i = 1, 2, 3$. In this case, the expression of ρ is more complicated than with $n = 2$, we have

$$\begin{aligned} \rho^3 &= \frac{\prod_{j=1}^4 (8+j)^3}{12 \cdot 2^{12} + \sum_{i=1}^3 i 2^{3(4-i)} (12-i) \prod_{j=5-i}^4 (8+j)^3 + 294 \prod_{j=1}^3 (8+j)^3} \\ &= \frac{1091586375}{321251750258} \cong 0.003398 < \frac{1}{294}. \end{aligned}$$

Moreover, we have

$$\frac{\Lambda(14)}{\left(2^4 + \sum_{i=1}^3 i 2^{4-i} + 8 \cdot 1176\right) \rho^3} = \frac{42833567}{1047915804474} (e^{14} - e^3) \cong 491.556 > \frac{4}{3} e^3.$$

Then, we can apply Theorem 4.3 to conclude that for each

$$\lambda \in \left(\frac{6479956}{27(e^{14} - e^3)}, \frac{428335667}{5821794 e^3} \right) = (0.2, 3.66),$$

problem (4.6) has at least three solutions.

REMARK 4.5. Finally, we realize that the bound given in (4.2) is a good one for the both considered examples. Indeed, we have

$$\lim_{T \rightarrow +\infty} \rho^q = \frac{1}{V_0}.$$

5. Conclusions and remarks

This final section is devoted to showing some of conclusions which we deduce from this work and several remarks which can be used in future to improve the results shown here.

The first thing that we want to point out is the differences between the two studied problems. Both problems are a generalization of previously known

results for second order problems, however the main difficulties which we found are as follows:

In the first problem, related to the existence of homoclinic solution, the main difficulty of the generalization remains in proving that the constructed functional is well defined and the hypotheses needed for f are the same on the different orders. Moreover, the obtained approach is applicable to higher order problems.

On the other hand, if we focus on the boundary value problem studied in Section 3, we can see directly that the obtained hypotheses are different for each order.

Furthermore, in such a generalization we need to consider the same index q in all the q -Laplacians considered. This fact is due to the construction of the Banach space and the corresponding norm in order to obtain the bound given in Lemma 3.1. Thus, if we can construct a suitable norm in X which allows us to obtain a similar result for different values of q , the result might be extended for such cases.

In [15] there are studied higher order periodic problems by means of the Linking Theorem. It can also be interesting to study such problems or to study problems with different boundary conditions by using the Averna and Bonnano theorem. It would be necessary to construct the correspondent Banach space coupled with the norm and see if we can obtain the appropriate bounds which give the existence results.

Finally, we have not studied the problem by means of using topological methods and fixed point theorems. Maybe, by using a different technique the obtained interval for λ could be different.

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