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ASYMPTOTIC BEHAVIOR FOR NONAUTONOMOUS FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH MEASURES OF NONCOMPACTNESS

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Abstract. We study the asymptotic behavior of nonautonomous differential inclusions with delays in Banach spaces by analyzing their pullback attractors. Our aim is to give a recipe expressed by measures of noncompactness to prove the asymptotic compactness of the process generated by our system. This approach is effective for various differential systems regardless of the compactness of the semigroup governed by linear part.

1. Introduction

We consider the following problem:

$$u'(t) \in Au(t) + F(t, u(t), u_t) \quad \text{for } t \geq \tau, \tag{1.1}$$

$$u(t) = \varphi^\tau(t - \tau) \quad \text{for } t \in [\tau - h, \tau], \tag{1.2}$$

where the state function u takes values in a separable Banach space X, A is a closed linear operator which generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X, F is a multivalued function defined on $[\tau,\infty)\times X\times C([-h,0];X)$, u_t is the history of the state function up to the time t, i.e. $u_t(s) = u(t+s)$ for $s \in [-h, 0]$, and φ^τ is an element of $C([-h,0];X)$.

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Differential inclusions of the form (1.1) emerge from a number of problems. In the monograph [20], Filippov presented a useful way to deal with differential equations with discontinuous right-hand sides, in which a regularized procedure leads to differential inclusions. Differential inclusions appear also in the control problems whose control factor is taken in the form of multivalued feedback. The presence of delayed terms in these problems is an inherent feature.

One of the most important questions concerning system (1.1)–(1.2) is to figure out the behavior of its solutions at large time, i.e. when $t - \tau \to +\infty$. In dealing with asymptotic behavior of differential equations without uniqueness or differential inclusions in autonomous form, there have been introduced and investigated such notions as generalized semiflows due to Ball [5], [6], multivalued semiflows due to Melnik and Valero [26]. A comparison of these two approaches was given in [14]. Thanks to the framework of Melnik and Valero, there have been many works devoted to the investigation of asymptotics for various classes of partial differential equations (PDEs) without uniqueness (see, e.g. [2], [3], [23], [30], [31]). We also refer to the theory of trajectory attractors developed by Chepyzov and Vishik [16] which is a fruitful way to study the long-time behavior of solutions of PDEs for which the uniqueness is unavailable. In order to study asymptotic behavior of nonautonomous differential systems, Melnik and Valero [27] proposed the framework of uniform global attractors for multivalued semiprocesses. Alternatively, the theory of pullback attractors has been developed for both nonautonomous and random dynamical systems in multivalued case by Caraballo et al. [8], [9] and [10].

In all frameworks, an essential step in formulating global attractors is to verify the asymptotic compactness condition for corresponding semiflows/processes. This condition holds if the semigroup governed by principal parts (i.e. $S(t) = e^{tA}$) is compact. However, for PDEs in unbounded domains the latter requirement is unrealistic. In these cases, one can use a nice condition expressed by measures of noncompactness (MNC), namely the ω-limit compact condition. We mention some typical works [24], [25], [36], [37] for single-valued dynamical systems, and [18], [35], [34] for multivalued ones, in which the ω-limit compactness was employed as a crucial condition. In concrete models formed by PDEs without delays, the testing of the ω-limit compact condition is usually replaced by checking the flattening condition, which is possible if one can construct a basis in phase spaces (see, e.g. [18], [25], [36], [37]). Unfortunately, it is impractical to check the latter condition for PDEs with delays since the corresponding phase spaces have complicated structure, i.e. it is impossible to find their basis. So our objective in this paper is to propose an effective way to verify the asymptotic compactness of multivalued nonautonomous dynamical systems (MNDS) generated by differential systems with delays. Let us give a brief description for our implementation. Denote by $\{\mathcal{U}(t,\tau,\cdot)\}_{t\geq\tau}$ the MNDS generated by (1.1)–(1.2), that is $\mathcal{U}(t,\tau,\varphi^\tau) = \{u_t : u(\cdot,\tau,\varphi^\tau)$ is an integral solution to (1.1)–(1.2)$\}$. Putting $\mathcal{G}_{T,t} = \mathcal{U}(t,t-T,\cdot)$ with $T > h$, we will show that $\mathcal{G}_{T,t}$ is condensing on $C([-h,0];X)$ by using the technique of MNC's estimates. Then the condensivity of $\mathcal{G}_{T,t}$ ensures the asymptotic compactness of the MNDS $\mathcal{U}$. It should be mentioned that, this approach is effective for various differential systems, especially for retarded ones, since one just has to test an MNC's estimate on the nonlinearity function (see concrete problems in the last section).

The rest of our work is organized as follows. In the next section, we recall some notions and facts related to MNC and MNDS. We also collect some results on existence and property of solution multimap for (1.1)–(1.2), which were proved in [17], [28]. In Section 3, we show that the MNDS generated by (1.1)–(1.2) admits a compact invariant pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in $C([-h,0];X)$. The last section presents applications of the abstract results to a polytope partial differential equation and a lattice differential system.

2. Preliminaries

Let E be a separable Banach space. We denote by 2^E the collection of all subsets of E and use the following notations:

$$\begin{aligned}
\mathcal{P}(E) &= \{A \in 2^E : A \neq \emptyset\},\\
\mathcal{B}(E) &= \{A \in \mathcal{P}(E) : A \text{ is bounded}\},\\
\mathcal{P}_{\rm c}(E) &= \{A \in \mathcal{P}(E) : A \text{ closed, convex and compact}\},\\
B_E[a,r] &= \{x \in E : \|x-a\| \leq r\}.
\end{aligned}$$

The function $\chi\colon \mathcal{B}(E) \to \mathbb{R}^+$ defined by

$$\chi(B) = \inf\{\varepsilon > 0 : B \text{ has a finite } \varepsilon\text{-net}\}$$

is called the Hausdorff measure of noncompactness on E. For $\mathcal{T} \in \mathcal{L}(E)$, the space of bounded linear operators on E, we define the χ-norm of $\mathcal{T}$ as follows (see, e.g. [1]):

$$\|\mathcal{T}\|_\chi = \inf\big\{\beta > 0 : \chi(\mathcal{T}(B)) \leq \beta\chi(B) \text{ for all } B \in \mathcal{B}(E)\big\}.$$

Then $\|\cdot\|_\chi$ is a semi-norm in $\mathcal{L}(E)$ and $\|\mathcal{T}\|_\chi \leq \|\mathcal{T}\|$. Obviously, $\mathcal{T}$ is a compact operator if and only if $\|\mathcal{T}\|_\chi = 0$.

Definition 2.1. Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on E. It is said to be:

(a) exponentially stable if there exist positive numbers M, α such that

$$\|S(t)\| \leq Me^{-\alpha t}, \quad \text{for all } t \geq 0;$$

(b) compact if $S(t)$ is a compact operator for each $t > 0$;

(c) χ-decreasing if there exist $N, \beta > 0$ such that

$$\|S(t)\|_\chi \leq N e^{-\beta t}, \quad \text{for all } t \geq 0;$$

(d) norm continuous if $t \mapsto S(t)$ is continuous in $\mathcal{L}(E)$ for $t > 0$.

Notice that for the C_0-semigroup $S(\,\cdot\,)$, the exponential stability implies the χ-decreasing property. In addition, if $S(\,\cdot\,)$ is compact then it is χ-decreasing with $\beta = +\infty$.

The following property of χ will be used in the sequel.

Proposition 2.2 ([22]). *If $D \subset L^1(\tau, T; E)$ is such that*

$$\sup\{\|\xi(t)\| : \xi \in D\} \leq \nu(t), \qquad \chi(D(t)) \leq q(t),$$

for some $\nu, q \in L^1(\tau, T; \mathbb{R}^+)$, then

$$\chi\bigg(\int_\tau^t D(s)\,ds\bigg) \leq \int_\tau^t q(s)\,ds$$

for $t \in [\tau, T]$, here

$$\int_\tau^t D(s)\,ds = \bigg\{\int_\tau^t \xi(s)\,ds : \xi \in D\bigg\}.$$

We now recall the definition of MNDS and pullback attractors (see, e.g. [9]).

Definition 2.3. A multivalued map $\mathcal{U}\colon \mathbb{R}^2_d \times E \to \mathcal{P}_{\rm c}(E)$, where $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : t \geq \tau\}$, is called a multivalued nonautonomous dynamical system (MNDS) on E if and only if

(a) $\mathcal{U}(t,t,x) = \{x\}$ for all $t \in \mathbb{R}$, $x \in E$;
(b) $\mathcal{U}(t,\tau,x) \subset \mathcal{U}(t,s,\mathcal{U}(s,\tau,x))$ for all $\tau \leq s \leq t, x \in E$.

The MNDS $\mathcal{U}$ is said to be strict if $\mathcal{U}(t,\tau,x) = \mathcal{U}(t,s,\mathcal{U}(s,\tau,x))$ for all $\tau \leq s \leq t$, $x \in E$.

A multivalued map $D\colon \mathbb{R} \to \mathcal{P}(E)$ is called a multifunction. Let $\mathcal{D}$ be a family of multifunctions taking values in $\mathcal{B}(E)$ and having the inclusion-closed property: if $D \in \mathcal{D}$ and D' is a multifunction such that $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, then $D' \in \mathcal{D}$. The family $\mathcal{D}$ is said to be a universe.

Definition 2.4. A multifunction $B \in \mathcal{D}$ is said to be pullback $\mathcal{D}$-absorbing if for every $D \in \mathcal{D}$, there exists $T = T(t, D) > 0$ such that

$$\mathcal{U}(t, t-s, D(t-s)) \subset B(t), \quad \text{for all } s \geq T.$$

We say that a multifunction $B \in \mathcal{D}$ is pullback $\mathcal{D}$-attracting (with respect to the MNDS $\mathcal{U}$) if for every $D \in \mathcal{D}$

$$\lim_{s \to +\infty} \operatorname{dist}_E(\mathcal{U}(t, t-s, D(t-s)), B(t)) = 0,$$

for all $t \in \mathbb{R}$. Here $\mathrm{dist}_E(\,\cdot\,,\,\cdot\,)$ is the Hausdorff semidistance between two subsets in E, i.e.

$$\mathrm{dist}_E(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|.$$

DEFINITION 2.5. A multifunction $A \in \mathcal{D}$ is said to be a global pullback $\mathcal{D}$-attractor for the MNDS $\mathcal{U}$ if it satisfies:

(a) $A(t)$ is compact for any $t \in \mathbb{R}$;
(b) A is pullback $\mathcal{D}$-attracting;
(c) A is negatively invariant, that is $A(t) \subset \mathcal{U}(t,\tau,A(\tau))$ for all $(t,\tau) \in \mathbb{R}^2_d$.

The pullback $\mathcal{D}$-attractor A is called strict if the invariance property in the third item is strict.

For a multifunction D, we define the pullback ω-limit set of D as a t-dependent set

$$\Lambda(t,D) = \bigcap_{\tau\geq 0}\overline{\bigcup_{s\geq\tau}\mathcal{U}(t,t-s,D(t-s))}.$$

LEMMA 2.6 ([9]). *Let $\mathcal{U}$ be a u.s.c.* MNDS *on E, i.e. $\mathcal{U}(t,\tau,\,\cdot\,)$ is u.s.c. for each $(t,\tau) \in \mathbb{R}^2_d$. Assume that B is a multifunction such that $\mathcal{U}$ is asymptotically compact with respect to B, i.e. for every sequence $s_n \to +\infty$, $t \in \mathbb{R}$, every sequence $y_n \in \mathcal{U}(t,t-s_n,B(t-s_n))$ is relatively compact. Then for $t \in \mathbb{R}$, the pullback ω-limit set $\Lambda(t,B)$ is nonempty, compact, and*

$$\lim_{s\to+\infty}\mathrm{dist}_E(\mathcal{U}(t,t-s,B(t-s)),\Lambda(t,B)) = 0,$$
$$\Lambda(t,B) \subset \mathcal{U}(t,s,\Lambda(s,B)), \quad \textit{for all } (t,s) \in \mathbb{R}^2_d.$$

The last lemma derives a sufficient condition ensuring the existence of pullback $\mathcal{D}$-attractor as follows.

THEOREM 2.7 ([9]). *Let $\mathcal{U}$ be a u.s.c.* MNDS *on E, and $B \in \mathcal{D}$ be a pullback $\mathcal{D}$-absorbing set for $\mathcal{U}$ such that $\mathcal{U}$ is asymptotically compact with respect to B. Then the multifunction A given by $A(t) = \Lambda(t,B)$ is a pullback $\mathcal{D}$-attractor for $\mathcal{U}$, and A is the unique element with these properties in $\mathcal{D}$. Moreover, if $\mathcal{U}$ is a strict* MNDS *then A is strictly invariant.*

We are in a position to collect some results on solvability and properties of solution set for problem (1.1)–(1.2). Put

$$J = [\tau,T],\ \mathcal{C}_h = C([-h,0];X),$$
$$C_{\varphi^\tau} = \{v \in C(J;X) : v(\tau) = \varphi^\tau(0)\}, \quad \text{for given } \varphi^\tau \in \mathcal{C}_h.$$

For $v \in C_{\varphi^\tau}$, we denote the function $v[\varphi^\tau] \in C([\tau-h,T];X)$ as follows:

$$v[\varphi^\tau](t) = \begin{cases} v(t) & \text{if } t \in [\tau,T], \\ \varphi^\tau(t-\tau) & \text{if } t \in [\tau-h,\tau]. \end{cases}$$

In the formulation of our problem, we make use of the following assumptions on A and F.

(A) The semigroup $S(\,\cdot\,)$ generated by A is norm continuous.

(F) The multimap $F\colon J\times X\times\mathcal{C}_h\to\mathcal{P}_{\rm c}(X)$ satisfies:

(1) *$t\mapsto F(t,x,y)$ admits a measurable selection for each $(x,y)\in X\times\mathcal{C}_h$ and $(x,y)\mapsto F(t,x,y)$ is u.s.c. for almost every $t\in J$;*

(2) *there exist nonnegative numbers a,b and a function $g\in L^1_{\rm loc}(\mathbb{R};\mathbb{R}^+)$ such that*

$$\|F(t,x,y)\|\le a\|x\|+b\|y\|_{\mathcal{C}_h}+g(t),\quad \textit{for all } x\in X,\ y\in\mathcal{C}_h,$$

here $\|F(t,x,y)\|=\sup\{\|\xi\|:\xi\in F(t,x,y)\}$;

(3) *if the semigroup $S(\,\cdot\,)$ is non-compact, then there exist functions $p,q\in L^1_{\rm loc}(\mathbb{R};\mathbb{R}^+)$ such that*

$$\chi(F(t,B,C))\le p(t)\chi(B)+q(t)\sup_{\theta\in[-h,0]}\chi(C(\theta))$$

for all bounded sets $B\subset X$, $C\subset\mathcal{C}_h$.

Remark 2.8. The assumptions on F are similar to the ones given in [17] and [28], where $\alpha(t)=g(t)$, $\beta(t)=a+b$ and $k(t)=p(t)+q(t)$.

Putting $\mathcal{P}_F(v)=\{f\in L^1(J;X):f(t)\in F(t,v(t),v[\varphi^\tau]_t)$ for a.e. $t\in J\}$, $v\in C_{\varphi^\tau}$, we have the following definition of integral solution to (1.1)–(1.2).

Definition 2.9. A function $u\colon[\tau-h,T]\to X$ is called an integral solution to problem (1.1)–(1.2) if $u\in C([\tau-h,T];X)$, $u(t)=\varphi^\tau(t-\tau)$ for $t\in[\tau-h,\tau]$ and there exists $f\in\mathcal{P}_F(u|_{[\tau,T]})$ such that

$$u(t)=S(t-\tau)\varphi^\tau(0)+\int_\tau^t S(t-s)f(s)\,ds \tag{2.1}$$

for any $t\in[\tau,T]$.

We define the multivalued operator $\mathcal{F}\colon C_{\varphi^\tau}\to\mathcal{P}(C_{\varphi^\tau})$ as follows:

$$\mathcal{F}(v)(t)=\bigg\{S(t-\tau)\varphi^\tau(0)+\int_\tau^t S(t-s)f(s)\,ds : f\in\mathcal{P}_F(v)\bigg\}.$$

Put

$$\mathcal{W}(f)(t)=\int_\tau^t S(t-s)f(s)\,ds,\quad \text{for } f\in L^1(J;X), \tag{2.2}$$

then

$$\mathcal{F}(v)(t)=S(t-\tau)\varphi^\tau(0)+\mathcal{W}\circ\mathcal{P}_F(v)(t).$$

It is obvious that $v\in C_{\varphi^\tau}$ is a fixed point of $\mathcal{F}$ if and only if $u=v[\varphi^\tau]$ is an integral solution of (1.1)–(1.2). By this reason, in the sequel we will refer to $\mathcal{F}$ as the solution operator.

The following existence result was proved in [28].

Theorem 2.10. *Let hypotheses* (A) *and* (F) *hold. Then problem* (1.1)–(1.2) *has at least one integral solution for each initial datum* $\varphi^\tau \in \mathcal{C}_h$. *Moreover, the set of all integral solutions is compact.*

Let π_T, $T > \tau$, be the truncate operator to $[\tau, T]$ acting on $C([\tau,+\infty);X)$, that is, for $z \in C([\tau,+\infty);X)$, $\pi_T(z)$ is the restriction of z on interval $[\tau,T]$. Denote

$$\Sigma(\varphi^\tau) = \big\{u \in C([\tau,+\infty);X) : \ u[\varphi^\tau] \text{ is an integral solution of (1.1)–(1.2) on } [\tau - h, T] \text{ for any } T > \tau\big\}.$$

Obviously,

$$\pi_T \circ \Sigma(\varphi^\tau) = S(\,\cdot - \tau)\varphi^\tau(0) + \mathcal{W} \circ \mathcal{P}_F(\pi_T \circ \Sigma(\varphi^\tau)), \tag{2.3}$$

for all $T > \tau$, and $\pi_T \circ \Sigma(\varphi^\tau) = \mathrm{Fix}(\mathcal{F})$, the fixed point set of the solution operator $\mathcal{F}$ of (1.1)–(1.2) in C_{φ^τ}. The following result was proved in [28].

Lemma 2.11. *The correspondence* $\varphi^\tau \mapsto \pi_T \circ \Sigma(\varphi^\tau)[\varphi^\tau]$ *is u.s.c. as a multimap from* $\mathcal{C}_h$ *to* $C([\tau - h, T];X)$.

Now we can define the MNDS $\mathcal{U}$ generated by problem (1.1)–(1.2):

$$\begin{aligned}
&\mathcal{U}\colon \mathbb{R}^2_d \times \mathcal{C}_h \to \mathcal{P}(\mathcal{C}_h),\\
&\mathcal{U}(t,\tau,\varphi^\tau) = \big\{u_t : \ u[\varphi^\tau] \text{ is an integral solution of (1.1)–(1.2)}\big\}\\
&\qquad\qquad\;\; = \{u_t : u \in \Sigma(\varphi^\tau)\}.
\end{aligned}$$

One can prove the MNDS properties of $\mathcal{U}$, including strictness one, by the same reasoning as that in [9]. In addition, we have following property.

Lemma 2.12. *Under assumptions* (A) *and* (F) (1)–(F) (3), $\mathcal{U}(t,\tau,\cdot\,)$ *is u.s.c. with compact values for each* $(t,\tau) \in \mathbb{R}^2_d$.

Proof. The conclusion is easily deduced from Lemma 2.11. □

3. Main results

In this section, we need the following assumptions:

(A*) The semigroup $S(t) = e^{tA}$ is norm-continuous, exponentially stable and χ-decreasing, that is

$$\|S(t)\| \le e^{-\alpha t}, \qquad \|S(t)\|_\chi \le N e^{-\beta t}, \quad \text{for all t } > 0, \tag{3.1}$$

where $N \ge 1, \alpha, \beta > 0$.

(F*) The nonlinearity F satisfies (F) with $a + b < \alpha$; $p, q \in L^\infty(\mathbb{R};\mathbb{R}^+)$ are such that $N(\|p\|_\infty + \|q\|_\infty) < \beta$, and $g \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^+)$ is such that

$$\varrho(t) := \operatorname*{ess\,sup}_{\tau \le t} g(\tau) < \infty.$$

It should be noted that, in general, condition (3.1) reads $\|S(t)\| \le Me^{-\alpha t}$ for $M \ge 1$. But we can take $M = 1$ since the norm in X can be replaced by the equivalent one $|||x||| = \sup\{e^{\alpha t}\|S(t)x\| : t \ge 0\}$ and we have

$$\|x\| \le |||x||| \le M\|x\|,$$
$$|||S(t)x||| = e^{-\alpha t}\sup\{e^{\alpha(t+s)}\|S(t+s)x\| : s \ge 0\} \le e^{-\alpha t}|||x|||.$$

In this section, we only consider the case when the semigroup $S(\cdot)$ is non-compact. In the opposite case, one can assign $\beta = +\infty$.

Denote by χ_C the Hausdorff MNC on $\mathcal{C}_h$. We have the following properties of χ_C (see, e.g. [1]):

(1) $\sup\limits_{s\in[-h,0]} \chi(D(s)) \le \chi_C(D)$ for all $D \subset \mathcal{C}_h$;

(2) if D is equicontinuous then $\chi_C(D) = \sup\limits_{s\in[-h,0]} \chi(D(s))$.

For fixed $T > h$ and $t \in \mathbb{R}$, we define the so-called translation multioperator $\mathcal{G}_{T,t}$ as follows:

$$\mathcal{G}_{T,t}\colon \mathcal{C}_h \to \mathcal{P}(\mathcal{C}_h), \qquad \mathcal{G}_{T,t}(\phi) = \mathcal{U}(t, t-T, \phi).$$

We will prove the condensivity property of $\mathcal{G}_{T,t}$. To this end we make use of the following result (see [21, § 4.5], or [33] for a generalized version).

Proposition 3.1 (Halanay's inequality). *Let the continuous function $f\colon [t_0 - h, T) \to \mathbb{R}^+$, $t_0 < T < +\infty$, satisfy the functional differential inequality*

$$f'(t) \le -\gamma f(t) + \nu \sup_{s\in[t-h,t]} f(s),$$

for $t \ge t_0$, where $\gamma > \nu > 0$. Then

$$f(t) \le \kappa e^{-\ell(t-t_0)}, \quad t \ge t_0,$$

where $\kappa = \sup\limits_{s\in[t_0-h,t_0]} f(s)$ and ℓ is the solution of the equation $\gamma = \ell + \nu e^{\ell h}$.

Using Halanay's inequality, we obtain the following result.

Lemma 3.2. *Let hypotheses* (A*) *and* (F*) *hold. Then there exist $T > h$ and $\zeta \in (0, 1)$ such that*

$$\chi_C(\mathcal{G}_{T,t}(B)) \le \zeta \cdot \chi_C(B), \quad \textit{for all } B \in \mathcal{B}(\mathcal{C}_h).$$

Proof. Putting $D = \Sigma(B)$, we recall that

$$D(s) = S(s-\tau)B(0) + \int_\tau^s S(s-r)\mathcal{P}_F(D)(r)\,dr, \quad \text{for all } (s,\tau) \in \mathbb{R}^2_d. \tag{3.2}$$

It is readily seen that $D(s)$ is bounded. Define a function v as follows:

$$v(r) = \begin{cases} \chi(D(r)) & \text{if } r \ge \tau, \\ \chi(B(r-\tau)) & \text{if } r \in [\tau - h, \tau]. \end{cases} \tag{3.3}$$

Then by (3.2),

$$v(s) \le \chi(S(s-\tau)B(0)) + \chi\bigg(\int_\tau^s S(s-r)\mathcal{P}_F(D)(r)\,dr\bigg).$$

By (A*) and (F*), we have

$$\chi(S(s-\tau)B(0)) \le Ne^{-\beta(s-\tau)}\chi(B(0)),$$
$$\chi(S(s-r)\mathcal{P}_F(D)(r)) \le Ne^{-\beta(s-r)}\Big(\|p\|_\infty\chi(D(r)) + \|q\|_\infty \sup_{\theta\in[r-h,r]}\chi(D[B](\theta))\Big),$$

where

$$D[B](\theta) = \begin{cases} D(\theta) & \text{if } \theta \ge \tau, \\ B(\theta-\tau) & \text{if } \theta \in [\tau-h,\tau]. \end{cases}$$

Thus, by Proposition 2.2, we get

$$v(s) \le e^{-\beta s}\bigg[Ne^{\beta\tau}\chi(B(0)) + N\int_\tau^s e^{\beta r}\Big(\|p\|_\infty\chi(D(r)) + \|q\|_\infty \sup_{\theta\in[r-h,r]}\chi(D[B](\theta))\Big)\,dr\bigg],$$

Denoting by $z(s)$ the right-hand side of the last inequality and setting $z(r) = Nv(r)$ for $r \in [\tau-h,\tau]$, we have $v(s) \le z(s)$, for all $s \ge \tau - h$ and

$$\begin{aligned} z'(s) &= -\beta z(s) + N\Big(\|p\|_\infty\, v(s) + \|q\|_\infty \sup_{r\in[s-h,s]} v(r)\Big) \\ &\le -(\beta - N\|p\|_\infty)z(s) + N\|q\|_\infty \sup_{r\in[s-h,s]} z(r), \end{aligned}$$

for $s \ge \tau$. Applying Halanay's inequality for z, we have

$$z(s) \le \sup_{r\in[\tau-h,\tau]} z(r)e^{-\ell(s-\tau)} = N\sup_{r\in[\tau-h,\tau]} v(r)e^{-\ell(s-\tau)}, \quad s \ge \tau,$$

where ℓ is the solution of the equation $\beta - N\|p\|_\infty = \ell + N\|q\|_\infty e^{\ell h}$. Therefore

$$v(s) \le z(s) \le N\sup_{r\in[\tau-h,\tau]} \chi(B(r-\tau))\, e^{-\ell(s-\tau)} \le Ne^{-\ell(s-\tau)}\chi_C(B),$$

for $s \ge \tau$, thanks to the definition of v in (3.3). Now for $s > h + \tau$ we have

$$\sup_{\theta\in[-h,0]} v(s+\theta) \le Ne^{-\ell(s-h-\tau)}\chi_C(B). \tag{3.4}$$

Taking into account (3.2), one has

$$D_s(\theta) = S(s+\theta-\tau)B(0) + \int_\tau^{s+\theta} S(s+\theta-r)\mathcal{P}_F(D)(r)\,dr, \tag{3.5}$$

for $\theta \in [-h,0]$, where $D_s = \{u_s : u \in D\} \subset \mathcal{C}_h$. Since $s - \tau > h$ and $S(\cdot)$ is norm continuous, the set of function Ξ_1 defined by $\Xi_1(\theta) = S(s-\tau+\theta)B(0)$ is equicontinuous in $\mathcal{C}_h$. Moreover, the set of functions Ξ_2 given by

$$\Xi_2(\theta) = \int_\tau^{s+\theta} S(s+\theta-r)\mathcal{P}_F(D)(r)\,dr$$

is also equicontinuous in $\mathcal{C}_h$. Accordingly, $D_s = \Xi_1 + \Xi_2$ is equicontinuous in $\mathcal{C}_h$ and then

$$\chi_C(D_s) = \sup_{\theta\in[-h,0]} \chi(D(s+\theta)) = \sup_{\theta\in[-h,0]} v(s+\theta) \le Ne^{-\ell(s-h-\tau)}\chi_C(B),$$

thanks to (3.4). Choosing $\tau = t-T$ with $T > T^* = h + (1/\ell)\ln N$, we get

$$\mathcal{G}_{T,t}(B) = \mathcal{U}(t,t-T,B) = \{u_t : u \in \Sigma(B)\} = D_t,$$

and then $\chi_C(\mathcal{G}_{T,t}(B)) = \chi_C(D_t) \le \zeta\cdot\chi_C(B)$, with $\zeta = Ne^{-\ell(T-h)} < 1$. □

Take $d \in (b, \alpha - a)$ and let ℓ be the solution of the equation

$$\alpha - a = \ell + d\,e^{\ell h}. \tag{3.6}$$

We consider the universe

$$\mathcal{D} = \Big\{D : D(\tau) = B_{\mathcal{C}_h}[0, r(\tau)],\ \lim_{\tau\to-\infty} r(\tau)e^{\ell\tau} = 0\Big\}.$$

We are ready to figure out the behavior of the MNDS $\mathcal{U}$.

Lemma 3.3. *Assume* (A*) *and* (F*). *Then the* MNDS $\mathcal{U}$ *admits a pullback* $\mathcal{D}$*-absorbing set.*

Proof. Let $D \in \mathcal{D}$, $D(\tau) = B_{\mathcal{C}_h}[0, r(\tau)]$. For $T > \tau$ and $\varphi^\tau \in D(\tau)$, we consider the solution $u[\varphi^\tau]$ given by

$$u(t) = S(t-\tau)\varphi^\tau(0) + \int_\tau^t S(t-s)f(s)\,ds, \quad \text{for } t \in [\tau, T],$$

where $f \in \mathcal{P}_F(u)$. Using (F) (2) and (A*), we have

$$\|u(t)\| \le e^{-\alpha(t-\tau)}\|\varphi^\tau(0)\| + \int_\tau^t e^{-\alpha(t-s)}\big[a\|u(s)\| + b\|u_s\|_{\mathcal{C}_h} + g(s)\big]\,ds. \tag{3.7}$$

Take $R = R(T)$ such that $b + \varrho(T)/R(T) = d < \alpha - a$, where ϱ is defined in (F*). Our aim is to show that there exists $T^* = T^*(D) > 0$ such that $\|u_T\|_{\mathcal{C}_h} \le R(T)$ for all $u \in \Sigma(\varphi^\tau)$ whenever $T - \tau \ge T^*$, which implies

$$\mathcal{U}(T,\tau,\varphi^\tau) \subset B_{\mathcal{C}_h}[0, R(T)], \quad \text{for all } \varphi^\tau \in D(\tau),\ \tau \le T - T^*.$$

We first observe that if $\|u_t\|_{\mathcal{C}_h} > R(T)$ for all $t \in [\tau, T]$, then

$$b\|u_s\|_{\mathcal{C}_h} + g(s) \le \|u_s\|_{\mathcal{C}_h}\bigg(b + \frac{\varrho(T)}{R(T)}\bigg) = d\|u_s\|_{\mathcal{C}_h}, \quad \text{for all } s \in [\tau, T].$$

Thus (3.7) implies

$$\|u(t)\| \le e^{-\alpha(t-\tau)}\|\varphi^\tau(0)\| + \int_\tau^t e^{-\alpha(t-s)}[\|u(s)\| + d\|u_s\|_{\mathcal{C}_h}]\,ds, \quad t \in [\tau, T].$$

Let

$$v(t) = \begin{cases} e^{-\alpha(t-\tau)}\|\varphi^\tau(0)\| + \displaystyle\int_\tau^t e^{-\alpha(t-s)}[a\|u(s)\| + d\|u_s\|_{\mathcal{C}_h}]\,ds & \text{if } t \geq \tau, \\ \|u(t)\| & \text{if } t \in [\tau - h, \tau]. \end{cases}$$

Then we have $\|u(t)\| \leq v(t)$ for all $t \in [\tau - h, T]$, and the following estimate holds:

$$v'(t) \leq -(\alpha - a)v(t) + d \sup_{s\in[t-h,t]} v(s), \quad t \geq \tau.$$

Application of Halanay's inequality yields

$$\|u(t)\| \leq \|\varphi^\tau\|_{\mathcal{C}_h} e^{-\ell(t-\tau)} \leq r(\tau)e^{-\ell(t-\tau)}, \quad \text{for all } t \in [\tau, T],$$

where ℓ is defined by (3.6). The last inequality tells us that $\|u_t\|_{\mathcal{C}_h}$ tends to zero as $\tau \to -\infty$, hence one can find $t_1 \in (\tau, T]$ such that $\|u_{t_1}\|_{\mathcal{C}_h} < R(T)$. This contradiction proves the existence of $t_0 \in [\tau, T]$ ensuring $\|u_{t_0}\|_{\mathcal{C}_h} \leq R(T)$.

If $t_0 = T$ then our proof is done. Otherwise, we claim that $\|u_t\|_{\mathcal{C}_h} \leq R(T)$ for all $t \in [t_0, T]$. Indeed, in the opposite case, there exists $t_1 \in [t_0, T)$ such that

$$\|u_{t_1}\|_{\mathcal{C}_h} \leq R(T) \quad \text{but} \quad \|u_t\|_{\mathcal{C}_h} > R(T), \quad \text{for all } t \in (t_1, t_1 + \theta),$$

where $\theta > 0$, $t_1 + \theta < T$. Regarding the solution $u[\varphi^\tau]$ on $[t_1, t_1 + \theta)$, we have

$$u(t) = S(t - t_1)u(t_1) + \int_{t_1}^t S(t - s)f(s)\,ds.$$

Then, for $t \in [t_1, t_1 + \theta)$,

$$\|u(t)\| \leq e^{-\alpha(t-t_1)}\|u(t_1)\| + \int_{t_1}^t e^{-\alpha(t-s)}[a\|u(s)\| + d\|u_s\|_{\mathcal{C}_h}]\,ds.$$

Using the same arguments as above, we see that, for all $t \in (t_1, t_1 + \theta)$

$$\|u(t)\| \leq \|u_{t_1}\|_{\mathcal{C}_h} e^{-\ell(t-t_1)} \leq \|u_{t_1}\|_{\mathcal{C}_h} \leq R(T).$$

Hence, for $t \in [t_1, t_1 + \theta)$, we have

$$\begin{aligned} \|u_t\|_{\mathcal{C}_h} &= \sup_{s\in[-h,0]} \|u(t+s)\| = \sup_{r\in[t-h,t]} \|u(r)\| \leq \sup_{r\in[t_1-h,t]} \|u(r)\| \\ &= \max\Big\{ \sup_{r\in[t_1-h,t_1]} \|u(r)\|; \sup_{r\in[t_1,t]} \|u(r)\| \Big\} \\ &= \max\Big\{ \|u_{t_1}\|_{\mathcal{C}_h}; \sup_{r\in[t_1,t]} \|u(r)\| \Big\} \leq R(T). \end{aligned}$$

This is a contradiction.

In summary, we designate $\widehat{B} = \{B_{\mathcal{C}_h}[0, R(t)] : t \in \mathbb{R}\}$ as a pullback absorbing set for the MNDS $\mathcal{U}$, where $R(t) = \varrho(t)/(d - b)$. Since $\varrho(\,\cdot\,)$ is non-decreasing, we see that $\lim_{\tau\to-\infty} \varrho(\tau)e^{\ell\tau} = 0$, which implies that $\widehat{B} \in \mathcal{D}$. In addition, $\widehat{B}$ is non-decreasing, i.e. $\widehat{B}(\tau) \subset \widehat{B}(t)$ for all $(t, \tau) \in \mathbb{R}^2_d$. □

Lemma 3.4. *Let hypotheses* (A*) *and* (F*) *hold. Then the* MNDS $\mathcal{U}$ *is asymptotically compact with respect to the absorbing set* $\widehat{B}$ *obtained by Lemma* 3.3.

Proof. We first claim that, for any $\varepsilon > 0$ one can find a number $T_\varepsilon(t, \widehat{B}) > 0$ such that

$$\chi_C(\mathcal{U}(t, t-s, \widehat{B}(t-s))) < \varepsilon, \quad \text{for all s } \geq T_\varepsilon(t, \widehat{B}).$$

Let $T > T^*$ and $\zeta \in (0,1)$ as in Lemma 3.2. Since $\widehat{B}$ is an absorbing set, one can take $\widehat{T} > 0$ such that

$$\mathcal{U}(t, t-s, \widehat{B}(t-s)) \subset \widehat{B}(t), \quad \text{for all } s \geq \widehat{T}. \tag{3.8}$$

Let $n \in \mathbb{N}$ be a number such that $\zeta^n \chi_C(\widehat{B}(t)) < \varepsilon$. For $s \geq T_\varepsilon(t, \widehat{B}) := nT + \widehat{T}$, we have

$$\begin{aligned}\mathcal{U}(t, t-s, &\widehat{B}(t-s)) \\ &= \mathcal{G}_{T,t} \circ \mathcal{G}_{T,t-T} \circ \ldots \circ \mathcal{G}_{T,t-(n-1)T}(\mathcal{U}(t-nT, t-s, \widehat{B}(t-s))) \\ &\subset \mathcal{G}_{T,t} \circ \mathcal{G}_{T,t-T} \circ \ldots \circ \mathcal{G}_{T,t-(n-1)T}(\widehat{B}(t-nT)),\end{aligned}$$

thanks to (3.8). Applying Lemma 3.2 iteratively, we get

$$\chi_C(\mathcal{U}(t, t-s, \widehat{B}(t-s))) \leq \zeta^n \chi_C(\widehat{B}(t-nT)) \leq \zeta^n \chi_C(\widehat{B}(t)) < \varepsilon.$$

Now let $s_k \to +\infty$ and $\xi_k \in \mathcal{U}(t, t-s_k, \widehat{B}(t-s_k))$. We will show that $\{\xi_k\}$ is relatively compact in $\mathcal{C}_h$. Since ε is arbitrarily small, this will be done if $\chi_C(\{\xi_k\}) < \varepsilon$.

Let $N \in \mathbb{N}$ be a fixed number such that $s_k \geq T_\varepsilon(t, \widehat{B}) + \widehat{T}$ for all $k \geq N$. Then we have

$$\begin{aligned}\mathcal{U}(t, t-s_k, \widehat{B}(t-s_k)) &= \mathcal{U}(t, t-T_\varepsilon, \mathcal{U}(t-T_\varepsilon, t-s_k, \widehat{B}(t-s_k))) \\ &\subset \mathcal{U}(t, t-T_\varepsilon, \widehat{B}(t-T_\varepsilon)),\end{aligned}$$

for all $k \geq N$, thanks to (3.8) again, here T_ε stands for $T_\varepsilon(t, \widehat{B})$. Thus

$$\{\xi_k : k \geq N\} \subset \mathcal{U}(t, t-T_\varepsilon, \widehat{B}(t-T_\varepsilon)),$$

and then

$$\chi_C(\{\xi_k : k \geq N\}) \leq \chi_C(\mathcal{U}(t, t-T_\varepsilon, \widehat{B}(t-T_\varepsilon)) < \varepsilon.$$

Since the set $\{\xi_k : k < N\}$ is finite, we have

$$\chi_C(\{\xi_k\}) \leq \chi_C(\{\xi_k : k < N\}) + \chi_C(\{\xi_k : k \geq N\}) = \chi_C(\{\xi_k : k \geq N\}) < \varepsilon. \quad \square$$

Combining Lemmas 2.12, 3.3 and 3.4, we arrive at the conclusion.

Theorem 3.5. *Let hypotheses* (A*) *and* (F*) *hold. Then the* MNDS $\mathcal{U}$ *generated by system* (1.1)–(1.2) *admits a global pullback* $\mathcal{D}$*-attractor in* $\mathcal{C}_h$.

4. Application

4.1. Polytope functional partial differential equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following problem:

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(t,x), \quad x \in \Omega,\ t > \tau, \tag{4.1}$$

$$f(t,x) \in \mathrm{co}\big\{f_i(t,x,u(t,x),u(t-\rho(t),x)) : i = 1,\dots,m\big\}, \tag{4.2}$$

$$u(t,x) = 0, \qquad x \in \partial\Omega,\ t > \tau, \tag{4.3}$$

$$u(\tau+s,x) = \varphi^\tau(x,s), \qquad x \in \Omega,\ s \in [-h,0], \tag{4.4}$$

where $\rho\colon \mathbb{R} \to [0,h]$, $f_i\colon \mathbb{R}\times\Omega\times\mathbb{R}^2 \to \mathbb{R}$, $i = 1,\dots,m$, are continuous functions,

$$\mathrm{co}\,\{f_1,\dots,f_m\} = \bigg\{\sum_{i=1}^m \eta_i f_i : \eta_i \geq 0,\ \eta_1 + \dots + \eta_m = 1\bigg\}.$$

Let $A = \Delta$ with $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$, $X = L^2(\Omega)$ and $\mathcal{C}_h = C([-h,0];L^2(\Omega))$. Then it is known that A is the infinitesimal generator of a compact, contraction semigroup on X (see [19]). Moreover, the semigroup $S(t) = e^{tA}$ is exponentially stable, that is $\|S(t)\| \leq e^{-\lambda_1 t}, t \geq 0$, where $\lambda_1 > 0$ is the first eigenvalue of $-A$. So one gets (A*) with $\alpha = \lambda_1$ and $\beta = +\infty$.

Regarding the nonlinearities f_i, we assume, in addition, that

(P) $|f_i(t,x,y,z)| \leq a|y| + b|z| + g(t,x)$ for all $(t,x,y,z) \in \mathbb{R}\times\Omega\times\mathbb{R}^2$, here a, b are non-negative numbers and $g\colon \mathbb{R}\times\Omega \to \mathbb{R}$ is continuous,

$$\int_\Omega |g(t,x)|^2\,dx \leq C(1+t^2)^\gamma e^{\omega t} \quad \text{with } \gamma \in \mathbb{R};\ C,\omega > 0.$$

Let $\widehat{f}_i\colon \mathbb{R}\times X\times\mathcal{C}_h \to X$ be the function given by

$$\widehat{f}_i(t,v,w)(x) = f_i(t,x,v(x),w(-\rho(0),x)).$$

Put $F(t,v,w) = \mathrm{co}\,\{\widehat{f}_i(t,v,w) : i = 1,\dots,m\}$. Then $F\colon \mathbb{R}\times X\times\mathcal{C}_h \to \mathcal{P}(X)$ is a multimap with closed, convex values. One observe that for a fixed (t,v,w), $F(t,v,w)$ is a bounded set in the finite dimensional space span $\{\widehat{f}_1,\dots,\widehat{f}_m\} \subset X$, so F has compact values. We point out that $F(t,\,\cdot\,,\,\cdot\,)$ is u.s.c. Indeed, let $\{v_n,w_n\} \subset X\times\mathcal{C}_h$ converge to (v,w). Then by the continuity of f_i and the Lebesgue dominated convergence theorem, $\widehat{f}_i(t,v_n,w_n) \to \widehat{f}_i(t,v,w)$ in X. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\widehat{f}_i(t,v_n,w_n) \in \widehat{f}_i(t,v,w) + \varepsilon B_X[0,1], \quad \text{for all } n \geq N,\ i = 1,\dots,m.$$

This implies

$$F(t,v_n,w_n) \subset F(t,v,w) + \varepsilon B_X[0,1], \quad \text{for all } n \geq N.$$

Since F has compact values, the last inclusion guarantees the upper-semicontinuity of $F(t,\cdot,\cdot)$. Now let $z\in F(t,v,w)$, then by (P) and the definition of F we have

$$|z(x)|\le\sum_{i=1}^{m}\eta_i|f_i(t,x,v(x),w(-\rho(0),x))|\le a|v(x)|+b|w(-\rho(0),x)|+|g(t,x)|.$$

So it follows from Minkowskiĭ's inequality that

$$\|z\|\le a\|v\|+b\|w\|_{\mathcal{C}_h}+\sqrt{C}(1+t^2)^{\gamma/2}e^{\omega t/2}.$$

Therefore (F*) is satisfied if $a+b<\lambda_1$. By Theorem 3.5 the MNDS governed by (4.1)–(4.4) has a global pullback $\mathcal{D}$-attractor in $C([-h,0];L^2(\Omega))$.

4.2. Lattice functional differential system. Consider the following infinite differential system:

$$\frac{du_i}{dt}(t)=u_{i+1}(t)-(2+\alpha)u_i(t)+u_{i-1}(t)+f_i(t,u_i(t),u_i(t-h)),\quad t>\tau,\tag{4.5}$$

$$u_i(\tau+s)=\phi_i^\tau(s),\quad s\in[-h,0],\ i\in\mathbb{Z},\tag{4.6}$$

where $u=(u_i)_{i\in\mathbb{Z}}$ is the state function, $\alpha>0$, $f_i\colon\mathbb{R}^3\to\mathbb{R}$, $i\in\mathbb{Z}$, are continuous functions. This model comes from a number of problems concerning image processing, pattern recognition, electrical engineering, etc. On the other hand, it is a result of spatial discretization of partial differential equations. Regardless of exhaustive references, we refer the reader to [7], [12], [15], [32], [38] for some recent results on asymptotic behavior of lattice differential systems.

Let ℓ^2 be the space of real sequences $x=(x_i)_{i\in\mathbb{Z}}$ satisfying

$$\|x\|^2=\sum_{i\in\mathbb{Z}}x_i^2<\infty.$$

Then ℓ^2, with the scalar product $(x,y)=\sum_{i\in\mathbb{Z}}x_iy_i$, becomes a separable Hilbert space with the basis $\{e_k\}_{k\in\mathbb{Z}}$ where $e_k=(\delta_{ki})_{i\in\mathbb{Z}}$ is the sequence of zeros but for a 1 in the k^{th} entry. Let $R_n\colon\ell^2\to\ell^2$ be the linear operator defined by

$$R_n(x)=\sum_{|i|>n}x_ie_i.$$

Then we recall that the Hausdorff MNC in ℓ^2 is given by (see [4, Theorem 4.2])

$$\chi(B)=\limsup_{n\to\infty}\sup_{x\in B}\|R_n(x)\|.\tag{4.7}$$

Define $A,B\colon\ell^2\to\ell^2$ as follows:

$$(Ax)_i=x_{i+1}-2x_i+x_{i-1},\qquad(Bx)_i=x_{i+1}-x_i.$$

Then the operator B^* given by $(B^*x)_i = x_{i-1} - x_i$ is the adjoint operator of B and $-A = BB^* = B^*B$. The linear part of (4.5) can be written as

$$\frac{du}{dt} = Au - \alpha u, \quad t > \tau.$$

This implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 &= (Au, u) - \alpha\|u\|^2 \\ &= -(B^*Bu, u) - \alpha\|u\|^2 = -\|Bu\|^2 - \alpha\|u\|^2 \le -\alpha\|u\|^2.\end{aligned}$$

Then $\|u(t)\| \le e^{-\alpha(t-\tau)}\|u(\tau)\|$. Therefore the C_0-semigroup $S(t) = e^{t(A-\alpha I)}$ is exponential stable, i.e. $\|S(t)\| \le e^{-\alpha t}$. In addition, since $A - \alpha I$ is a bounded operator on ℓ^2, $S(\cdot)$ can be extended to a differentiable C_0-group. Hence $\{S(t) : t \in \mathbb{R}\}$ is norm-continuous but non-compact (since $I = S(t)S(-t)$ is non-compact). At this point, (A*) is verified with $\beta = \alpha$, $N = 1$.

Regarding the nonlinearities f_i, $i \in \mathbb{Z}$, we assume that

(Q) There exist $a, b > 0$, $g = (g_i)\colon \mathbb{R} \to \ell^2$ such that

$$|g_i(t)| \le C_i(1+t^2)^\gamma e^{\omega t}, \quad (C_i)_{i\in\mathbb{Z}} \in \ell^2,\ \gamma \in \mathbb{R},\ \omega > 0,$$
$$|f_i(t,x,y)|^2 \le ax^2 + by^2 + |g_i(t)|^2.$$

Now, for $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, $w = (w_i)_{i\in\mathbb{Z}} \in C([-h,0];\ell^2)$, put

$$F(t,v,w) = (f_i(t, v_i, w_i(-h)))_{i\in\mathbb{Z}}.$$

Then it is easily seen that $F\colon \mathbb{R} \times \ell^2 \times \mathcal{C}_h \to \ell^2$ is a continuous function, here $\mathcal{C}_h = C([-h,0];\ell^2)$. Moreover, by assumption (Q)

$$\begin{aligned}\|F(t,v,w)\|^2 &= \sum_{i\in\mathbb{Z}} |f_i(t,v_i,w_i(-h))|^2 \\ &\le a\sum_{i\in\mathbb{Z}}|v_i|^2 + b\sum_{i\in\mathbb{Z}}|w_i(-h)|^2 + \sum_{i\in\mathbb{Z}}|g_i(t)|^2 \\ &= a\|v\|^2 + b\|w(-h)\|^2 + \|g(t)\|^2 \\ &\le a\|v\|^2 + b\sup_{s\in[-h,0]}\|w(s)\|^2 + \|g(t)\|^2.\end{aligned}$$

Thus

$$\|F(t,v,w)\| \le \sqrt{a}\|v\| + \sqrt{b}\|w\|_{\mathcal{C}_h} + \|g(t)\|. \tag{4.8}$$

On the other hand, in view of (4.7), for any bounded sets $V \subset \ell^2$, $W \subset \mathcal{C}_h$ one has

$$\begin{aligned}\chi(F(t,V,W)) &= \limsup_{n\to\infty}\sup_{(v,w)\in V\times W}\bigg(\sum_{|i|>n}|f_i(t,v_i,w_i(-h))|^2\bigg)^{1/2} \\ &\le \limsup_{n\to\infty}\sup_{(v,w)\in V\times W}\bigg(a\sum_{|i|>n}|v_i|^2 + b\sum_{|i|>n}|w_i(-h)|^2 + \sum_{|i|>n}|g_i(t)|^2\bigg)^{1/2}\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\le \limsup_{n\to\infty} \sup_{(v,w)\in V\times W} \big(\sqrt{a}\|R_n(v)\| + \sqrt{b}\|R_n(w(-h))\| + \|R_n(g(t))\|\big)\\
&\le \limsup_{n\to\infty} \Big(\sqrt{a}\sup_{v\in V}\|R_n(v)\| + \sqrt{b}\sup_{w\in W}\|R_n(w(-h))\| + \|R_n(g(t))\|\Big)\\
&= \sqrt{a}\,\chi(V) + \sqrt{b}\,\chi(W(-h)) \le \sqrt{a}\,\chi(V) + \sqrt{b}\sup_{s\in[-h,0]}\chi(W(s)).
\end{aligned}$$

Taking into account (4.8)–(4.9), hypothesis (F*) is fulfilled if $\sqrt{a}+\sqrt{b}<\alpha$. Consequently, the MNDS generated by (4.5)–(4.6) admits a global pullback $\mathcal{D}$-attractor in $C([-h,0];\ell^2)$.

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