

**MULTIPLICITY THEOREMS
FOR RESONANT AND SUPERLINEAR
NONHOMOGENEOUS ELLIPTIC EQUATIONS**

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ABSTRACT. We consider nonlinear elliptic equations driven by the sum of a p -Laplacian ($p > 2$) and a Laplacian. We consider two distinct cases. In the first one, the reaction $f(z, \cdot)$ is $(p-1)$ -linear near $\pm\infty$ and resonant with respect to a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. We prove a multiplicity theorem producing three nontrivial solutions. In the second case, the reaction $f(z, \cdot)$ is $(p-1)$ -superlinear but does not satisfy the Ambrosetti–Rabinowitz condition. We prove two multiplicity theorems. In the first main result we produce six nontrivial solutions all with sign information and in the second theorem we have five nontrivial solutions. Our approach uses variational methods combined with the Morse theory, truncation methods, and comparison techniques.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous Dirichlet problem:

$$(1.1) \quad -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad 2 < p < \infty.$$

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Here Δ_p denotes the p -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

If $p = 2$, then $\Delta_2 = \Delta =$ the Laplace differential operator.

The reaction f is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). Additional regularity conditions on $f(z, \cdot)$ are introduced in order to produce extra solutions. We consider two distinct cases. In the first (Section 3), we assume that $f(z, \cdot)$ is $(p - 1)$ -linear near $\pm\infty$ and interacts with a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (resonant problem). We prove a multiplicity theorem, producing three nontrivial solutions two of which have constant sign. In the second case (Section 4), we deal with reaction $f(z, \cdot)$ which is $(p - 1)$ -superlinear near $\pm\infty$ but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short). We also assume that $f(z, \cdot)$ has z -dependent zeros of constant sign. We prove a multiplicity theorem producing six nontrivial solutions, four of constant sign and two nodal (sign changing).

Recently nonhomogeneous nonlinear equations driven by the sum of a p -Laplacian and a Laplacian (a $(p, 2)$ -equation for short), were studied by Cingolani and Degiovanni [7], Papageorgiou and Rădulescu [21], Papageorgiou and Smyrlis [22] and Sun [24]. All these works deal with equations that have a $(p - 1)$ -linear reaction and either they do not allow resonance (see [7]) or the resonance is with respect to the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see [21], [22], [24]). Recall that for $p \neq 2$, we do not have a complete knowledge of the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$, the eigenspaces are not linear subspaces and we do not have a direct sum decomposition of $W_0^{1,p}(\Omega)$ in terms of them. All these facts, make problems resonant at higher parts of the spectrum difficult to deal with. On the other hand, problems with reactions which have zeros of constant sign, were investigated only in the context of p -Laplacian equations, by Bartsch, Liu and Weth [5] (constant zeros) and by Iturriaga, Massa, Sanchez and Ubilla [17] (variable zeros for a class of parametric equations). None of the aforementioned works produces six nontrivial solutions all with sign information.

Our approach is a combination of variational methods based on the critical point theory together with the Morse theory (critical groups) and truncation and comparison techniques. In the next section, for the convenience of the reader, we recall the main mathematical tools which we will use in the sequel.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*) . Let $\varphi \in C^1(X)$. We say that φ satisfies the Cerami condition (the C-condition for short), if the following holds:

“Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \rightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence”.

This is a compactness type condition on the functional φ which compensates for the fact that the underlying space X is not locally compact (being in general infinite dimensional). The C-condition is more general than the more familiar Palais–Smale condition (PS-condition for short) and the two are equivalent if φ is bounded below. The C-condition is a basic tool in proving a deformation theorem from which we can derive the minimax theory of certain critical values of φ . Prominent in that theory is the so-called “mountain pass theorem” due to Ambrosetti and Rabinowitz [3]. Here the theorem is presented in a slightly more general form with the C-condition replacing the PS-condition (see, for example, Gasinski and Papageorgiou [15, p. 648]).

THEOREM 2.1. *Assume that $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geq m_\rho$ and c is a critical value of φ .

In what follows by $|\cdot|$ we denote the norm of \mathbb{R}^N and by $\|\cdot\|$ the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $1 < p < \infty$ and let $A_p: W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$, $1/p + 1/p' = 1$, be the nonlinear map defined by

$$\langle A_p(u), y \rangle = \int_\Omega |Du|^{p-2} (Du, Dy)_{\mathbb{R}^N} dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega).$$

When $p = 2$, then we write $A_2 = A$ and this map is linear. The next proposition summarizes the main properties of this map (see, for example, Papageorgiou and Kyritsi [20, p. 314]).

PROPOSITION 2.2. *The map $A_p: W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), demicontinuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle \leq 0,$$

then $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$.

Also, we can view A_p defined on the bigger Sobolev space $W^{1,p}(\Omega)$, that is $A_p: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$. With a slight abuse of notation, we still denote the duality brackets by $\langle \cdot, \cdot \rangle$ and we have

$$\langle A(u), y \rangle = \int_{\Omega} |Du|^{p-2} (Du, Dy)_{\mathbb{R}^N} dz \quad \text{for all } u, y \in W^{1,p}(\Omega).$$

This map too is bounded, demicontinuous, maximal monotone and of type $(S)_+$.

In addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone $C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior, given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

By p^* we denote the critical Sobolev exponent. So,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

Let $f_0: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function with subcritical growth in the x -variable, that is,

$$|f_0(z, x)| \leq a_0(z)(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$, $1 < r < p^*$. Let $F_0(z, x) = \int_0^x f_0(z, s) ds$ and $\varphi_0: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ ($2 < p < \infty$) be the C^1 -functional defined by

$$\varphi_0(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} F_0(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The following proposition is a special case of a more general result of Gasinski and Papageorgiou [16] and essentially is a consequence of the regularity results of Lieberman [19].

PROPOSITION 2.3. *Assume that $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\bar{\Omega})$ -minimizer of φ_0 , that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\bar{\Omega}), \|h\|_{C_0^1(\bar{\Omega})} \leq \rho_0.$$

Then $u_0 \in C_0^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$ -minimizer of φ_0 , that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega), \|h\| \leq \rho_1.$$

Let $\eta, \hat{\eta} \in L^\infty(\Omega)$. We say $\eta \prec \hat{\eta}$ if and only if for every compact $K \subseteq \Omega$, we can find $\varepsilon = \varepsilon_K > 0$ such that

$$\eta(z) + \varepsilon \leq \hat{\eta}(z) \quad \text{for a.a. } z \in K.$$

Clearly, if $\eta, \hat{\eta} \in C(\Omega)$ and $\eta(z) < \hat{\eta}(z)$ for all $z \in \Omega$, then $\eta \prec \hat{\eta}$.

A simple modification of the proof of Proposition 2.6 of Arcoya and Ruiz [4], in order to accommodate the presence of the extra linear term $-\Delta u$, leads to the following strong comparison property.

PROPOSITION 2.4. *Assume that $\beta \geq 0$, $\eta, \hat{\eta} \in L^\infty(\Omega)$, $\eta \prec \hat{\eta}$ and $u \in C_0^1(\bar{\Omega})$, $v \in \text{int } C_+$ satisfy*

$$\begin{aligned} -\Delta_p u(z) - \Delta u(z) + \beta|u(z)|^{p-2}u(z) &= \eta(z) \quad \text{in } \Omega, \\ -\Delta_p v(z) - \Delta v(z) + \beta v(z)^{p-1} &= \hat{\eta}(z) \quad \text{in } \Omega. \end{aligned}$$

Then $v - u \in \text{int } C_+$.

As we have already mentioned in Introduction, our approach will use also the Morse theory (critical groups). So, next we recall some basic definitions and facts from Morse theory. Given a topological pair (Y_1, Y_2) with $Y_2 \subseteq Y_1 \subseteq X$, for every integer $k \geq 0$ by $H_k(Y_1, Y_2)$ we denote the k^{th} -relative singular homology group with integer coefficients for the pair (Y_1, Y_2) . Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leq c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

If $u \in K_\varphi^c$ is isolated, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \geq 0,$$

where U is a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighbourhood U .

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [15, p.628]) implies that the above definition of critical groups at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We introduce

$$\begin{aligned} M(t, u) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi, \\ P(t, \infty) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

The Morse relation says that

$$(2.1) \quad \sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t)Q(t),$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Next, let us recall some basic facts concerning the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we consider the following nonlinear eigenvalue problem:

$$(2.2) \quad -\Delta_p u(z) = \widehat{\lambda} |u(z)|^{p-2} u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

A number $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2.2) admits a nontrivial solution \widehat{u} . The nontrivial solution \widehat{u} is an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. The smallest eigenvalue is denoted by $\widehat{\lambda}_1(p)$ and has the following properties:

- $\widehat{\lambda}_1(p) > 0$ and it is isolated;
- $\widehat{\lambda}_1(p)$ is simple (that is, if \widehat{u}, \widehat{v} are eigenfunctions corresponding to the eigenvalue $\widehat{\lambda}_1(p)$, then $\widehat{u} = \xi \widehat{v}$ for some $\xi \neq 0$);
- we have

$$(2.3) \quad \widehat{\lambda}_1(p) = \inf \left[\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right].$$

In (2.3) the infimum is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(p) > 0$. In what follows, by $\widehat{u}_1(p)$ we denote the L^p -normalized (that is, $\|\widehat{u}_1(p)\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(p)$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [15, pp. 737–738]), imply that $\widehat{u}_1(p) \in \text{int } C_+$. Since the spectrum $\sigma(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ is closed and $\widehat{\lambda}_1(p) > 0$ is isolated, then the second eigenvalue is well-defined by

$$\widehat{\lambda}_2(p) = \inf \left[\widehat{\lambda} \in \sigma(p) : \widehat{\lambda} > \widehat{\lambda}_1(p) \right].$$

Let $\text{ind}(A)$ be the \mathbb{Z}_2 -cohomological index introduced by Fadell and Rabinowitz [13]. Employing the Lusternik–Schnirelmann minimax scheme, we can define a whole divergent sequence $\{\widehat{\lambda}_k(p)\}_{k \geq 1}$ of distinct eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$ by setting

$$\widehat{\lambda}_k(p) = \inf \left[\sup_{u \in A} \|Du\|_p^p : A \subseteq M \text{ symmetric, } \text{ind}(A) \geq k \right], \quad k \geq 1,$$

where $M = \{u \in W_0^{1,p}(\Omega) : \|u\|_p = 1\}$ is a C^1 -Banach manifold. For $k = 1, 2$, these eigenvalues are as above. We do not know if the sequence $\{\widehat{\lambda}_k(p)\}_{k \geq 1}$ exhausts $\sigma(p)$. This is the case if $N = 1$ (ordinary differential equation) or if

$p = 2$ and $N \geq 2$ (linear eigenvalue problem). In the latter case, we have

$$H_0^1(\Omega) = \overline{\bigoplus_{k \geq 1} E(\widehat{\lambda}_k(2))},$$

where $E(\widehat{\lambda}_k(2))$ denotes the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(2)$. We know that for every $k \geq 1$, $E(\widehat{\lambda}_k(2))$ is finite dimensional, $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ and it has the ‘‘Unique Continuation Property’’ (UCP for short), that is, if $u \in E(\widehat{\lambda}_k(2))$ and vanishes on a set of positive measure, then $u \equiv 0$.

We mention that if \widehat{u} is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda}_m(p)$ with $m \geq 2$, then $\widehat{u} \in C_0^1(\overline{\Omega})$ (nonlinear regularity theory) and \widehat{u} is nodal (that is, sign changing). In fact, if $p = 2$, $\{\widehat{u}_n\}_{n \geq 1} \subseteq H_0^1(\Omega)$ are the eigenfunctions corresponding to the eigenvalues $\{\widehat{\lambda}_k(2)\}_{k \geq 1}$ and $S_n = \{z \in \Omega : \widehat{u}_n(z) = 0\}$, then $\Omega \setminus S_n$ has at most n -components (the Courant nodal domain theorem, see, for example, Gasinski and Papageorgiou [15, p. 797]).

For $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on \mathbb{R}^N . Finally, if $h: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

the Nemytskiĭ (superposition) operator corresponding to the function h . Evidently, $z \mapsto N_h(u)(z) = h(z, u(z))$ is measurable.

3. Resonant problems

In this section we study problem (1.1) when the reaction $f(z, \cdot)$ is $(p - 1)$ -linear near $\pm\infty$ and resonant with respect to a nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we impose the following conditions on the function f :

(H₁) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)(1 + |x|^{p-1})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) $\lim_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = \widehat{\lambda}_m(p)$ uniformly for almost all $z \in \Omega$ and with $m \geq 2$;

(iii) if $F(z, x) = \int_0^x f(z, s)ds$, then there exists $\tau \in (2, p)$ such that

$$0 < \eta_0 \leq \liminf_{x \rightarrow \pm\infty} \frac{pF(z, x) - f(z, x)x}{|x|^\tau} \quad \text{uniformly for almost all } z \in \Omega;$$

(iv) there exists $\vartheta \in L^\infty(\Omega)_+$ such that

$$\vartheta(z) \leq \widehat{\lambda}_1(2) \quad \text{a.e. in } \Omega, \quad \vartheta \not\equiv \widehat{\lambda}_1(2),$$

$$\limsup_{x \rightarrow 0} \frac{2F(z, x)}{x^2} \leq \vartheta(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that $f(z, x)x + \xi_\rho|x|^p \geq 0$ for almost all $z \in \Omega$, all $|x| \leq \rho$.

REMARK 3.1. If there exists $\delta > 0$ such that $f(z, x)x \geq 0$ for almost all $z \in \Omega$, all $|x| \leq \delta$, then hypothesis (H₁) (v) is automatically satisfied.

EXAMPLE 3.2. The following function satisfies hypotheses (H₁). For the sake of simplicity we drop the z -dependence:

$$f(x) = \begin{cases} \vartheta x + \widehat{\lambda}_m(p)|x|^{q-2}x & \text{if } |x| \leq 1, \\ \widehat{\lambda}_m(p)|x|^{p-2}x + \vartheta|x|^{\tau-2}x & \text{if } |x| > 1, \end{cases}$$

with $\tau, q > 2$, $\tau < p$ and $\vartheta \in (0, \widehat{\lambda}_1(2))$.

We set $f_\pm(z, x) = f(z, \pm x^\pm)$. These are Carathéodory functions. Let

$$F_\pm(z, x) = \int_0^x f_\pm(z, s) ds$$

and let $\varphi_\pm: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functionals defined by

$$\varphi_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_\pm(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, let $\varphi: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi \in C^1(W_0^{1,p}(\Omega))$.

PROPOSITION 3.3. *Assume that hypotheses (H₁) (i), (ii) hold. Then the functionals φ_\pm satisfy the C-condition.*

PROOF. We do the proof for φ_+ , the proof for φ_- being similar. So, let $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ be such that

$$(3.1) \quad |\varphi_+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \geq 1,$$

$$(3.2) \quad (1 + \|u_n\|)\varphi'(u_n) \rightarrow 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \rightarrow \infty.$$

From (3.2) we have

$$(3.3) \quad \left| \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle - \int_\Omega f_+(z, u_n)h dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \rightarrow 0^+$.

In (3.3) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$(3.4) \quad \|Du_n^-\|_p^p + \|Du_n^-\|_2^2 \leq \varepsilon_n \quad \text{for all } n \geq 1$$

$$\Rightarrow u_n^- \rightarrow 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty.$$

CLAIM. $\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

We argue indirectly. So, suppose that $\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. We may assume that $\|u_n^+\| \rightarrow \infty$. Let $y_n = u_n^+/\|u_n^+\|$, $n \geq 1$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \geq 1$. So, by passing to a subsequence if necessary, we may assume that

$$(3.5) \quad y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \rightarrow y \quad \text{in } L^p(\Omega), \text{ with } y \geq 0.$$

From (3.3) and (3.4), we have

$$(3.6) \quad \left| \langle A_p(y_n), h \rangle + \frac{1}{\|u_n^+\|^{p-2}} \langle A(y_n), h \rangle - \int_{\Omega} \frac{f_+(z, u_n)}{\|u_n^+\|^{p-1}} h \, dz \right| \leq \varepsilon'_n \|h\|$$

with $\varepsilon'_n \rightarrow 0^+$ as $n \rightarrow \infty$.

In (3.6) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.5). Then

$$(3.7) \quad \lim_{n \rightarrow \infty} \langle A_p(y_n), y_n - y \rangle = 0 \quad (\text{recall } p > 2),$$

$$\Rightarrow y_n \rightarrow y \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{see Proposition 2.2), hence } \|y\| = 1, y \geq 0.$$

Hypothesis (H₁) (i) implies that

$$\left\{ \frac{N_{f_+}(u_n)}{\|u_n^+\|^{p-1}} = \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n \geq 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$(3.8) \quad \frac{N_{f_+}(u_n)}{\|u_n^+\|^{p-1}} \xrightarrow{w} g \quad \text{in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty.$$

By virtue of hypothesis (H₁) (ii), we have

$$(3.9) \quad g = \widehat{\lambda}_m(p) y^{p-1}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 30). Then, if in (3.6) we pass to the limit as $n \rightarrow \infty$ and use (3.7)–(3.9) and the fact that $p > 2$, we obtain

$$(3.10) \quad \langle A_p(y), h \rangle = \widehat{\lambda}_m(p) \int_{\Omega} y^{p-1} h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow A_p(y) = \widehat{\lambda}_m(p) y^{p-1}.$$

Since $m \geq 2$, from (3.10) we infer that $y \equiv 0$ or y is nodal, both cases contradict (3.7). This proves Claim.

From (3.4) and Claim, it follows that $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so we may assume that

$$(3.11) \quad u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow u \text{ in } L^p(\Omega).$$

In (3.3) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.11). Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u_n), u_n - u \rangle] = 0, \\ & \Rightarrow \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u), u_n - u \rangle] \leq 0 \quad (\text{since } A \text{ is monotone}), \\ & \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle \leq 0, \\ & \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) \quad (\text{see Proposition 2.2}). \end{aligned}$$

This proves that the functional φ_+ satisfies the C-condition. Similarly for the functional φ_- . \square

PROPOSITION 3.4. *If hypotheses (H₁) (i)–(iii) hold, then the functional φ satisfies the C-condition.*

PROOF. Let $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$(3.12) \quad |\varphi(u_n)| \leq M_2 \quad \text{for some } M_2 > 0, \text{ all } n \geq 1,$$

$$(3.13) \quad (1 + \|u_n\|)\varphi'(u_n) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \rightarrow \infty.$$

From (3.13) we have

$$(3.14) \quad \left| \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle - \int_{\Omega} f(z, u_n) h \, dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \rightarrow 0^+$. In (3.14) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and obtain

$$(3.15) \quad \|Du_n\|_p^p + \|Du\|_2^2 - \int_{\Omega} f(z, u_n) u_n \, dz \leq \varepsilon_n \quad \text{for all } n \geq 1.$$

Also, from (3.12) we have

$$(3.16) \quad -\|Du_n\|_p^p - \frac{p}{2} \|Du_n\|_2^2 + \int_{\Omega} pF(z, u_n) \, dz \leq pM_2 \quad \text{for all } n \geq 1.$$

Adding (3.15) and (3.16), we obtain

$$(3.17) \quad \int_{\Omega} [pF(z, u_n) - f(z, u_n)u_n] \, dz \leq M_3 + \left(\frac{p}{2} - 1\right) \|Du_n\|_2^2$$

for some $M_3 > 0$, all $n \geq 1$.

By virtue of hypotheses (H₁) (i),(iii), we can find $\eta_1 \in (0, \eta_0)$ and $c_1 > 0$ such that

$$\begin{aligned} \eta_1 |x|^\tau - c_1 &\leq pF(z, x) = f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \\ (3.18) \quad \Rightarrow \eta_1 \|u_n\|_\tau^\tau &\leq M_4 + \left(\frac{p}{2} - 1\right) \|Du_n\|_2^2 \quad \text{for some } M_4 > 0, \text{ all } n \geq 1 \end{aligned}$$

(see (3.17)).

CLAIM $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

Arguing by contradiction, suppose that Claim is not true. Then by passing to a subsequence if necessary, we may assume that $\|u_n\| \rightarrow \infty$. Let $y_n = u_n/\|u_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(3.19) \quad y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \rightarrow y \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty.$$

From (3.14) we have

$$(3.20) \quad \left| \langle A_p(y_n), h \rangle + \frac{1}{\|u_n\|^{p-2}} \langle A(y_n), h \rangle - \int_\Omega \frac{f(z, u_n)}{\|u_n\|^{p-1}} h \, dz \right| \leq \frac{\varepsilon_n \|h\|}{\|u_n\|^{p-1} (1 + \|u_n\|)}$$

for all $h \in W_0^{1,p}(\Omega)$ and with $\varepsilon_n \rightarrow 0^+$. Hypothesis (H₁) (i) implies that $\{N_f(u_n)/\|u_n\|^{p-1}\}_{n \geq 1} \subseteq L^{p'}(\Omega)$ is bounded. So, if in (3.20) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.19), then

$$(3.21) \quad \lim_{n \rightarrow \infty} \langle A_p(y_n), y_n - y \rangle = 0 \text{ (recall } p > 2) \Rightarrow y_n \rightarrow y \text{ in } W_0^{1,p}(\Omega),$$

hence $\|y\| = 1$. From (3.18), we have

$$\begin{aligned} \eta_1 \|y_n\|_\tau^\tau &\leq \frac{M_4}{\|u_n\|^\tau} + \frac{c_2}{\|u_n\|^{\tau-2}} \quad \text{with } c_2 = \left(\frac{p}{2} - 1\right) > 0 \quad \text{(recall } p > 2) \\ &\Rightarrow \eta_1 \|y\|_\tau^\tau \leq 0, \end{aligned}$$

hence $y = 0$, contradicting (3.21). This proves Claim.

By virtue of Claim, we may assume that

$$(3.22) \quad u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow u \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty.$$

In (3.14) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.22). Then

$$\begin{aligned} \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u_n), u_n - u \rangle] &= 0, \\ \Rightarrow \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u), u_n - u \rangle] &\leq 0 \quad \text{(since } A \text{ is monotone)} \\ \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle &\leq 0 \\ \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) &\quad \text{(see Proposition 2.2).} \end{aligned}$$

Therefore φ satisfies the C-condition. □

PROPOSITION 3.5. *If hypotheses (H₁) (i)–(iv) hold, then $u = 0$ is a local minimizer for the functionals φ_{\pm} and φ .*

PROOF. We do the proof for the functional φ_+ , the proofs for φ_- and φ being similar. By virtue of hypotheses (H₁) (i),(iv) given $\varepsilon > 0$ we can find $c_3 = c_3(\varepsilon) > 0$ such that

$$(3.23) \quad F(z, x) \leq \frac{1}{2}(\vartheta(z) + \varepsilon)x^2 + c_3|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then, for all $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \varphi_+(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_{\Omega} F_+(z, u) \, dz \\ &\geq \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{1}{2} \int_{\Omega} (\vartheta(z) + \varepsilon)(u^+)^2 \, dz - c_3\|u^+\|_p^p \\ &\hspace{15em} \text{(see (3.23))} \\ &\geq \frac{1}{2} \left[\left(\|Du\|_2^2 - \int_{\Omega} \vartheta(z)u^2 \, dz \right) - \frac{\varepsilon}{\widehat{\lambda}_1(2)} \|Du\|_2^2 \right] - c_4\|u\|^p \\ &\hspace{15em} \text{for some } c_4 > 0 \text{ (see (2.3))} \\ &\geq \frac{1}{2} \left[c_5 - \frac{\varepsilon}{\widehat{\lambda}_1(2)} \right] \|u\|^2 - c_4\|u\|^p \hspace{5em} \text{for some } c_5 > 0 \end{aligned}$$

(see Papageorgiou and Kyritsi [20, p. 356]). Choosing $\varepsilon \in (0, \widehat{\lambda}_1(2)c_5)$, we infer that

$$(3.24) \quad \varphi_+(u) \geq c_6\|u\|^2 - c_4\|u\|^p \quad \text{for some } c_6 > 0.$$

Since $p > 2$, from (3.24) we see that for $\rho \in (0, 1)$ small, we have

$$\begin{aligned} \varphi_+(u) \geq 0 = \varphi_+(0) \quad &\text{for all } u \in W_0^{1,p}(\Omega) \text{ with } \|u\| \leq \rho, \\ \Rightarrow u = 0 \quad &\text{is a local minimizer for the functional } \varphi_+. \end{aligned}$$

Similarly we show that $u = 0$ is also a local minimizer for the functionals φ_- and φ . □

Now we are ready to produce constant sign solutions.

THEOREM 3.6. *If hypotheses (H₁) hold, then problem (1.1) admits at least two nontrivial constant sign solutions*

$$u_0 \in \text{int } C_+ \quad \text{and} \quad v_0 \in -\text{int } C_+.$$

PROOF. First we produce a positive solution. From Proposition 3.5 we see that we can find $\rho \in (0, 1)$ small such that

$$(3.25) \quad \varphi_+(0) = 0 < \inf[\varphi_+(u) : \|u\| = \rho] = m_{\rho}^+$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29). By virtue of hypotheses (H₁) (i),(ii), given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$(3.26) \quad F(z, x) \geq \frac{1}{p}(\widehat{\lambda}_m(p) - \varepsilon)|x|^p - c_7 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then, for $t > 0$, we have

$$\begin{aligned} \varphi_+(t\widehat{u}_1(p)) &= \frac{t^p}{p}\widehat{\lambda}_1(p) + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 = \int_{\Omega} F(z, t\widehat{u}_1) dz \\ &\leq \frac{t^p}{p}[\widehat{\lambda}_1(p) - \widehat{\lambda}_m(p) + \varepsilon] + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 + c_7|\Omega|_N \end{aligned}$$

(see (3.26)). Choose $\varepsilon \in (0, \widehat{\lambda}_m(p) - \widehat{\lambda}_1(p))$ (recall that $m \geq 2$). Then

$$\begin{aligned} (3.27) \quad \varphi_+(t\widehat{u}_1(p)) &\leq -\frac{c_8 t^p}{p} + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 + c_7|\Omega|_N \quad (\text{for some } c_8 > 0) \\ &\Rightarrow \frac{\varphi_+(t\widehat{u}_1(p))}{t^p} \leq -\frac{c_8}{p} + \frac{1}{2t^{p-2}}\|D\widehat{u}_1(p)\|_2^2 + \frac{c_7}{t^p}|\Omega|_N \\ &\Rightarrow \limsup_{t \rightarrow +\infty} \frac{\varphi_+(t\widehat{u}_1(p))}{t^p} \leq -\frac{c_8}{p} < 0 \quad (\text{recall } p > 2) \\ &\Rightarrow \lim_{t \rightarrow +\infty} \varphi_+(t\widehat{u}_1(p)) = -\infty. \end{aligned}$$

From (3.25), (3.27) and Proposition 3.3, we see that we can apply Theorem 2.1 (the mountain pass theorem) and find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$(3.28) \quad u_0 \in K_{\varphi_+} \quad \text{and} \quad m_{\rho}^+ \leq \varphi_+(u_0).$$

From (3.25) and (3.28) it follows that $u_0 \neq 0$. Also

$$(3.29) \quad \varphi'_+(u_0) = 0 \quad (\text{see (3.28)}) \quad \Rightarrow \quad A_p(u_0) + A(u_0) = N_{f_+}(u_0).$$

On (3.29) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\|Du_0^-\|_p^p + \|Du_0^-\|_2^2 = 0 \quad \Rightarrow \quad u_0 \geq 0, \quad u_0 \neq 0.$$

Then from (3.29) we see that u_0 is a nontrivial solution of (1.1). From Ladyzhenskaya and Uraltseva [18, p.286], we have $u_0 \in L^\infty(\Omega)$. Then we can apply Theorem 1 of Lieberman [19] and conclude that $u_0 \in C_+ \setminus \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis (H₁) (v). Then

$$\begin{aligned} (3.30) \quad -\Delta_p u_0(z) - \Delta u_0(z) + \xi_\rho u_0(z)^{p-1} &= f(z, u_0(z)) + \xi_\rho u_0(z)^{p-1} \geq 0 \\ &\quad \text{for a.a. } z \in \Omega \\ &\Rightarrow \Delta_p u_0(z) + \Delta u_0(z) \leq \xi_\rho u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned}$$

Then from (3.30) and Pucci and Serrin [23, pp.111,120], we infer that $u_0 \in \text{int } C_+$.

Similarly, working with the functional φ_- , via the mountain pass theorem, we produce $v_0 \in -\text{int } C_+$ a negative solution of problem (1.1). □

To produce a third nontrivial solution, we need to strengthen our hypotheses on the reaction $f(z, \cdot)$ and employ also tools from the Morse theory. The new hypotheses on f , are the following:

(H₂) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that, for almost all $z \in \Omega$ $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq a(z)(1 + |x|^{p-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$ and with $a \in L^\infty(\Omega)_+$;

(ii) $\lim_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = \widehat{\lambda}_m(p)$ uniformly for almost all $z \in \Omega$ and with $m \geq 2$;

(iii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists $\tau \in (2, p)$ such that

$$\lim_{x \rightarrow \pm\infty} \frac{pF(z, x) - f(z, x)x}{|x|^\tau} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) $f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x}$ uniformly for almost all $z \in \Omega$ and

$$f'_x(z, 0) \leq \widehat{\lambda}_1(2) \quad \text{for a.a. } z \in \Omega, \quad f'_x(\cdot, 0) \not\equiv \widehat{\lambda}_1(2);$$

(v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that

$$f(z, x)x + \xi_\rho|x|^p \geq 0 \quad \text{for a.a. } z \in \Omega, \quad \text{all } |x| \leq \rho.$$

Under these stronger conditions on the reaction, we can produce a third nontrivial solution and have the following multiplicity theorem.

THEOREM 3.7. *If hypotheses (H₂) hold, then problem (1.1) has at least three nontrivial solutions*

$$u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+ \quad \text{and} \quad y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}.$$

PROOF. The two nontrivial constant sign solutions $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$ are guaranteed by Theorem 3.6. From the proof of Theorem 3.6, we know that $u_0 \in \text{int } C_+$ is a critical point of mountain pass type for the functional φ_+ , while $v_0 \in -\text{int } C_+$ is a critical point of mountain pass type for the functional φ_- . Therefore we have

$$(3.31) \quad C_1(\varphi_+, u_0) \neq 0 \quad \text{and} \quad C_1(\varphi_-, v_0) \neq 0.$$

We consider the homotopy

$$h(t, u) = (1 - t)\varphi_+(u) + t\varphi(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

We assume that K_φ is finite (otherwise we already have infinitely many distinct solutions for problem (1.1)). Suppose that we can find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$(3.32) \quad t_n \rightarrow t, \quad u_n \rightarrow u_0 \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad h'_u(t_n, u_n) = 0 \quad \text{for all } n \geq 1.$$

From (3.32), we have

$$\begin{aligned} A_p(u_n) + A(u_n) &= (1 - t_n)N_{f_+}(u_n) + t_nN_f(u_n) \quad \text{for all } n \geq 1 \\ \Rightarrow -\Delta_p u_n(z) - \Delta u_n(z) &= (1 - t_n)f_+(z, u_n(z)) + t_nf(z, u_n(z)) \\ &\quad \text{for a.a. } z \in \Omega, \text{ all } n \geq 1. \end{aligned}$$

From Ladyzhenskaya and Uraltseva [18, p. 286], we know that we can find $M_5 > 0$ such that $\|u_n\|_\infty \leq M_5$ for all $n \geq 1$ (see also (3.32)). Then Theorem 1 of Lieberman [19] implies that there exist $\alpha \in (0, 1)$ and $M_6 > 0$ such that

$$(3.33) \quad u_n \in C_0^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq M_6 \quad \text{for all } n \geq 1.$$

From (3.33) and the compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, we have $u_n \rightarrow u_0$ in $C_0^1(\bar{\Omega})$ (see (3.32)). Since $u_0 \in \text{int } C_+$ (see Theorem 3.6), we infer that $u_n \in \text{int } C_+$ for all $n \geq 1$ and so $\{u_n\}_{n \geq n_0}$ is a sequence of distinct positive solutions of problem (1.1), a contradiction to our hypothesis that K_φ is finite. So, (3.32) cannot occur and then the homotopy invariance of critical groups, implies that

$$(3.34) \quad C_k(\varphi_+, u_0) = C_k(\varphi, u_0) \quad \text{for all } k \geq 0.$$

Similarly we show that

$$(3.35) \quad C_k(\varphi_-, v_0) = C_k(\varphi, v_0) \quad \text{for all } k \geq 0 \quad (\text{recall } v_0 \in -\text{int } C_+).$$

From (3.31), (3.34) and (3.35), we have

$$(3.36) \quad C_1(\varphi, u_0) \neq 0, \quad C_1(\varphi, v_0) \neq 0.$$

Since $\varphi \in C^2(W_0^{1,p}(\Omega))$ (see hypotheses (H₂)), from (3.36) and Papageorgiou and Smyrlis [22] (see also Papageorgiou and Rădulescu [21]), it follows that

$$(3.37) \quad C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Also, Proposition 3.5 implies that

$$(3.38) \quad C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Hypothesis (H₂) (iii) implies that given $\xi > 0$, we can find $M_7 = M_7(\xi) > 0$ such that

$$(3.39) \quad pF(z, x) - f(z, x)x \geq \xi|x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_7.$$

For almost all $z \in \Omega$ and for $s \neq 0$, we have

$$\begin{aligned} \frac{d}{ds} \frac{F(z, s)}{|s|^p} &= \frac{f(z, s)|s|^p - p|s|^{p-2}sF(z, s)}{|s|^{2p}} \\ &\leq \frac{|s|^{p-2}s[f(z, s)s - pF(z, s)]}{|s|^{2p}} = \frac{f(z, s)s - pF(z, s)}{|s|^p s}. \end{aligned}$$

Then for $s > 0$, we have

$$(3.40) \quad \frac{d}{ds} \frac{F(z, s)}{s^p} = \frac{f(z, s)s - pF(z, s)}{s^{p+1}} \leq -\xi s^{\tau-p-1}$$

for a.a. $z \in \Omega$, all $s \geq M_7$ (see (3.39))

$$\Rightarrow \frac{F(z, y)}{y^p} - \frac{F(z, x)}{x^p} \leq \frac{\xi}{p-\tau} \left[\frac{1}{y^{p-\tau}} - \frac{1}{x^{p-\tau}} \right] \quad \text{for a.a. } z \in \Omega, \text{ all } y \geq x \geq M_7.$$

Note that hypothesis (H₂) (ii) implies that

$$(3.41) \quad \lim_{s \rightarrow \pm\infty} \frac{pF(z, s)}{|s|^p} = \widehat{\lambda}_m(p) \quad \text{uniformly for a.a. } z \in \Omega.$$

Recall that $\tau < p$ (see hypothesis (H₂) (iii)). So, if in (3.40) we let $y \rightarrow +\infty$ and use (3.41), then

$$\begin{aligned} \frac{\widehat{\lambda}_m(p)}{p} - \frac{F(z, x)}{x^p} &\leq -\frac{\xi}{p-\tau} \frac{1}{x^{p-\tau}} && \text{for a.a. } z \in \Omega, \text{ all } x \geq M_7 \\ \Rightarrow \frac{\widehat{\lambda}_m(p)}{p} x^p - F(z, x) &\leq -\frac{\xi}{p-\tau} x^\tau && \text{for a.a. } z \in \Omega, \text{ all } x \geq M_7 \\ \Rightarrow \frac{\widehat{\lambda}_m(p) x^p - pF(z, x)}{x^\tau} &\leq -\xi \frac{p}{p-\tau} && \text{for a.a. } z \in \Omega, \text{ all } x \geq M_7. \end{aligned}$$

Since $\xi > 0$ is arbitrary, it follows that

$$\lim_{x \rightarrow +\infty} \frac{\widehat{\lambda}_m(p) x^p - pF(z, x)}{x^\tau} = -\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

For $s < 0$, we have

$$\begin{aligned} \frac{d}{ds} \frac{F(z, s)}{|s|^p} &= \frac{f(z, s)s - pF(z, s)}{|s|^p s} \geq -\frac{\xi}{|s|^{p-\tau} s} \\ &\text{for a.a. } z \in \Omega, \text{ all } s \leq -M_7 \text{ (see (3.39) and recall that } s < 0) \\ \Rightarrow \frac{F(z, x)}{|x|^p} - \frac{F(z, y)}{|y|^p} &\geq -\frac{\xi}{p-\tau} \left[\frac{1}{|y|^{p-\tau}} - \frac{1}{|x|^{p-\tau}} \right] \\ &\text{for a.a. } z \in \Omega, \text{ all } y \leq x \leq -M_7. \end{aligned}$$

Letting $y \rightarrow -\infty$ and using (3.41), we obtain

$$\begin{aligned} \frac{F(z, x)}{|x|^p} - \frac{\widehat{\lambda}_m(p)}{p} &\geq \frac{\xi}{p-\tau} \frac{1}{|x|^{p-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq -M_7 \text{ (see (3.41))} \\ \Rightarrow \frac{\widehat{\lambda}_m(p) |x|^p - pF(z, x)}{|x|^\tau} &\leq -\xi \frac{p}{p-\tau} \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq -M_7. \end{aligned}$$

Since $\xi > 0$ is arbitrary, we infer that

$$\lim_{x \rightarrow -\infty} \frac{\widehat{\lambda}_m(p) |x|^p - pF(z, x)}{|x|^\tau} = -\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Therefore we have proved that

$$(3.42) \quad \lim_{x \rightarrow \pm\infty} \frac{\widehat{\lambda}_m(p)|x|^p - pF(z, x)}{|x|^\tau} = -\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

We introduce the following two sets:

$$S_r = \{u \in W_0^{1,p}(\Omega) : \|u\|^p = r^p, \|Du\|_p^p \leq \widehat{\lambda}_m(p)\|u\|_p^p\},$$

$$A = \{u \in W_0^{1,p}(\Omega) : \|Du\|_p^p \geq \widehat{\lambda}_{m+1}(p)\|u\|_p^p\}.$$

Note that S_r is a C^1 -Banach manifold with boundary.

From Degiovanni and Lancelotti [9, Theorem 3.2], we have

$$\text{ind}(S_r) = \text{ind}(W_0^{1,p}(\Omega) \setminus A) = m.$$

Also, from Cingolani and Degiovanni [7, Theorem 3.6] we know that the sets S and A homologically link in dimension m .

Hypotheses (H_2) (i),(ii) imply that given $\varepsilon > 0$, we can find $c_9 = c_9(\varepsilon) > 0$ such that

$$F(z, x) \leq \frac{\widehat{\lambda}_m(p) + \varepsilon}{p}|x|^p + c_9 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Recall that $\widehat{\lambda}_m(p) < \widehat{\lambda}_{m+1}(p)$. Then, for all $u \in A$, we have

$$\varphi(u) \geq \frac{\widehat{\lambda}_{m+1}(p) - \widehat{\lambda}_m(p) - \varepsilon}{p}\|u\|_p^p - c_9|\Omega|_N.$$

Choosing $\varepsilon \in (0, \widehat{\lambda}_{m+1}(p) - \widehat{\lambda}_m(p))$, we obtain $\inf_A \varphi > -\infty$. Also, we claim that $\sup_{S_r} \varphi < \inf_A \varphi$ for $r > 0$ big enough. Arguing by contradiction, suppose that we can find $\{u_n\}_{n \geq 1} \subseteq S$ and $\xi > 0$ such that

$$(3.43) \quad -\xi \leq \varphi(u_n) \quad \text{for all } n \geq 1 \text{ and } \|u_n\| \rightarrow +\infty.$$

Let $y_n = u_n/\|u_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$. So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \rightarrow y \quad \text{in } L^p(\Omega) \text{ as } n \rightarrow \infty.$$

We have

$$1 = \|Dy_n\|_p^p \leq \widehat{\lambda}_m(p)\|y_n\|_p^p \quad (\text{recall } u_n \in S \text{ for all } n \geq 1)$$

$$\Rightarrow 1 \leq \widehat{\lambda}_m(p)\|y\|_p^p \quad (\text{see (3.42)})$$

$$\Rightarrow y \neq 0.$$

Therefore $|u_n(z)| \rightarrow +\infty$ for almost all $z \in \Omega_0 = \{z \in \Omega : y(z) \neq 0\}$ and $|\Omega_0|_N > 0$. We have (see (3.43))

$$-\frac{p\xi}{\|u_n\|^\tau} \leq \int_\Omega \frac{\widehat{\lambda}_m(p)|u_n|^p - pF(z, u_n)}{|u_n|^\tau} |y_n|^\tau dz + \frac{p}{2} \frac{1}{\|u_n\|^{\tau-2}} \|Dy_n\|_2^2.$$

Since $\tau > 2$, passing to the limit as $n \rightarrow \infty$ and since $\|u_n\| \rightarrow \infty$, we reach a contradiction to (3.42). This proves the claim, namely that $\varphi(u) \rightarrow -\infty$ as

$\|u\| \rightarrow +\infty$ with $u \in S$. Therefore choosing $r > 0$ big enough, we can have $\sup_S \varphi < \inf_A \varphi$. So, we can apply Theorem 3.2 of Cingolani and Degiovanni [7] and infer that there exists $y_0 \in W_0^{1,p}(\Omega)$ such that $y_0 \in K_\varphi$ and $C_m(\varphi, y_0) \neq 0$. Comparing with (3.37) and (3.38), we obtain $y_0 \notin \{0, u_0, v_0\}$. Also, since $y_0 \in K_\varphi$, y_0 is a nontrivial solution of problem (1.1) and as before the nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{\Omega})$. \square

REMARK 3.8. Is it possible to show that y_0 is nodal?

4. Superlinear problems

In this section, we assume that the reaction $f(z, \cdot)$ is $(p-1)$ -superlinear near $\pm\infty$ without satisfying the AR-condition and has z -dependent zeros of constant sign. For such problems, we prove a multiplicity theorem producing six nontrivial solutions all with sign information and a second multiplicity theorem producing five solutions.

The hypotheses on the reaction f are the following:

(H₃) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$,

$f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq a(z)(1 + |x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$ and $r \in (p, p^*)$;

(ii) $\lim_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) there exist $\tau \in (\max\{1, (r-p)N/p\}, r)$ and $\eta_0 > 0$ such that

$$\eta_0 \leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\tau} \quad \text{uniformly for almost all } z \in \Omega;$$

(iv) $f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x}$ uniformly for almost all $z \in \Omega$ and for some $m \geq 2$, we have

$$f'_x(z, 0) \in [\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)] \quad \text{for almost all } z \in \Omega,$$

$$f'_x(\cdot, 0) \neq \widehat{\lambda}_m(2), \quad f'_x(\cdot, 0) \neq \widehat{\lambda}_{m+1}(2);$$

(v) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$w_-(z) \leq c_- < 0 < c_+ \leq w_+(z) \quad \text{for all } z \in \overline{\Omega},$$

$$f(z, w_+(z)) \leq 0 \leq f(z, w_-(z)) \quad \text{for a.a. } z \in \Omega,$$

and

$$\langle A_p(w_-), \vartheta \rangle + \langle A(w_-), \vartheta \rangle \leq 0 \leq \langle A_p(w_+), \vartheta \rangle + \langle A(w_+), \vartheta \rangle$$

for all $\vartheta \in W^{1,p}(\Omega)$ with $\vartheta \geq 0$;

(vi) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for almost all $z \in \Omega$ the map $x \rightarrow f(z, x) + \xi_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

REMARK 4.1. We stress that now the regularity condition on $f(z, \cdot)$ is stronger, namely $f(z, \cdot) \in C^1(\mathbb{R})$. Hypothesis (H₃) (ii) implies that the potential $F(z, \cdot)$ is p -superlinear near $\pm\infty$. This hypothesis together with (H₃) (iii) imply that the reaction $f(z, \cdot)$ is $(p - 1)$ -superlinear near $\pm\infty$. Note that, unlike most works on superlinear problems, we do not employ the AR-condition. We recall that this condition says that there exist $q > p$ and $M > 0$ such that

$$\begin{aligned} 0 < qF(z, x) &\leq f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \\ 0 < \operatorname{ess\,inf}_\Omega F(\cdot, \pm M). \end{aligned}$$

The AR-condition implies the following weaker unilateral growth condition:

$$c_{10}|x|^q \leq F(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M \text{ and some } c_{10} > 0.$$

From this, we have the much weaker condition

$$\lim_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Here we employ the above asymptotic condition and hypothesis (H₃) (iii), which together are weaker than the AR-condition and incorporate in our framework superlinear reactions with “slower” growth near $\pm\infty$ which fail to satisfy the AR-condition (see the example below). Hypothesis (H₃) (iv) implies that asymptotically at 0 we have nonuniform nonresonance with respect to the spectral interval $[\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)]$. Finally hypothesis (H₃) (v) implies the existence of z -dependent zeros of constant sign. Evidently this hypothesis is satisfied if there exist $c_- < 0 < c_+$ such that

$$f(z, c_+) \leq 0 \leq f(z, c_-) \quad \text{for a.a. } z \in \Omega.$$

As a final remark concerning hypotheses (H₃), we mention that by virtue of hypotheses (H₃) (i),(iv), we can find $\xi_0 > \widehat{\lambda}_1(2)$ and $\xi_1 > 0$ such that

$$(4.1) \quad f(z, x)x \geq \xi_0 x^2 - \xi_1 |x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

EXAMPLE 4.2. The following function satisfies hypotheses (H₃). For the sake of simplicity we drop the z -dependence

$$f(x) = |x|^{p-2}x \left(\ln |x| + \frac{1}{p} \right) + \vartheta x - c|x|^{\tau-2}x$$

with $\tau \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ ($m \geq 2$), $c \geq 1/p + \vartheta$, $2 < \tau < p$. Note that this function does not satisfy the AR-condition.

First we will produce constant sign solutions. As before $\varphi: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ is the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} F(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

To produce the first two constant sign solutions, the asymptotic condition on $f(z, \cdot)$ at $\pm\infty$ is irrelevant (that is, we do not need hypotheses (H₃) (ii),(iii)).

PROPOSITION 4.3. *If hypotheses (H₃) (i), (iv)–(vi) hold, then problem (1.1) has two nontrivial constant sign solutions*

$$u_0 \in \text{int } C_+ \quad \text{and} \quad v_0 \in -\text{int } C_+$$

such that $-c_- \leq w_-(z) \leq v_0(z) \leq 0 \leq u_0(z) \leq w_+(z) \leq c_+$ for all $z \in \bar{\Omega}$.

PROOF. First we produce the positive solution. To this end, we introduce the following truncation of $f(z, \cdot)$:

$$(4.2) \quad \hat{f}_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z, x) & \text{if } 0 \leq x \leq w_+(z), \\ f(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases}$$

This is a Carathéodory function. Let

$$\hat{F}_+(z, x) = \int_0^x \hat{f}_+(z, s) ds$$

and consider the C^1 -functional $\hat{\varphi}_+: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} \hat{F}_+(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (4.2) that $\hat{\varphi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$(4.3) \quad \hat{\varphi}_+(u_0) = \inf[\hat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)] = \hat{\eta}_+.$$

Hypothesis (H₃) (iv) implies that, given $\varepsilon > 0$, we can find $\delta \in (0, c_+)$ such that

$$(4.4) \quad \begin{aligned} f(z, x) &\geq (f'_x(z, 0) - \varepsilon)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta] \\ \Rightarrow F(z, x) &\geq \frac{1}{2}(f'_x(z, 0) - \varepsilon)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta]. \end{aligned}$$

Let $\hat{u}_1(2) \in \text{int } C_+$ be the L^2 -normalized principal eigenfunction of $(-\Delta, H_0^1(\Omega))$ and let $t \in (0, 1)$ be small such that $t\hat{u}_1(2)(z) \in [0, \delta]$ for all $z \in \bar{\Omega}$. Then

$$\hat{\varphi}_+(t\hat{u}_1(2)) \leq \frac{t^p}{p} \|D\hat{u}_1(2)\|_p^p + \frac{t^2}{2} \left[\int_{\Omega} (\hat{\lambda}_1(2) - f'_x(z, 0)) \hat{u}_1(2)^2 dz + \varepsilon \right]$$

(see (4.4) and recall that $\|\widehat{u}_1(2)\|_2 = 1$). Note that

$$\xi_* = \int_{\Omega} [f'_x(z, 0) - \widehat{\lambda}_1(2)] \widehat{u}_1(2)^2 dz > 0$$

(see hypothesis (H₃) (iv)). We have

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) \leq \frac{t^p}{p} \|D\widehat{u}_1(2)\|_p^p - \frac{t^2}{2} [\xi_* - \varepsilon].$$

Choosing $\varepsilon \in (0, \xi_*)$ and since $p > 2$, for $t \in (0, 1)$ even smaller if necessary, we have

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) < 0 \quad \Rightarrow \quad \widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+(0)$$

(see (4.2)), hence $u_0 \neq 0$. From (4.2), we have

$$(4.5) \quad \widehat{\varphi}'_+(u_0) = 0 \quad \Rightarrow \quad A_p(u_0) + A(u_0) = N_{\widehat{f}_+}(u_0).$$

On (4.5) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain $\|Du_0^-\|_p^p + \|Du_0^-\|_2^2 = 0$ (see (4.2)), hence $u_0 \geq 0$, $u_0 \neq 0$. Also, on (4.4) we act with $(u_0 - w_+)^+$ in $W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \langle A_p(u_0), (u_0 - w_+)^+ \rangle + \langle A(u_0), (u_0 - w_+)^+ \rangle &= \int_{\Omega} \widehat{f}_+(z, u_0)(u_0 - w_+)^+ dz \\ &= \int_{\Omega} f(z, w_+)(u_0 - w_+)^+ dz && \text{(see (4.2))} \\ &\leq \langle A_p(w_+), (u_0 - w_+)^+ \rangle + \langle A(w_+), (u_0 - w_+)^+ \rangle && \text{(see hypothesis (H}_3\text{) (v))} \\ &\Rightarrow \langle A_p(u_0) - A_p(w_+), (u_0 - w_+)^+ \rangle + \langle A(u_0) - A(w_+), (u_0 - w_+)^+ \rangle \leq 0 \\ &\Rightarrow |\{u_0 > w_+\}|_N = 0, \end{aligned}$$

hence $u_0 \leq w_+$. So, we have proved that

$$u_0 \in [0, w_+] = \{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq w_+(z) \text{ a.e. in } \Omega\}.$$

Then from (4.2) and (4.5) it follows that u_0 is a positive solution of problem (1.1). As before the nonlinear regularity theory (see Lieberman [19]) and the nonlinear maximum principle (see Pucci and Serrin [23, pp. 111, 120]), imply $u_0 \in \text{int } C_+$.

Similarly, we consider the following truncation of $f(z, \cdot)$:

$$\widehat{f}_-(z, x) = \begin{cases} f(z, w_-(z)) & \text{if } x < w_-(z), \\ f(z, x) & \text{if } w_-(z) \leq x \leq 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

This is a Carathéodory function. We set

$$\widehat{F}_-(z, x) = \int_0^x \widehat{f}_-(z, s) ds$$

and consider the C^1 -functional $\widehat{\varphi}_- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} \widehat{F}_-(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Reasoning as above, via the direct method, we produce a negative solution $v_0 \in -\text{int } C_+$ with $c_- \leq w_-(z) \leq v_0(z) \leq 0$ for all $z \in \overline{\Omega}$. \square

Using these two constant sign solutions and the asymptotic conditions at $\pm\infty$ (see hypotheses (H_3) (ii),(iii)), we can generate two more nontrivial solutions of constant sign.

PROPOSITION 4.4. *If hypotheses (H_3) hold, then problem (1.1) has two additional constant sign solutions*

$$\widehat{u} \in \text{int } C_+ \quad \text{and} \quad \widehat{v} \in -\text{int } C_+$$

such that $\widehat{u} - u_0 \in \text{int } C_+$ and $v_0 - \widehat{v} \in \text{int } C_+$.

PROOF. Again, first we show the existence of the second positive solution. To this end, let $u_0 \in \text{int } C_+$ be the positive solution produced in Proposition 4.3. We introduce the following truncation of the reaction $f(z, \cdot)$:

$$(4.6) \quad g_+(z, x) = \begin{cases} f(z, u_0(z)) & \text{if } x \leq u_0(z), \\ f(z, x) & \text{if } u_0(z) < x. \end{cases}$$

This is a Carathéodory function. We set

$$G_+(z, x) = \int_0^x g_+(z, s) \, ds$$

and consider the C^1 -functional $\psi_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} G_+(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From Aizicovici, Papageorgiou and Staicu [2] (proof of Proposition 4), we know that

$$(4.7) \quad \psi_+ \text{ satisfies the C-condition.}$$

CLAIM. *We may assume that $u_0 \in \text{int } C_+$ is a local minimizer of ψ_+ .*

We consider the following truncation of $g_+(z, \cdot)$ (recall that $u_0 \leq w_+$, see Proposition 4.3):

$$(4.8) \quad \widehat{g}_+(z, x) = \begin{cases} g_+(z, x) & \text{if } x \leq w_+(z), \\ g_+(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases}$$

This is a Carathéodory function. We set

$$\widehat{G}_+(z, x) = \int_0^x \widehat{g}_+(z, s) \, ds$$

and consider the C^1 -functional $\widehat{\psi}: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_{\Omega} \widehat{G}_+(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (4.6) and (4.8) it is clear that $\widehat{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$(4.9) \quad \begin{aligned} \widehat{\psi}_+(\widehat{u}_0) &= \inf[\widehat{\psi}_+(u) : u \in W_0^{1,p}(\Omega)] \\ &\Rightarrow \widehat{\psi}'_+(\widehat{u}_0) = 0 \Rightarrow A_p(\widehat{u}_0) + A(\widehat{u}_0) = N_{\widehat{g}_+}(\widehat{u}_0). \end{aligned}$$

On (4.9) first we act with $(u_0 - \widehat{u}_0)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \langle A_p(\widehat{u}_0), (u_0 - \widehat{u}_0)^+ \rangle + \langle A(\widehat{u}_0), (u_0 - \widehat{u}_0)^+ \rangle &= \int_{\Omega} \widehat{g}_+(z, \widehat{u}_0)(u_0 - \widehat{u}_0)^+ dz \\ &= \int_{\Omega} f(z, u_0)(u_0 - \widehat{u}_0)^+ dz \quad (\text{see (4.8) and (4.6)}) \\ &= \langle A_p(u_0), (u_0 - \widehat{u}_0)^+ \rangle + \langle A(u_0), (u_0 - \widehat{u}_0)^+ \rangle \quad (\text{see Proposition 4.3}) \\ &\Rightarrow \langle A_p(u_0) - A_p(\widehat{u}_0), (u_0 - \widehat{u}_0)^+ \rangle + \langle A(u_0) - A(\widehat{u}_0), (u_0 - \widehat{u}_0)^+ \rangle = 0 \\ &\Rightarrow |\{u_0 > \widehat{u}_0\}|_N = 0, \end{aligned}$$

hence $u_0 \leq \widehat{u}_0$. Also, on (4.9) we act with $(\widehat{u}_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \langle A_p(\widehat{u}_0), (\widehat{u}_0 - w_+)^+ \rangle + \langle A(\widehat{u}_0), (\widehat{u}_0 - w_+)^+ \rangle &= \int_{\Omega} \widehat{g}_+(z, \widehat{u}_0)(\widehat{u}_0 - w_+)^+ dz \\ &= \int_{\Omega} f(z, w_+)(\widehat{u}_0 - w_+)^+ dz \quad (\text{see (4.8) and (4.6)}) \\ &\leq \langle A_p(w_+), (\widehat{u}_0 - w_+)^+ \rangle + \langle A(w_+), (\widehat{u}_0 - w_+)^+ \rangle \quad (\text{see hypothesis (H}_3\text{) (v)}) \\ &\Rightarrow \langle A_p(\widehat{u}_0) - A_p(w_+), (\widehat{u}_0 - w_+)^+ \rangle + \langle A(\widehat{u}_0) - A(w_+), (\widehat{u}_0 - w_+)^+ \rangle \leq 0 \\ &\Rightarrow |\{\widehat{u}_0 > w_+\}|_N = 0, \end{aligned}$$

hence $\widehat{u}_0 \leq w_+$. So, we have proved that

$$\widehat{u}_0 \in [u_0, w_+] = \{u \in W_0^{1,p}(\Omega) : u_0(z) \leq u(z) \leq w_+(z) \text{ a.e. in } \Omega\}.$$

Then from (4.6), (4.8) and (4.9) it follows that \widehat{u}_0 is a positive solution of problem (1.1) and as before the nonlinear regularity theory and the nonlinear maximum principle imply $\widehat{u}_0 \in \text{int } C_+$.

If $\widehat{u}_0 \neq u_0$, then this is the desired second positive solution of (1.1).

If $\widehat{u}_0 = u_0 \in \text{int } C_+$, then for $\rho = \|w_+\|_{\infty}$, let $\xi_{\rho} > 0$ be as postulated by hypothesis (H₃) (vi). We have

$$\begin{aligned} -\Delta_p u_0(z) - \Delta u_0(z) + \xi_{\rho} u_0(z)^{p-1} &= f(z, u_0(z)) + \xi_{\rho} u_0(z)^{p-1} \\ &\leq f(z, w_+(z)) + \xi_{\rho} w_+(z)^{p-1} \end{aligned}$$

(recall that $u_0 \leq w_+$ and see hypothesis (H₃ (vi))). It follows that

$$(4.10) \quad \langle A_p(u_0), \vartheta \rangle + \langle A(u_0), \vartheta \rangle - \int_{\Omega} f(z, u_0) \vartheta \, dz \leq 0 \\ \leq \langle A_p(w_+), \vartheta \rangle + \langle A(w_+), \vartheta \rangle - \int_{\Omega} f(z, w_+) \vartheta \, dz$$

for all $\vartheta \in W^{1,p}(\Omega)$ with $\vartheta \geq 0$ (see hypothesis (H₃ (v))).

Let $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. Then $a \in C^1(\mathbb{R}^N)$ (recall $p > 2$) and

$$\nabla a(y) = |y|^{p-2} \left(I + (p-2) \frac{y \otimes y}{|y|^2} \right) + I \quad \text{for all } y \in \mathbb{R}^N \\ \Rightarrow (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq (1 + |y|^{p-2})|\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N.$$

So, by virtue of (4.10), we can apply Theorem 2.5.2 of Pucci and Serrin [23, p. 35] (the tangency principle) and infer that

$$(4.11) \quad u_0(z) < w_+(z) \text{ for all } z \in \bar{\Omega} \quad \Rightarrow \quad u_0 \in \text{int}_{C_0^1(\bar{\Omega})}[0, w_+].$$

But from (4.6) and (4.8) it is clear that $\widehat{\psi}_+|_{[0, w_+]} = \psi_+|_{[0, w_+]}$. So, (4.11) implies that $u_0 \in \text{int } C_+$ is a local $C_0^1(\bar{\Omega})$ -minimizer of ψ_+ , hence it is also a local $W_0^{1,p}(\Omega)$ -minimizer of ψ_+ (see Proposition 2.3). This proves Claim.

By virtue of Claim, we can find $\rho \in (0, 1)$ small such that

$$(4.12) \quad \psi_+(u_0) < \inf[\psi_+(u) : \|u - u_0\| = \rho] = m_+$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29).

Hypothesis (H₃) (ii) implies that for any $\tilde{u} \in \text{int } C_+$, we have

$$(4.13) \quad \psi_+(t\tilde{u}) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

Then (4.7), (4.12) and (4.13) permit the use of Theorem 2.1 (the mountain pass theorem) and obtain $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$(4.14) \quad \widehat{u} \in K_{\psi_+} \quad \text{and} \quad m_+ \leq \psi_+(\widehat{u}).$$

From (4.12) and (4.14), we see that $\widehat{u} \neq u_0$. Also, since $\widehat{u} \in K_{\psi_+}$ (see (4.14)), we have that

$$(4.15) \quad A_p(\widehat{u}) + A(\widehat{u}) = N_{g_+}(\widehat{u}).$$

Acting on (4.15) with $(u_0 - \widehat{u})^+ \in W_0^{1,p}(\Omega)$ and using (4.6), as before we obtain that $u_0 \leq \widehat{u}$ and so \widehat{u} is a positive solution of problem (1.1) (see (4.6) and (4.15)). As before the nonlinear regularity theory implies that $\widehat{u} \in \text{int } C_+$.

Let $\rho = \|\widehat{u}\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis (H₃) (vi). Then for $\widehat{\xi}_\rho > \xi_\rho$ we have

$$\begin{aligned}
 (4.16) \quad & -\Delta_p u_0(z) - \Delta u_0(z) + \widehat{\xi}_\rho u_0(z)^{p-1} = f(z, u_0(z)) + \widehat{\xi}_\rho u_0(z)^{p-1} \\
 & \leq f(z, \widehat{u}(z)) + \widehat{\xi}_\rho \widehat{u}(z)^{p-1} \quad (\text{since } u_0 \leq \widehat{u}, \text{ see hypothesis (H}_3\text{) (vi)}) \\
 & = -\Delta_p \widehat{u}(z) - \Delta \widehat{u}(z) + \widehat{\xi}_\rho \widehat{u}(z)^{p-1} \quad \text{a.e. in } \Omega.
 \end{aligned}$$

Once again the tangency principle of Pucci and Serrin [23, Theorem 2.5.2, p. 35] implies that $u_0(z) < \widehat{u}(z)$ for all $z \in \Omega$. Hence, from (4.16) and Proposition 2.4 (recall $\widehat{\xi}_\rho > \xi_\rho$) we infer that $\widehat{u} - u_0 \in \text{int } C_+$.

Similarly, for the second negative solution we introduce the truncation

$$g_-(z, x) = \begin{cases} f(z, x) & \text{if } x \leq v_0(z), \\ f(z, v_0(z)) & \text{if } v_0(z) < x. \end{cases}$$

This is a Carathéodory function. We set

$$G_-(z, x) = \int_0^x g_-(z, s) \, ds$$

and introduce the C^1 -functional $\psi_- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_-(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega G_-(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working as above, this time using the functional ψ_- , we produce a second negative solution $\widehat{v} \in -\text{int } C_+$ such that $v_0 - \widehat{v} \in \text{int } C_+$. □

Next we look for nodal (sign changing) solutions. We will employ tools from the Morse theory.

Let $d_m = \dim \bigoplus_{i=1}^m E(\widehat{\lambda}_i(2))$ (recall that $E(\widehat{\lambda}_i(2))$ denotes the eigenspace of $(-\Delta, H_0^1(\Omega))$ corresponding to the eigenvalue $\widehat{\lambda}_i(2)$; see Section 2).

PROPOSITION 4.5. *If hypotheses (H₃) hold, then $C_k(\varphi, 0) = \delta_{k,d_m} \mathbb{Z}$ for all $k \geq 0$.*

PROOF. Let $\tau : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^2 -functional defined by

$$\tau(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \frac{1}{2} \int_\Omega f'_x(z, 0) u(z)^2 \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We consider the affine homotopy

$$h(t, u) = (1-t)\varphi(u) + t\tau(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$(4.17) \quad t_n \rightarrow t \in [0, 1], \quad u_n \rightarrow 0 \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad h'_u(t_n, u_n) = 0 \text{ for all } n \geq 1.$$

We have

$$(4.18) \quad A_p(u_n) + A(u_n) = (1-t_n)N_f(u_n) + t_n f'_x(\cdot, 0)u_n \quad \text{for all } n \geq 1.$$

Let $y_n = u_n/\|u_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(4.19) \quad y_n \rightarrow y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \rightarrow y \text{ in } L^r(\Omega).$$

From (4.18) we have

$$(4.20) \quad \|u_n\|^{p-2} A_p(y_n) + A(y_n) = (1 - t_n) \frac{N_f(u_n)}{\|u_n\|} + t_n f'_x(\cdot, 0) y_n$$

for all $n \geq 1$. From hypotheses (H₃) (i),(iv), we have

$$|f(z, x)| \leq c_{10}(|x| + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{10} > 0.$$

Note that (4.18) and Ladyzhenskaya and Uraltseva [18, p.286], imply that $\{u_n\}_{n \geq 1} \subseteq L^\infty(\Omega)$ is bounded. So, it follows that $\{N_f(u_n)/\|u_n\|\}_{n \geq 1} \subseteq L^r(\Omega)$ is bounded. Thus by passing to a subsequence if necessary and using hypothesis (H₃) (iv), we have

$$(4.21) \quad \frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} f'_x(\cdot, 0) y \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.$$

Since $\{A_p(y_n)\}_{n \geq 1} \subseteq W^{-1,p'}(\Omega)$ is bounded (see Proposition 2.2) and $\|u_n\|^{p-2} \rightarrow 0$ (recall $p > 2$ and see (4.17)), we have

$$(4.22) \quad \|u_n\|^{p-2} A_p(y_n) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \rightarrow \infty.$$

So, if in (4.20) we pass to the limit as $n \rightarrow \infty$ and use (4.21) and (4.22), then

$$A(y) = f'_x(\cdot, 0) y \quad \Rightarrow \quad -\Delta y(z) = f'_x(z, 0) y(z) \text{ a.e. in } \Omega, \quad y|_{\partial\Omega} = 0.$$

Hypothesis (H₃) (iv) implies that $y = 0$. On the other hand, from (4.20) and Theorem 2.1 of Lieberman [19], we know that there exist $\alpha \in (0, 1)$ and $M > 0$ such that

$$(4.23) \quad y_n \in C_0^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|y_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq M \quad \text{for all } n \geq 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$ and using (4.19), we have

$$y_n \rightarrow y \text{ in } C_0^1(\bar{\Omega}) \quad \Rightarrow \quad \|y\| = 1,$$

contradiction since $y = 0$. So, (4.17) cannot occur and then the homotopy invariance of the critical groups implies that

$$(4.24) \quad C_k(\varphi, 0) = C_k(\tau, 0) \quad \text{for all } k \geq 0.$$

From Theorem 1 of Cingolani and Vannella [8], we have

$$C_k(\tau, 0) = \delta_{k,d_m} \mathbb{Z} \text{ for all } k \geq 0 \quad \Rightarrow \quad C_k(\varphi, 0) = \delta_{k,d_m} \mathbb{Z} \text{ for all } k \geq 0$$

(see (4.24)). □

PROPOSITION 4.6. *Assume that hypotheses (H₃) (i)–(iii) hold and $\inf \varphi(K_\varphi) > -\infty$. Then $C_k(\varphi, \infty) = 0$ for all $k \geq 0$.*

PROOF. Note that φ satisfies the C-condition (see Aizicovici, Papageorgiou and Staicu [2]). Hypotheses (H₃) (i),(ii) imply that given $\xi > 0$, we can find $c_{11} = c_{11}(\xi) > 0$ such that

$$(4.25) \quad F(z, x) \geq \frac{\xi}{p}|x|^p - c_{11} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Let $u \in \partial B_1 = \{u \in W_0^{1,p}(\Omega) : \|u\| = 1\}$ and $t > 0$. Then

$$\varphi(tu) \leq \frac{t^p}{p}[1 - \xi\|u\|_p^p] + \frac{t^2}{2}\|Du\|_2^2 + c_{11}|\Omega|_N.$$

Choosing $\xi > 1/\|u\|_p^p$ and since $p > 2$, we infer that

$$(4.26) \quad \varphi(tu) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

By virtue of hypotheses (H₃) (i),(iii), we can find $\eta_1 \in (0, \eta_0)$ and $c_{12} > 0$ such that

$$(4.27) \quad f(z, x)x - pF(z, x) \geq \eta_1|x|^\tau - c_{12} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then for all $v \in W_0^{1,p}(\Omega)$, we have (see (4.27))

$$(4.28) \quad \int_{\Omega} [pF(z, v) - f(z, v)v] dz \leq -\eta_1\|v\|_\tau^\tau + c_{12}|\Omega|_N.$$

Let $\vartheta < \min[-c_{12}|\Omega|_N - 1, \inf \varphi(K_\varphi)]$. From (4.26), we see that for every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ we can find $t \geq 1$ such that

$$(4.29) \quad \varphi(tu) \leq \frac{\vartheta}{p}.$$

Then, for such big $t \geq 1$, we have

$$\begin{aligned} (4.30) \quad \frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu), u \rangle \\ &= \frac{1}{t} \left[t^p \|Du\|_p^p + t^2 \|Du\|_2^2 - \int_{\Omega} f(z, tu)(tu) dz \right] \\ &\leq \frac{1}{t} \left[t^p \|Du\|_p^p + t^2 \|Du\|_2^2 - \int_{\Omega} pF(z, tu) dz + c_{12}|\Omega|_N \right] \\ &\hspace{15em} \text{(see (4.28))} \\ &\leq \frac{1}{t} [p\varphi(tu) + c_{12}|\Omega|_N] \hspace{10em} \text{(since } p > 2) \\ &\leq \frac{1}{t} [\vartheta + c_{12}|\Omega|_N] < 0. \end{aligned}$$

Given $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, let $\tau(u) \geq 1$ be the first $t \geq 1$ such that (4.29) holds. Then by virtue of (4.30) and the implicit function theorem, we have $\tau \in C^1(W_0^{1,p}(\Omega) \setminus \{0\})$. Let $h \in C([0, 1] \times (W_0^{1,p}(\Omega) \setminus \{0\}), W_0^{1,p}(\Omega) \setminus \{0\})$ be defined by

$$h(t, u) = (1 - t + t\tau(u))u \quad \text{for all } (t, u) \in [0, 1] \times (W_0^{1,p}(\Omega) \setminus \{0\}).$$

Note that

$$\begin{aligned} h(0, u) &= u, & h(1, u) &= \tau(u)u \in \varphi^\vartheta & \text{for all } u \in W_0^{1,p}(\Omega) \setminus \{0\}, \\ h(t, u) &= u & & & \text{for all } (t, u) \in [0, 1] \times \varphi^\vartheta. \end{aligned}$$

These facts imply that φ^ϑ is a strong deformation retract of $W_0^{1,p}(\Omega) \setminus \{0\}$. Using the radial retraction and Dugundji [11, Theorem 6.5, p. 325], we see that ∂B_1 is a deformation retract of $W_0^{1,p}(\Omega) \setminus \{0\}$. It follows that the sets φ^ϑ and ∂B_1 are homotopy equivalent and so we have

$$(4.31) \quad H_k(W_0^{1,p}(\Omega), \partial B_1) = H_k(W_0^{1,p}(\Omega), \varphi^\vartheta) \quad \text{for all } k \geq 0.$$

The choice of ϑ implies that

$$(4.32) \quad H_k(W_0^{1,p}(\Omega), \varphi^\vartheta) = C_k(\varphi, \infty) \quad \text{for all } k \geq 0.$$

Also, since $W_0^{1,p}(\Omega)$ is infinite dimensional, the boundary ∂B_1 is contractible in itself (see, for example, Gasinski and Papageorgiou [15, p. 691]). Therefore

$$(4.33) \quad H_k(W_0^{1,p}(\Omega), \partial B_1) = 0 \quad \text{for all } k \geq 0.$$

From (4.31), (4.32) and (4.33) we conclude that $C_k(\varphi, \infty) = 0$ for all $k \geq 0$. \square

Next we compute the critical groups of the functional φ at $\hat{u} \in K_\varphi$ and at $\hat{v} \in K_\varphi$.

PROPOSITION 4.7. *If hypotheses (H₃) hold and K_φ is finite, then $C_k(\varphi, \hat{u}) = C_k(\varphi, \hat{v}) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$.*

PROOF. We do the proof for the pair (φ, \hat{u}) , the proof for the pair (φ, \hat{v}) being similar. Let $\psi_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the functional introduced in the proof of Proposition 4.4. We consider the homotopy $h_+ : [0, 1] \times W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)$ defined by

$$h_+(t, u) = (1-t)\varphi(u) + t\psi_+(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose we can find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$(4.34) \quad t_n \rightarrow t \in [0, 1], \quad u_n \rightarrow \hat{u} \text{ in } W_0^{1,p}(\Omega) \text{ and } (h_+)'_u(t_n, u_n) = 0 \text{ for all } n \geq 1.$$

From (4.34) we have

$$(4.35) \quad -\Delta_p u_n(z) - \Delta u_n(z) = (1-t_n)f(z, u_n(z)) + t_n g_+(z, u_n(z))$$

for almost all $z \in \Omega$. From Ladyzhenskaya and Uraltseva [18, p. 286], we know that there exists $M_8 > 0$ such that $\|u_n\|_\infty \leq M_8$ for all $n \geq 1$. Then applying Theorem 2.1 of Lieberman [19], we can find $\alpha \in (0, 1)$ and $M_9 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq M_9 \quad \text{for all } n \geq 1.$$

The compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$ and (4.34) imply

$$u_n \rightarrow \hat{u} \quad \text{in } C_0^1(\bar{\Omega}) \text{ as } n \rightarrow \infty.$$

From Proposition 4.4 we know that $\widehat{u} - u_0 \in \text{int } C_+$. Therefore we can find $n_0 \geq 1$ such that $u_n \geq u_0$ for all $n \geq n_0$. Then from (4.6) and (4.35) it follows that $\{u_n\}_{n \geq n_0} \subseteq K_\varphi$, which contradicts our hypothesis that K_φ is finite. Therefore, we can find $\rho \in (0, 1)$ small such that $\widehat{u} \in \text{int } C_+$ is the only critical point of $\{h_+(t, \cdot)\}_{t \in [0,1]}$ in $\overline{B}_\rho(\widehat{u}) = \{u \in W_0^{1,p}(\Omega) : \|u - \widehat{u}\| \leq \rho\}$. Then the homotopy invariance of critical groups implies that

$$(4.36) \quad C_k(\varphi, \widehat{u}) = C_k(\psi_+, \widehat{u}) \quad \text{for all } k \geq 0.$$

From the proof of Proposition 4.4, we know that $\widehat{u} \in \text{int } C_+$ is a critical point of mountain pass type for the functional ψ_+ . Therefore

$$(4.37) \quad C_1(\psi_+, \widehat{u}) \neq 0 \quad \Rightarrow \quad C_1(\varphi, \widehat{u}) \neq 0$$

(see (4.36)). Since $\varphi \in C^2(W_0^{1,p}(\Omega))$ (see hypotheses (H₃)), from (4.37) and Papageorgiou and Smyrlis [22] (see also Papageorgiou and Rădulescu [21]), we have

$$C_k(\varphi, \widehat{u}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

In a similar fashion, we show that $C_k(\varphi, \widehat{v}) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. □

Our strategy in order to generate nodal solutions is the following. First we show that problem (1.1) admits extremal constant sign solutions, that is there exist a smallest positive solution $u_+ \in \text{int } C_+$ and a biggest negative solution $v_- \in -\text{int } C_+$. Having the extremal constant sign solutions, we focus on the order interval $[v_-, u_+] = \{u \in W_0^{1,p}(\Omega) : v_-(z) \leq u(z) \leq u_+(z) \text{ almost everywhere in } \Omega\}$. Through a combination of variational methods and of Morse theory, we show that the problem has a nontrivial solution in $[v_-, u_+]$ distinct from v_- and u_+ . The extremality of v_- and u_+ , implies that this solution must be nodal. The nonhomogeneity of the differential operator, is the source of difficulties in the execution of this solution strategy. In this respect, the unilateral growth estimate in (4.1), will help us to overcome these difficulties. So, motivated by (4.1), we consider the following auxiliary Dirichlet problem:

$$(4.38) \quad -\Delta_p u(z) - \Delta u(z) = \xi_0 u(z) - \xi_1 |u(z)|^{r-2} u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

PROPOSITION 4.8. *Problem (4.38) admits a unique positive solution $u_* \in \text{int } C_+$ and since the equation in (4.38) is odd, $v_* = -u_* \in -\text{int } C_+$ is the unique negative solution of (4.38).*

PROOF. Let $\mu_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functional defined by

$$\mu_+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \frac{\xi_0}{2} \|u^+\|_2^2 + \frac{\xi_1}{p} \|u^+\|_r^r \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $r > p > 2$, it is clear that μ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_* \in W_0^{1,p}(\Omega)$ such that

$$(4.39) \quad \mu_+(u_*) = \inf[\mu_+(u) : u \in W_0^{1,p}(\Omega)].$$

Since $\xi_0 > \widehat{\lambda}_1(2)$ and $p > 2$, we see that for $t \in (0, 1)$ small enough, we have

$$\mu_+(t\widehat{u}_1(2)) < 0 \quad \Rightarrow \quad \mu_+(u_*) < 0 = \mu_+(0)$$

(see (4.39)), hence $u_* \neq 0$. From (4.39) we have

$$(4.40) \quad \mu'_+(u_*) = 0 \quad \Rightarrow \quad A_p(u_*) + A(u_*) = \xi_0 u_*^+ - \xi_1 (u_*^+)^{r-1}.$$

On (4.40) we act with $-u_*^- \in W_0^{1,p}(\Omega)$ and obtain $u_* \geq 0$, $u_* \neq 0$. So, it follows that u_* is a positive solution of the auxiliary problem (4.38). The nonlinear regularity theory (see [19]) and the nonlinear maximum principle (see Pucci and Serrin [23, pp. 111, 120]) imply that $u_* \in \text{int } C_+$.

We need to show the uniqueness of this positive solution. To this end, let $G_0(t) = t^p/p + t^2/2$ for all $t \geq 0$. We set $G(y) = G_0(|y|)$ and consider the integral functional $j: L^1(\Omega) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_{\Omega} G(Du^{1/2}) \, dz & \text{if } u \geq 0, \, u^{1/2} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We claim that $j(\cdot)$ is convex. So, let $u_1, u_2 \in \text{dom } j = \{u \in W_0^{1,p}(\Omega) : j(u) < +\infty\}$ and let $v_1 = u_1^{1/2}$, $v_2 = u_2^{1/2} \in W_0^{1,p}(\Omega)$. We define

$$v_3 = ((1-t)u_1 + tu_2)^{1/2} \quad \text{with } t \in [0, 1].$$

From Lemma 1 of Benguria, Brezis and Lieb [6] (see also Diaz and Saa [10, Lemma 1]), we have

$$|Dv_3(z)| \leq [(1-t)|Dv_1(z)|^2 + t|Dv_2(z)|^2]^{1/2} \quad \text{for a.a. } z \in \Omega.$$

Note that $t \rightarrow G_0(t)$ is increasing and $t \rightarrow G_0(t^{1/2})$ is convex on $[0, +\infty]$. So, we obtain

$$\begin{aligned} & G_0(|Dv_3(z)|) \\ & \leq G_0([(1-t)|Dv_1(z)|^2 + t|Dv_2(z)|^2]^{1/2}) \quad (\text{the monotonicity of } G_0(\cdot)) \\ & \leq (1-t)G_0(|Dv_1(z)|^2) + tG_0(|Dv_2(z)|^2) \quad (\text{the convexity of } t \rightarrow G_0(t^{1/2})) \\ & \Rightarrow G(Dv_3(z)) \leq (1-t)G(Dv_1(z)) + tG(Dv_2(z)) \quad \text{a.e. in } \Omega \\ & \Rightarrow j(\cdot) \quad \text{is convex.} \end{aligned}$$

Let u, v be two positive solutions of auxiliary problem (4.38). Then from the first part of the proof we have $u, v \in \text{int } C_+$. So, for all $h \in C_0^1(\overline{\Omega})$ and for $t \in [0, 1]$ small, we have $u^2 \pm th$, $v^2 \pm th \in \text{dom } j$. Then by the chain rule, we have

$$\begin{aligned} j'(u^2)(h) &= \left\langle A_p(u) + A(u), \frac{h}{u} \right\rangle, \\ j'(v^2)(h) &= \left\langle A_p(v) + A(v), \frac{h}{v} \right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega) \end{aligned}$$

(recall $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$). The convexity of j , implies the monotonicity of $j'(y)(\cdot)$ for all $y \in \text{dom } j$. Hence

$$\begin{aligned} 0 &\leq \langle j'(u^2) - j'(v^2), u^2 - v^2 \rangle \\ &= \int_{\Omega} \left(\frac{-\Delta_p u - \Delta u}{u} - \frac{-\Delta_p v - \Delta v}{v} \right) (u^2 - v^2) dz \\ &= \int_{\Omega} \xi_1 (v^{r-2} - u^{r-2})(u^2 - v^2) dz \leq 0 \\ &\Rightarrow u = v \Rightarrow u_* \in \text{int } C_+ \text{ is the unique positive solution of (4.38).} \end{aligned}$$

Since the equation is odd, we deduce that $v_* = -u_* \in -\text{int } C_+$ is the unique negative solution of problem (4.38). \square

Now let S_+ (resp. S_-) denote the set of positive (resp. negative) solutions of problem (1.1) belonging in the order interval $[0, w_+]$ (resp. $[w_-, 0]$). From Proposition 4.3 we know that

$$S_+, S_- \neq \emptyset \quad \text{and} \quad S_+ \subseteq \text{int } C_+, \quad S_- \subseteq -\text{int } C_+.$$

PROPOSITION 4.9. *If hypotheses (H₃) hold and $u \in S_+$ (resp. $v \in -S_-$), then $u_* \leq u$ and $v \leq v_*$.*

PROOF. We do the proof for $u \in S_+$, the proof for $v \in S_-$ being similar. We introduce the following Carathéodory function:

$$(4.41) \quad \gamma_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \xi_0 x - \xi_1 x^{r-1} & \text{if } 0 \leq x \leq u(z), \\ \xi_0 u(z) - \xi_1 u(z)^{r-1} & \text{if } u(z) < x. \end{cases}$$

Let

$$\Gamma_+(z, x) = \int_0^x \gamma_+(z, s) ds$$

and consider the C^1 -functional $\xi_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\xi_+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} \Gamma_+(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (4.41) that ξ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_* \in W_0^{1,p}(\Omega)$ such that

$$(4.42) \quad \xi_+(\tilde{u}_*) = \inf[\xi_+(u) : u \in W_0^{1,p}(\Omega)].$$

As before (see the proof of Proposition 4.8), for $t \in (0, 1)$ small (at least such that $t\widehat{u}_1(2) \leq u$; recall that $u \in \text{int } C_+$ and see Lemma 3.3 of Filippakis, Kristaly and Papageorgiou [14]), we have $\xi_+(t\widehat{u}_1(2)) < 0$. Therefore $\xi_+(\tilde{u}_*) < 0$ (see (4.42)), hence $\tilde{u}_* \neq 0$. Also, from (4.42) we have

$$(4.43) \quad \xi'_+(\tilde{u}_*) = 0 \quad \Rightarrow \quad A_p(\tilde{u}_*) + A(\tilde{u}_*) = N_{\gamma_+}(\tilde{u}_*).$$

On (4.43) we act with $-\tilde{u}_*^- \in W_0^{1,p}(\Omega)$ and obtain $\tilde{u}_* \geq 0$, $\tilde{u}_* \neq 0$. Also, we act with $(\tilde{u}_* - u)^+ \in W_0^{1,p}(\Omega)$ and have

$$\begin{aligned} \langle A_p(\tilde{u}_*), (\tilde{u}_* - u)^+ \rangle + \langle A(\tilde{u}_*), (\tilde{u}_* - u)^+ \rangle &= \int_{\Omega} \gamma_+(z, \tilde{u}_*)(\tilde{u}_* - u)^+ dz \\ &= \int_{\Omega} [\xi_0 u - \xi_1 u^{r-1}](\tilde{u}_* - u)^+ dz && \text{(see (4.41))} \\ &\leq \int_{\Omega} f(z, u)(\tilde{u}_* - u)^+ dz && \text{(see (4.1))} \\ &= \langle A_p(u), (\tilde{u}_* - u)^+ \rangle + \langle A(u), (\tilde{u}_* - u)^+ \rangle && \text{(since } u \in S_+), \\ &\Rightarrow \langle A_p(\tilde{u}_*) - A_p(u), (\tilde{u}_* - u)^+ \rangle + \langle A(\tilde{u}_*) - A(u), (\tilde{u}_* - u)^+ \rangle \leq 0 \\ &\Rightarrow |\{\tilde{u}_* > u\}|_N = 0, \end{aligned}$$

hence $\tilde{u}_* \leq u$. So, we have proved that

$$\begin{aligned} \tilde{u}_* \in [0, u] &= \{y \in W_0^{1,p}(\Omega) : 0 \leq y(z) \leq u(z) \text{ for a.a. } z \in \Omega\}, \quad \tilde{u}_* \neq 0 \\ &\Rightarrow \tilde{u}_* \text{ is a nontrivial positive solution of problem (4.38)} \\ &&& \text{(see (4.41) and (4.43))} \\ &\Rightarrow \tilde{u}_* = u_* \in \text{int } C_+ && \text{(see Proposition 4.8)} \\ &\Rightarrow u_* \leq u. \end{aligned}$$

Similarly we show that $v \leq v_*$. □

Now we can establish the existence of extremal constant sign solutions for problem (1.1).

PROPOSITION 4.10. *If hypotheses (H₃) hold, then problem (1.1) admits a smallest positive solution $u_+ \in \text{int } C_+$ and a biggest negative solution $v_- \in -\text{int } C_+$.*

PROOF. As in Filippakis, Kristaly and Papageorgiou [14], we can show that the set of positive solutions of (1.1) is downward directed, that is, if u, y are two positive solutions, then there exists a third positive solution of (1.1) such that $w \leq u$, $w \leq y$. So, in order to produce the smallest positive solution, we can restrict ourselves to the set S_+ . Then from Dunford and Schwartz [12, p. 336], we know that we can find $\{u_n\}_{n \geq 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n \geq 1} u_n.$$

We have

$$\begin{aligned} (4.44) \quad A_p(u_n) + A(u_n) &= N_f(u_n), \quad u_n \in [0, w_+] \quad \text{for all } n \geq 1 \\ &\Rightarrow \{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.} \end{aligned}$$

So, we may assume that

$$(4.45) \quad u_n \xrightarrow{w} u_+ \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow u_+ \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.$$

Acting on the equation in (4.44) with $u_n - u_+ \in W_0^{1,p}(\Omega)$, passing to the limit as $n \rightarrow \infty$ and using (4.45), we obtain

$$(4.46) \quad \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u_+ \rangle + \langle A(u_n), u_n - u_+ \rangle] = 0 \Rightarrow u_n \rightarrow u_+ \text{ in } W_0^{1,p}(\Omega)$$

(as before, see the proof of Proposition 3.3). So, if in (4.44) we pass to the limit as $n \rightarrow \infty$ and use (4.46), then

$$A_p(u_+) + A(u_+) = N_f(u_+).$$

From Proposition 4.9, we have $u_* \leq u_n$ for all $n \geq 1$, hence $u_* \leq u_+$ (see (4.46)) and so $u_+ \in S_+$ and $u_+ = \inf S_+$, that is $u_+ \in \text{int } C_+$ is the smallest positive solution of problem (1.1). Similarly we produce $v_- \in -\text{int } C_+$ the biggest negative solution of problem (1.1). \square

Now we are ready for the second multiplicity theorem. It generates six nontrivial solutions, all with sign information.

THEOREM 4.11. *If hypotheses (H₃) hold, then problem (1.1) admits at least six nontrivial solutions*

$$\begin{aligned} u_0, \hat{u} &\in \text{int } C_+, & \hat{u} - u_0 &\in \text{int } C_+, \\ v_0, \hat{v} &\in -\text{int } C_+, & v_0 - \hat{v} &\in \text{int } C_+, \end{aligned}$$

and $y_0, \hat{y} \in C_0^1(\bar{\Omega}) \setminus \{0\}$ nodal.

PROOF. The four nontrivial constant sign solutions

$$u_0, \hat{u} \in \text{int } C_+ \quad \text{and} \quad v_0, \hat{v} \in -\text{int } C_+, \quad \hat{u} - u_0 \in \text{int } C_+, \quad v_0 - \hat{v} \in \text{int } C_+$$

come from Propositions 4.3 and 4.4. So, we need to produce the two nodal solutions. By virtue of Proposition 4.10 without any loss of generality, we may assume that $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$ are extremal. Let g_* be the Carathéodory function defined by

$$(4.47) \quad g_*(z, x) = \begin{cases} f(z, v_0(z)) & \text{if } x < v_0(z), \\ f(z, x) & \text{if } v_-(z) \leq x \leq u_0(z), \\ f(z, u_0(z)) & \text{if } u_0(z) < x. \end{cases}$$

Let

$$G_*(z, x) = \int_0^x g_*(z, s) \, ds$$

and consider the C^1 -functional $\tau_* : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tau_*(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_{\Omega} G_*(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, we consider the positive and negative truncations of $g_*(z, \cdot)$, namely the Carathéodory functions

$$g_*^\pm(z, x) = g_*(z, \pm x^\pm).$$

We set

$$G_*^\pm(z, x) = \int_0^x g_*^\pm(z, s) ds$$

and consider the C^1 -functionals $\tau_*^\pm : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tau_*^\pm(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega G_*^\pm(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is easy to see that

$$K_{\tau_*} \subseteq [v_0, u_0], \quad K_{\tau_*^+} \subseteq [0, u_0], \quad K_{\tau_*^-} \subseteq [v_0, 0]$$

(see the proof of Proposition 4.3). The extremality of u_0 and v_0 , implies that

$$(4.48) \quad K_{\tau_*} \subseteq [v_0, u_0], \quad K_{\tau_*^+} = \{0, u_0\}, \quad K_{\tau_*^-} = \{v_0, 0\}.$$

CLAIM. $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$ are local minimizers of the functional τ_* .

From (4.47) it is clear that τ_*^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$(4.49) \quad \tau_*^+(\tilde{u}_0) = \inf[\tau_*^+(u) : u \in W_0^{1,p}(\Omega)].$$

As before (see the proof of Proposition 4.3), we have $\tau_*^+(\tilde{u}_0) < 0 = \tau_*^+(0)$, hence $\tilde{u}_0 \neq 0$. Since $\tilde{u}_0 \in K_{\tau_*^+}$ (see (4.49)), we have $\tilde{u}_0 = u_0$ (see (4.48)). Recall that $u_0 \in \text{int } C_+$. Because $\tau_*^+|_{C_+} = \tau_*|_{C_+}$, it follows that $u_0 \in \text{int } C_+$ is a local $C_0^1(\bar{\Omega})$ -minimizer of τ_* . Invoking Proposition 2.3, we infer that $u_0 \in \text{int } C_+$ is a local $W_0^{1,p}(\Omega)$ -minimizer of τ_* . Similarly, for $v_0 \in -\text{int } C_+$, using this time the functional τ_*^- . This proves Claim.

Without any loss of generality, we may assume that $\tau_*(v_0) \leq \tau_*(u_0)$ (the analysis is similar, if the opposite inequality holds). By virtue of Claim, we can find $\rho \in (0, 1)$ small enough ($\rho < \|u_0 - v_0\|$) such that

$$(4.50) \quad \tau_*(v_0) \leq \tau_*(u_0) < \inf[\tau_*(u) : \|u - u_0\| = \rho] = m_\rho$$

(see [1]). Since τ_* is coercive (see (4.47)), it satisfies the C-condition. This fact and (4.50), permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$(4.51) \quad y_0 \in K_{\tau_*} \quad \text{and} \quad m_\rho \leq \tau_*(y_0).$$

From (4.50) and (4.51) it follows that $y_0 \notin \{v_0, u_0\}$. If we can show that $y_0 \neq 0$, then we infer that y_0 is nodal. Reasoning as in the proof of Proposition 4.4, using hypothesis (H₃) (vi) and Proposition 2.4, we show that

$$(4.52) \quad u_0 - y_0 \in \text{int } C_+ \quad \text{and} \quad y_0 - v_0 \in \text{int } C_+.$$

Since y_0 is a critical point of τ_* of mountain pass type, we have

$$(4.53) \quad C_1(\tau_*, y_0) \neq 0.$$

Note that $\tau_*|_{[v_0, u_0]} = \varphi|_{[v_0, u_0]}$ (see (4.47)). From this fact and (4.52), via a homotopy invariance argument, as in the proof of Proposition 4.7, we obtain

$$(4.54) \quad C_k(\tau_*, y_0) = C_k(\varphi, y_0) \text{ for all } k \geq 0 \Rightarrow C_1(\varphi, y_0) \neq 0$$

(see (4.53)). But $\varphi \in C^2(W_0^{1,p}(\Omega))$. Hence

$$(4.55) \quad C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0$$

(see [22], [21]). From Proposition 4.5, we know that

$$(4.56) \quad C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0, \text{ with } d_m \geq 2$$

(recall $m \geq 2$). Comparing (4.55) and (4.56), we have $y_0 \neq 0$, hence y_0 is nodal and $y_0 \in C_0^1(\bar{\Omega})$ (nonlinear regularity theory). From (4.54)–(4.56), we have

$$(4.57) \quad C_k(\tau_*, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0,$$

$$(4.58) \quad C_k(\tau_*, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0.$$

By virtue of Claim, we have

$$(4.59) \quad C_k(\tau_*, u_0) = C_k(\tau_*, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$

Finally, the coercivity of τ_* implies

$$(4.60) \quad C_k(\tau_*, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$

Suppose $K_{\tau_*} = \{v_0, u_0, 0, y_0\}$. Then from (4.57)–(4.60) and the Morse relation with $t = -1$ (see (2.1)), we have

$$2(-1)^0 + (-1)^{d_m} + (-1)^1 = (-1)^0 \Rightarrow (-1)^{d_m} = 0,$$

a contradiction. So, there exists $\hat{y} \in K_{\tau_*}$, $\hat{y} \notin \{v_0, u_0, 0, y_0\}$. From (4.48) it follows that \hat{y} is a second nodal solution and by nonlinear regularity, we have $\hat{y} \in C_0^1(\bar{\Omega})$. \square

It is natural to ask what happens if in (H₃) (iv), $m \geq 1$. In this case the unilateral growth estimate (4.1) is not valid. Therefore, we cannot generate extremal solutions and so Theorem 4.11 fails. Nevertheless, we still have a multiplicity theorem producing five nontrivial solutions, but we are unable to determine the sign of the fifth solution.

The new hypotheses on the reaction f are the following:

- (H₄) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$, hypotheses (H₄) (i)–(v) are the same as the corresponding hypotheses (H₃) (i)–(iii),(v),(vi) and

(iv) $f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x}$ uniformly for almost all $z \in \Omega$ and for some $m \geq 1$, we have

$$f'_x(z, 0) \in [\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)] \quad \text{for a.a. } z \in \Omega$$

$$\text{and } f'_x(\cdot, 0) \not\equiv \widehat{\lambda}_m(2), \quad f'_x(\cdot, 0) \not\equiv \lambda_{m+1}(2).$$

THEOREM 4.12. *If hypotheses (H₄) hold, then problem (1.1) has at least five nontrivial solutions*

$$u_0, \widehat{u} \in \text{int } C_+ \quad \text{with } \widehat{u} - u_0 \in \text{int } C_+,$$

$$v_0, \widehat{v} \in -\text{int } C_+ \quad \text{with } v_0 - \widehat{v} \in \text{int } C_+,$$

$$y_0 \in C_0^1(\overline{\Omega}).$$

PROOF. A careful reading of the proofs of Propositions 4.3 and 4.4, reveals that by virtue of the UCP, the results remain valid under the new set of hypotheses (H₄). Therefore we can say that problem (1.1) has at least four nontrivial solutions of constant sign

$$u_0, \widehat{u} \in \text{int } C_+ \quad \text{with } \widehat{u} - u_0 \in \text{int } C_+,$$

$$v_0, \widehat{v} \in -\text{int } C_+ \quad \text{with } v_0 - \widehat{v} \in \text{int } C_+.$$

From the proof of Proposition 4.3, we know that u_0 is a minimizer of $\widehat{\varphi}_+$ (see (4.3)). From (4.11) we have $u_0 \in \text{int}_{C_0^1(\overline{\Omega})}[0, w_+]$. Since $\widehat{\varphi}_+|_{[0, w_+]} = \varphi|_{[0, w_+]}$ (see (4.2)), it follows that u_0 is a local $C_0^1(\overline{\Omega})$ -minimizer of φ , hence by Proposition 2.3 it is also a local $W_0^{1,p}(\Omega)$ -minimizer of φ . Similarly we show that $v_0 \in -\text{int } C_+$ too is a local minimizer of φ . Therefore, we have

$$(4.61) \quad C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0.$$

From the proof of Proposition 4.4, we know that $\widehat{u} \in \text{int } C_+$ is a critical point of ψ_+ of mountain pass type. Hence

$$(4.62) \quad C_1(\psi_+, \widehat{u}) \neq 0.$$

Since $\widehat{u} - u_0 \in \text{int } C_+$, via a homotopy invariance argument as in the proof of Proposition 4.7, we show that

$$(4.63) \quad C_k(\psi_+, \widehat{u}) = C_k(\varphi, \widehat{u}) \quad \text{for all } k \geq 0$$

$$\Rightarrow C_1(\varphi, \widehat{u}) \neq 0 \quad (\text{see (4.62)})$$

$$\Rightarrow C_k(\varphi, \widehat{u}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0 \quad (\text{see [22], [21]}).$$

Similarly we show that

$$(4.64) \quad C_k(\varphi, \widehat{v}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

From Propositions 4.5 and 4.6, we have

$$(4.65) \quad C_k(\varphi, 0) = \delta_{k,d_m} \mathbb{Z} \quad \text{for all } k \geq 0,$$

$$(4.66) \quad C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0.$$

Suppose $K_\varphi = \{v_0, u_0, \widehat{v}, \widehat{u}, 0\}$. Then from (4.61), (4.63)–(4.66) and the Morse relation with $t = -1$ (see (2.1)), we have

$$2(-1)^0 + 2(-1)^1 + (-1)^{d_m} = 0 \quad \Rightarrow \quad (-1)^{d_m} = 0,$$

a contradiction. So, there exists $y_0 \in K_\varphi$, $y_0 \notin \{v_0, u_0, \widehat{v}, \widehat{u}, 0\}$, hence y_0 is a fifth nontrivial solution for problem (1.1) and $y_0 \in C_0^1(\overline{\Omega})$ (nonlinear regularity theory). \square

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