

GENERALIZED TOPOLOGICAL TRANSITION MATRIX

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ABSTRACT. This article represents a major step in the unification of the theory of algebraic, topological and singular transition matrices by introducing a definition which is a generalization that encompasses all of the previous three. When this more general transition matrix satisfies the additional requirement that it covers flow-defined Conley-index isomorphisms, one proves algebraic and connection-existence properties. These general transition matrices with this covering property are referred to as generalized topological transition matrices and are used to consider connecting orbits of Morse–Smale flows without periodic orbits, as well as those in a continuation associated to a dynamical spectral sequence.

1. Introduction

A challenging question in the study of dynamical systems is that of the existence of global bifurcations. The difficulty in detecting such bifurcation orbits is the fact that one must analyze the dynamical system globally. Topological techniques for global analysis are, therefore, a perfect fit for such an investigation. In particular, Conley index theory has proven to be quite useful in this role, as

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can be seen by the ample use of connection and transition matrices in bifurcation-related results. See [3], [4], [7]–[10] and [17].

Connection matrices have been extensively studied and can be computed by numerical techniques, see [1], [2] and [6]. Their continuation properties have proven to be useful in detecting global bifurcations. In particular, the continuation theorem [10] states that the connection matrices of an admissible ordering are invariant under local continuation. Yet, under global continuation, sets of connection matrices can undergo change. For instance, if there is a continuation between parameters with unique but different connection matrices, then within the continuation there must be a parameter value with nonunique connection matrices. At such a parameter value the system typically has a global bifurcation.

In other words, Morse decompositions and connection matrices provide a supporting structure within which global bifurcations can be detected, particularly via changes in the associated algebraic structures. These differences that occur in connection matrices under continuation, which can naturally be identified algebraically, were the main motivation for the introduction of transition matrices as a combinatorial mechanism to keep track of these changes. These transition matrices have since appeared in the literature under several guises: singular [22], topological [17], algebraic [11], and directional [15]. These four types of matrices are defined differently (particularly under contrasting conditions) and have distinct properties. On the other hand, due to underlying similarities in the definitions and their corresponding properties, a unified theory for transition matrices has long been called for.

In this paper we briefly introduce the generalization which unifies the theory. We focus on an initial and important step toward understanding the properties of this newly defined and more general transition matrix, which has the additional property that it covers flow-defined Conley-index isomorphisms. We refer to these matrices as generalized topological transition matrices and prove several properties they possess.

In contrast to the classical case, in our definition of a (generalized) topological transition matrix in Section 2, we do not require that there are no connections at the initial and final parameters of a continuation. As a consequence, this theory can be applied to a much broader class of dynamical systems than the classical topological transition matrix. We also establish properties of the generalized topological transition matrices – including connecting orbit existence results – corresponding to those of the classical topological transition matrix. In Section 3, we apply this new theory to Morse–Smale flows without periodic orbits. In this setting one demonstrates uniqueness and provides a simple way to compute the generalized topological transition matrix. In the last section, we see how the

generalized topological transition matrices can be obtained from a continuation associated to a dynamical spectral sequence.

We assume that the reader is familiar with the basic ideas in Conley Index Theory, including Morse decompositions, homology index braids, connection matrices, etc. (see [3], [8]–[10], [19] and [24]).

In the next paragraphs we summarize some basic ideas in Conley Index Theory and connection matrix theory. We highly recommend the reader to see [3], [8]–[10], [19] and [24] for more detailed background.

Let φ be a continuous flow on a locally compact Hausdorff space and let S be a compact invariant set under φ . A *Morse decomposition* of S is a collection of mutually disjoint compact invariant subsets of S ,

$$\mathcal{M}(S) = \{M(\pi) \mid \pi \in \mathbf{P}\}$$

indexed by a finite set \mathbf{P} , where each set $M(\pi)$ is called a *Morse set*. A partial order $<$ on \mathbf{P} is called the *admissible ordering* if for $x \in S \setminus \bigcup_{\pi \in \mathbf{P}} M(\pi)$ there exists $p < q$ such that $\alpha(x) \subseteq M(q)$ and $\omega(x) \subseteq M(p)$. The flow defines an admissible ordering of M , called the *flow ordering* of M , denoted $<_F$, and such that $M(\pi) <_F M(\pi')$ if and only if there exists a sequence of distinct elements of \mathbf{P} : $\pi = \pi_0, \dots, \pi_n = \pi'$, where $C(M(\pi_j), M(\pi_{j-1}))$, the set of connecting orbits between $M(\pi_j)$ to $M(\pi_{j-1})$, is nonempty for each $j = 1, \dots, n$. Note that every admissible ordering of M is an extension of $<_F$.

In the Conley theory one begins with the Conley index for isolated invariant sets, i.e. $S \subseteq X$ is an *isolated invariant set* if there exists a compact set $N \subseteq X$ such that $S \subseteq \text{int } N$ and

$$S = \text{Inv}(N, \varphi) = \{x \in N \mid \mathcal{O}(x) \subseteq N\}.$$

The *homological Conley index* of S , $CH_*(S)$ is the homology of the pointed space (N/L) , where (N, L) is an index pair for S . Setting

$$M(\mathbf{I}) = \bigcup_{\pi \in \mathbf{I}} M(\pi) \cup \bigcup_{\pi, \pi' \in \mathbf{I}} C(M(\pi'), M(\pi)),$$

the Conley index of $M(\mathbf{I})$, $CH_*(M(\mathbf{I}))$, in short $H_*(\mathbf{I})$, is well defined, since $M(\mathbf{I})$ is an isolated invariant set for all $\mathbf{I} \in \mathbf{I}(<)$.

Given $\mathcal{M}(S)$, a Morse decomposition of S , the existence of an admissible ordering on $\mathcal{M}(S)$ implies that any recurrent dynamics in S must be contained within the Morse sets, thus the dynamics off the Morse sets must be gradient-like. For this reason, Conley index theory refers to the dynamics within a Morse set as local dynamics and off the Morse sets as global dynamics. We briefly introduce the connection matrix theory, which addresses this latter aspect.

DEFINITION 1.1. Given \mathcal{G} , a graded module braid over $<$, and a collection of graded modules $\mathcal{C} = \{C(\pi)\}_{\pi \in \mathbf{P}}$, let $\Delta: \bigoplus_{\pi \in \mathbf{P}} C(\pi) \rightarrow \bigoplus_{\pi \in \mathbf{P}} C(\pi)$ be a $<$ -upper triangular boundary map. If $\mathcal{H}\Delta$, the graded module braid generated by Δ , is isomorphic to \mathcal{G} , and $C(p)$ is isomorphic to $\mathcal{G}(p)$ then Δ is called a *connection matrix* of \mathcal{G} .

To simplify the notation, for $\mathbf{I} \in \mathbf{I}(<)$ we denote $\bigoplus_{\pi \in \mathbf{I}} C(\pi)$ by $C(\mathbf{I})$, and the corresponding homology module in $\mathcal{H}\Delta$ by $H(\mathbf{I})$. In particular, the homology index braid of an admissible ordering of a Morse decomposition $\mathcal{G} = \{H_*(\mathbf{I})\}_{\mathbf{I} \in \mathbf{I}(<)}$ is an example of a graded module braid. In this setting a $<$ -upper triangular boundary map

$$\Delta: \bigoplus_{\pi \in \mathbf{P}} CH_*(M(\pi)) \rightarrow \bigoplus_{\pi \in \mathbf{P}} CH_{*-1}(M(\pi))$$

satisfying Definition 1.1 for $\mathcal{C}\Delta = \{CH_*(M(\pi))\}_{\pi \in \mathbf{P}}$ is called a *connection matrix for a Morse decomposition*. Since in this paper, our aim is to work with topological transition matrices, we focus on connection matrices for Morse decompositions with coefficients in a PID. Thus, let $\mathcal{CM}(<)$ denote the set of all connection matrices for a given ($<$ -ordered) Morse decomposition $\mathcal{M}(S)$.

One of the key features in Conley theory is its invariance under continuation. Since the connection matrices for Morse decompositions are algebraically derived from the homology Conley index braid, this seems to indicate that connecting orbits that persist over open sets in parameter space are identified by connection matrices. We now define Conley index continuation.

Let Γ be a Hausdorff topological space, Λ a compact, locally contractible, connected metric space and X a locally compact metric space. Assume that $X \times \Lambda \subseteq \Gamma$ is a local flow. $X \times \Lambda$ is called parametrized flow if for each $\lambda \in \Lambda$, $X \times \lambda$ is a local flow.

Let $\phi: Z \times \Lambda \rightarrow X$ be a parametrization of a local flow X . Denote the restriction $\phi|_{(Z \times \{\lambda\})}$ by ϕ_λ and its image by X_λ .

LEMMA 1.2 (Salamon). *For any compact set $N \subseteq X$ the set $\Lambda(N) = \{\lambda \in \Lambda \mid N \times \lambda \text{ is an isolating neighbourhood in } X \times \lambda\}$ is open in Λ .*

DEFINITION 1.3. The space of isolated invariant sets is

$$\mathcal{S} = \mathcal{S}(\phi) = \{S \times \lambda \mid \lambda \in \Lambda$$

and $S \times \lambda$ is an isolated invariant compact set in $X \times \lambda\}$.

For all compact sets $N \subseteq X \times \lambda$ define the map $\varrho_N: \Lambda(N) \rightarrow \mathcal{S}$ by $\varrho_N(\lambda) = \text{Inv}(N \times \lambda)$. Then consider the topology on the space \mathcal{S} generated by the sets $\{\varrho_N(U) \mid N \subseteq X \text{ compact, } U \subseteq \Lambda(N) \text{ open}\}$.

A map $\gamma: \Lambda \rightarrow \mathcal{S}$ is called a *section* of the space of isolated invariant sets if $\Pi_\Lambda \circ \gamma = \text{id}|_\Lambda$.

We are interested in the situation where the homology index braids of admissible orderings of Morse decompositions at parameters λ and μ are isomorphic, see Theorem 1.7. That is, it is not enough that a Morse decomposition continues over Λ , it must also continue with a partial order, more specifically:

DEFINITION 1.4. Let $\mathcal{M}(S) = \{M(\pi) \mid \pi \in (\mathbf{P}, <)\}$ be an ordered Morse decomposition of the isolated invariant set $S \subseteq X \times \Lambda$. Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi \in (\mathbf{P}, <_\lambda)}$, $M_\mu = \{M_\mu(\pi)\}_{\pi \in (\mathbf{P}, <_\mu)}$, S_λ and S_μ be the sets obtained by intersection of $\mathcal{M}(S)$ and S by the fibers $X \times \{\lambda\}$ and $X \times \{\mu\}$, respectively, where $<_\nu$ is the order restricted to the order $<$ in the parameter $\nu \in \Lambda$.

- (a) We say that $\mathcal{M}(S)$ with its order $<$ *continues over* Λ if there exist sections σ and $\varsigma: \Lambda \rightarrow \mathcal{S}$ such that $\{\varsigma_\pi(\nu) \mid \pi \in (\mathbf{P}, <_\nu)\}$ is a Morse decomposition for $\sigma(\nu)$, for all $\nu \in \Lambda$.
- (b) If, furthermore, there exist a path $\omega: [0, 1] \rightarrow \Lambda$ from λ to μ ; $\sigma(\lambda) = S_\lambda$; $\sigma(\mu) = S_\mu$; $\varsigma_\pi(\lambda) = M_\lambda(\pi)$; $\varsigma_\pi(\mu) = M_\mu(\pi)$; and if $\mathcal{M}(S)$ continues at least over $\omega([0, 1])$, then we say that the admissible orderings $<_\lambda$ and $<_\mu$ *are related by continuation* or continue from one to the other. See Figure 1.

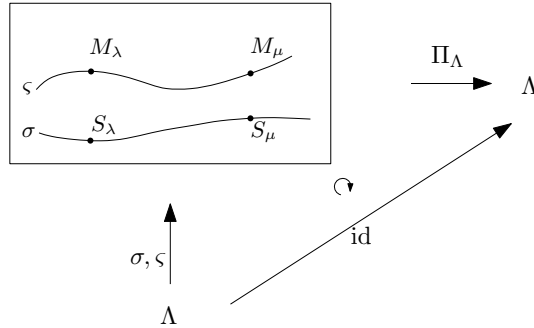


FIGURE 1. Sections from Definition 1.4.

The following Lemma 1.5 is a combination of Lemma 6.4 in [24] and of Proposition 2.9 in [17].

LEMMA 1.5 (McCord, Mischaikov, Salamon).

- (a) Let $\gamma: \Lambda \rightarrow \mathcal{S}$ be a section, then γ is continuous if and only if

$$S = \bigcup_{\lambda \in \Lambda} \gamma(\lambda)$$

is an isolated invariant set in $X \times \Lambda$.

(b) *Let*

$$S = \bigcup_{\lambda \in \Lambda} \sigma(\lambda), \quad M(\pi) = \bigcup_{\lambda \in \Lambda} \varsigma_{\pi}(\lambda) \quad \text{for any } \pi \in \mathbf{P}.$$

Then, S is an isolated invariant set in $X \times \Lambda$ under ϕ and $\mathcal{M}(S) = \{M(\pi) \mid \pi \in (\mathbf{P}, <)\}$ is its Morse decomposition if and only if $\mathcal{M}(S)$ with its order continues.

Note that, by Lemma 1.5, Definition 1.4 is equivalent to the definitions of continuation with order presented in [24], [10], [17] and [11].

By Lemma 1.5, the *minimal order* $<_m$ for $\mathcal{M}(S)$ that continues over Λ is the flow defined order for $\mathcal{M}(S)$. Note that if $\mathcal{M}(S)$ with order $<$ continues then $<$ extends $<_m$.

PROPOSITION 1.6. *If $p <_m q$ then there exist $s_1, \dots, s_n \in [0, 1]$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the set of connecting orbits $C(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i))$ is nonempty, where $\omega: [0, 1] \rightarrow \Lambda$ is a path between λ and μ . We call these connections, unordered chain connections, in short, ucc.*

PROOF. Since a Morse set of $\mathcal{M}(S)$ is

$$M(\pi) = \bigcup_{\lambda \in \Lambda} \varsigma_{\pi}(\lambda),$$

then $p <_m q$ implies that there is a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the set of connecting orbits $C(M(p_{i-1}), M(p_i))$ is nonempty. Note that the connecting orbit between $M(p_{i-1})$ and $M(p_i)$ occurs at some parameter in Λ . Therefore, we have the desired result whenever $\mathcal{M}(S)$ continues over a path $\omega: [0, 1] \rightarrow \Lambda$. \square

Now we have the necessary framework to state the following results in [10] and [11], which we use subsequently.

We have the following continuation theorem for homology index braids of admissible orderings of Morse decompositions.

THEOREM 1.7 (Franzosa). *If the admissible orderings $<_{\lambda}$ and $<_{\mu}$ are related by continuation, then $\mathcal{H}(<_{\lambda})$ and $\mathcal{H}(<_{\mu})$, the homology index braids of the admissible orderings, are isomorphic.*

The next global continuation theorem for connection matrices of admissible orderings of Morse decompositions follows from the previous theorem.

THEOREM 1.8 (Franzosa). *If the admissible orderings $<_{\lambda}$ and $<_{\mu}$ are related by continuation, then $\mathcal{CM}(<_{\lambda}) = \mathcal{CM}(<_{\mu})$.*

The following proposition describes the relationship between connection matrices of Morse decompositions if the flow ordering of one Morse decomposition

continues to an admissible ordering of another, since every admissible ordering is an extension of the flow ordering.

PROPOSITION 1.9 (Franzosa). *Let $<_1$ and $<_2$ be admissible orderings for $\mathcal{M}(S)$ and assume that $<_1$ is an extension of $<_2$. Then*

$$\mathcal{CM}(<_2) \subseteq \mathcal{CM}(<_1).$$

The collection of connection matrices of a Morse decomposition is upper semicontinuous over the space of Morse decompositions and over the parameter space Λ .

THEOREM 1.10 (Franzosa). *There exists a neighbourhood W of λ in Λ such that if $\mu \in W$, then M_λ is related by continuation with order to a Morse decomposition M_μ of an isolated invariant set in X_μ , and for such M_μ , $\mathcal{CM}(M_\mu) \subseteq \mathcal{CM}(M_\lambda)$.*

The next proposition is not hard to verify and it can be found in [11].

PROPOSITION 1.11 (Franzosa, Mischaikow). *Let $C = \{C(p)\}_{p \in \mathbf{P}}$ and $C' = \{C'(p)\}_{p \in \mathbf{P}}$ be collections of graded modules, and $\Delta: C(\mathbf{P}) \rightarrow C(\mathbf{P})$, $\Delta': C'(\mathbf{P}) \rightarrow C'(\mathbf{P})$ be $<$ -upper triangular boundary maps. If $T: C(\mathbf{P}) \rightarrow C'(\mathbf{P})$ is $<$ -upper triangular and such that $T\Delta = \Delta'T$, then $\{T(\mathbf{I})\}_{\mathbf{I} \in \mathbf{I}(<)}$ is a chain map from $\mathcal{C}\Delta$ to $\mathcal{C}\Delta'$.*

2. Generalization of the topological transition matrix

In this section we introduce the (generalized) topological transition matrix. As we mentioned previously, this provides a generalization of the classical topological transition matrix because we do not require that there are no connections at the initial and final parameters of a continuation, and therefore the general case applies to a much broader class of dynamical systems than the classical case. Furthermore, we show that there is no loss in the bifurcation information that can be obtained in the general case in comparison to the classical case (see Theorem 5 in comparison to the corresponding classical result presented here as Theorem 4). But first, we introduce a quick review of the classical topological transition matrix in [17] and [18].

Suppose that S_0 and S_1 are invariant sets related by continuation in X_{λ_0} and X_{λ_1} . Hence, there exist a map $\omega: [0, 1] \rightarrow \Lambda$ such that $\omega(0) = \lambda_0$ and $\omega(1) = \lambda_1$ and an isolated invariant set S over $\omega(I)$ such that $S_{\lambda_i} = S_i$. The inclusion $f_i: X_{\lambda_i} \rightarrow X \times \omega(I)$ induces an isomorphism $CH_*(S_i) \xrightarrow{f_{i*}} CH_*(S)$, where $CH_*(S_i)$ and $CH_*(S)$ indicate the Conley homology indices of S_i in X_{λ_i} and of S in $X \times \omega(I)$, respectively. Thus, there is an isomorphism

$$F_\omega: CH_*(S_0) \xrightarrow{f_{1*}^{-1} \circ f_{0*}} CH_*(S_1)$$

that depends on the endpoint-preserving homotopy class ω . If $\pi_1(\Lambda) = 0$ then F_ω is independent of the path ω and one writes F_{λ_1, λ_2} instead of F_ω . The flow-defined continuation isomorphism is well-behaved with respect to composition of paths: $F_{\lambda, \lambda} = \text{id}$, $F_{\mu, \nu} \circ F_{\lambda, \mu} = F_{\lambda, \nu}$ and $F_{\lambda, \mu} = F_{\mu, \lambda}^{-1}$. For more details see [18] and [24].

Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi \in \mathbf{P}}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi \in \mathbf{P}}$ be Morse decompositions, related by continuation, for the isolated invariant sets $S_\lambda \subseteq X_\lambda$ and $S_\mu \subseteq X_\mu$, respectively. In this setting, let

$$\Lambda' = \left\{ \lambda \in \Lambda \mid S_\lambda = \bigcup_{p \in \mathbf{P}} M_\lambda(p) \right\}$$

be a parameter set in which the corresponding Morse decomposition does not have connecting orbits. Choose a path ω from λ to μ , where $\lambda, \mu \in \Lambda'$. To simplify the notation, we denote $CH_*(M_\nu(\mathbf{I})) = H_{*, \nu}(\mathbf{I})$ or just $CH(M_\nu(\mathbf{I})) = H_\nu(\mathbf{I})$, where $\mathbf{I} \in \mathbf{I}(<_\nu)$ and $\nu \in \{\lambda, \mu\}$.

By Conley’s theory we have that there is an isomorphism $\Phi_\lambda: C_*\Delta_\lambda(\mathbf{P}) \rightarrow H_{*, \lambda}(\mathbf{P})$ for $\lambda \in \Lambda'$, where $C_*\Delta_\lambda(\mathbf{P}) = \bigoplus_{\pi \in \mathbf{P}} CH(M_\lambda(\pi))$ is the chain complex with the connection matrix Δ_λ .

Therefore, we can carry out the continuation along the path ω in two ways: first by continuing S_λ along the path ω using the isomorphism $F_{\lambda, \mu}$; secondly continuing $\bigcup_{p \in \mathbf{P}} M_\lambda(p)$ along the path ω by using isomorphism $E_{\lambda, \mu} = \bigoplus_{p \in \mathbf{P}} F_{\lambda, \mu}(M(p))$.

More precisely, we have the following diagram:

$$\begin{array}{ccc} C\Delta_\lambda(\mathbf{P}) & \xrightarrow{E_{\lambda, \mu}} & C\Delta_\mu(\mathbf{P}) \\ \Phi_\lambda \downarrow & & \downarrow \Phi_\mu \\ H_\lambda(\mathbf{P}) & \xrightarrow{F_{\lambda, \mu}} & H_\mu(\mathbf{P}). \end{array}$$

In general the diagram above is not commutative. Due to the lack of commutativity, one is able to obtain information about connection orbits. Fix a base \mathfrak{B}_λ in $C_*\Delta_\lambda$ and use the isomorphism $E_{\lambda, \mu}$ in order to define a base $E_{\lambda, \mu}(\mathfrak{B}_\lambda)$ in $C_*\Delta_\mu$. The composition $T_{\lambda, \mu} = \Phi_\mu^{-1} \circ F_{\lambda, \mu} \circ \Phi_\lambda$ can be represented as a matrix with respect to those bases and such matrix is called a *(classical) topological transition matrix*.

The following theorem in [17] summarizes some important properties for this matrix, which we refer to, from now on, as the classical topological transition matrix.

THEOREM 2.1 (McCord–Mischaikow). *Let $\Lambda' \subseteq \Lambda$ be such that for all λ and $\mu \in \Lambda'$ there are no connection orbits in M_λ and M_μ , and M_λ and M_μ are related by continuation. Then:*

- (a) $\Delta_\mu T_{\lambda,\mu} + T_{\lambda,\mu} \Delta_\lambda = 0$.
- (b) $T_{\lambda,\mu}$ is an isomorphism.
- (c) $T_{\lambda,\mu}$ is upper triangular matrix with respect to order $<$.
- (d) If $\nu \in \Lambda'$ then $T_{\lambda,\lambda} = \text{id}$, $T_{\lambda,\nu} = T_{\mu,\nu} \circ T_{\lambda,\mu}$ and $T_{\mu,\lambda} = T_{\lambda,\mu}^{-1}$.
- (e) If $T_{\lambda,\mu}(p, q) \neq 0$ and ω is a path between λ and μ , then there exist a finite sequence $0 < s_1 \leq \dots \leq s_n < 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the connecting orbit set $C(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i))$ is nonempty.

Item (a) of the theorem above is trivial since $\Delta_\lambda = \Delta_\mu = 0$, and $T_{\lambda,\mu}$ has the property of being unique, since $T_{\lambda,\mu}$ is a composition of isomorphisms.

DEFINITION 2.2. Given chain complex braids \mathcal{C} and \mathcal{C}' and graded module braids \mathcal{G} and \mathcal{G}' , a chain map $\mathcal{T}: \mathcal{C} \rightarrow \mathcal{C}'$ is said to *cover* an isomorphism θ (relative to Φ and Φ') if for all $\mathbf{I} \in \mathbf{I}(<)$, we have that the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{HC}(\mathbf{I}) & \xrightarrow{\mathcal{T}_*(\mathbf{I})} & \mathcal{HC}'(\mathbf{I}) \\
 \Phi(\mathbf{I}) \downarrow & & \downarrow \Phi'(\mathbf{I}) \\
 \mathcal{G}(\mathbf{I}) & \xrightarrow{\theta(\mathbf{I})} & \mathcal{G}'(\mathbf{I}),
 \end{array}$$

where $\mathcal{T}_*(\mathbf{I})$ is the homology map induced by the chain map $\mathcal{T}(\mathbf{I})$, $\Phi: \mathcal{HC} \rightarrow \mathcal{G}$ and $\Phi': \mathcal{HC}' \rightarrow \mathcal{G}'$ are isomorphisms from the homology of the chain complex braid to the graded module braid.

DEFINITION 2.3. If, in Definition 2.2, \mathcal{C} and \mathcal{C}' arise from connection matrices $\Delta: \bigoplus_{\mathbf{P}} C(p) \rightarrow \bigoplus_{\mathbf{P}} C(p)$, $\Delta': \bigoplus_{\mathbf{P}} C'(p) \rightarrow \bigoplus_{\mathbf{P}} C'(p)$, respectively, and \mathcal{T} arises from a matrix $T: \bigoplus_{\mathbf{P}} C(p) \rightarrow \bigoplus_{\mathbf{P}} C'(p)$ then T is called a *generalized transition matrix* for Δ and Δ' .

In a forthcoming paper we will further explore properties of the generalized transition matrix and demonstrate how it generalizes all four transition matrices, namely, singular [22], topological [17], algebraic [11] and directional [15]. In this paper we prove generalizations of the definition and properties of the classical topological transition matrices. With this in mind we restrict Definition 2.3 in order to obtain a new and broader definition for a topological transition matrix.

DEFINITION 2.4. If T is a generalized transition matrix that covers the flow-defined continuation isomorphism F , then we refer to T as a *generalized topological transition matrix*.

As we mentioned previously, an advantage of this definition over the definition of the classical topological transition is that here we do not need the

restrictive requirement that there are no connections between the Morse sets at the end parameter values in a continuation. In Example 2.11 in this section and in Sections 3 and 4 we provide some dynamical examples where we apply the generalized topological transition matrix in settings where it is not possible to apply the classical topological transition matrix.

We have the following characterization result.

PROPOSITION 2.5. *T is a generalized topological transition matrix related to the connection matrices $(\Delta_\lambda, \Phi_\lambda)$ and (Δ_μ, Φ_μ) if and only if*

$$T: \bigoplus_{p \in \mathcal{P}} CH_*(M_\lambda(p)) \rightarrow \bigoplus_{p \in \mathcal{P}} CH_*(M_\mu(p))$$

is a zero degree map such that

- (a) $\{T(\mathbf{I})\}_{\mathbf{I} \in I(<)}$ is a chain map from $\mathcal{C}\Delta_\lambda$ to $\mathcal{C}\Delta_\mu$;
- (b) the following diagram:

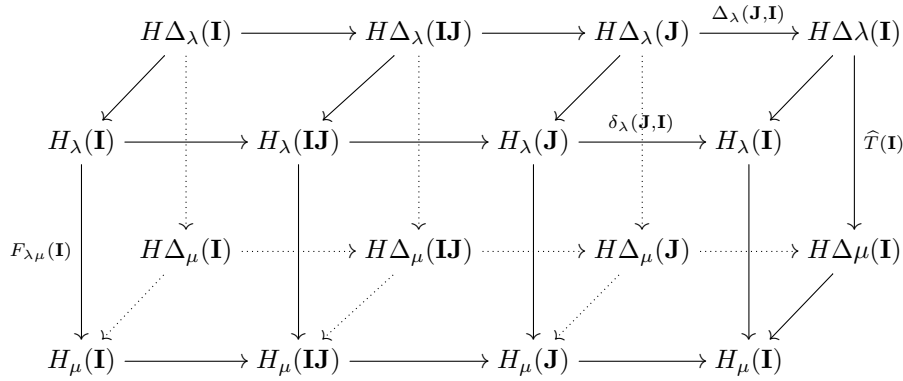


DIAGRAM 1

commutes for all adjacent pairs (\mathbf{I}, \mathbf{J}) , where $\widehat{T}(\cdot)$ is the induced homology map of $T(\cdot)$.

PROOF. By Definition 2.4, the diagram commuting in the transversal sections implies that the whole diagram commutes. This follows easily since the top and bottom diagrams commute by the definition of the connection matrix; the diagram in the background commutes because T is a chain map, and lastly the diagram in the foreground commutes by the continuation of the homology index braid (Theorem 1.7) which, in part, asserts that for all adjacent pairs (\mathbf{I}, \mathbf{J}) the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_\lambda(\mathbf{I}) & \longrightarrow & H_\lambda(\mathbf{IJ}) & \longrightarrow & H_\lambda(\mathbf{J}) \xrightarrow{\delta_\lambda(\mathbf{J},\mathbf{I})} H_\lambda(\mathbf{I}) \longrightarrow \dots \\ & & \downarrow F(\mathbf{I}) & & \downarrow F(\mathbf{IJ}) & & \downarrow F(\mathbf{J}) & & \downarrow F(\mathbf{I}) \\ \dots & \longrightarrow & H_\mu(\mathbf{I}) & \longrightarrow & H_\mu(\mathbf{IJ}) & \longrightarrow & H_\mu(\mathbf{J}) \xrightarrow{\delta_\mu(\mathbf{J},\mathbf{I})} H_\mu(\mathbf{I}) \longrightarrow \dots \quad \square \end{array}$$

Denote GTTM ($<$) as the set of all generalized topological transition matrices with the partial order $<$.

When there are no connections in the λ and μ parameters, then $\Delta_\lambda = 0 = \Delta_\mu$. Thus, the induced homology map $\widehat{T} = T$ and by choosing an adjacent pair (\mathbf{I}, \mathbf{J}) such that $\mathbf{P} = \mathbf{IJ}$, from Diagram 1 we have that the following diagram commutes:

$$\begin{array}{ccc} H_\lambda(\mathbf{P}) & \longrightarrow & C\Delta_\lambda(\mathbf{P}) \\ F(\mathbf{P}) \downarrow & & \downarrow T(\mathbf{P}) \\ H_\mu(\mathbf{P}) & \longrightarrow & C\Delta_\mu(\mathbf{P}). \end{array}$$

Therefore, $T(\mathbf{P})$ is a classical topological transition matrix. Thus, generalized topological transition matrices encompass the classical topological transition matrices.

PROPOSITION 2.6. *The classical topological transition matrix is a particular case of the generalized topological transition matrix.*

Although the next result is straightforward from Definition 2.5, it is worthwhile to emphasize its importance, since given two connection matrices related by continuation there exist some entries that are the same for the matrices Δ_λ and Δ_μ . More accurately,

PROPOSITION 2.7. *Let $p, q \in \mathbf{P}$ be such that either p and q are not related by order $<$ or $p < q$. If the pair $(\{p\}, \{q\})$ is an adjacent pair, i.e. there is no $p' \in \mathbf{P}$ such that $p < p' < q$, then $\Delta_{qp,\lambda} = T^{-1}(\{q\}) \circ \Delta_{qp,\mu} \circ T(\{p\})$.*

PROOF. By hypothesis, $(\{p\}, \{q\})$ is an adjacent pair, then by the definition of the generalized topological transition matrix we have that the following diagram commutes:

$$\begin{array}{ccc} H\Delta_\lambda(\{p\}) & \xrightarrow{\Delta_\lambda(\{q\}, \{p\})} & H\Delta_\lambda(\{q\}) \\ \widehat{T}(\{p\}) \downarrow & & \downarrow \widehat{T}(\{q\}) \\ H\Delta_\mu(\{p\}) & \xrightarrow{\Delta_\mu(\{q\}, \{p\})} & H\Delta_\mu(\{q\}) \end{array}$$

which can be rewritten as

$$\begin{array}{ccc} CH(M_\lambda(p)) & \xrightarrow{\Delta_{qp,\lambda}} & CH(M_\lambda(q)) \\ T(\{p\}) \downarrow & & \downarrow T(\{q\}) \\ CH(M_\mu(p)) & \xrightarrow{\Delta_{qp,\mu}} & CH(M_\mu(q)). \end{array}$$

Since $T(\{p\})$ and $T(\{q\})$ are isomorphisms it follows that

$$\Delta_{qp,\lambda} = T^{-1}(\{q\}) \circ \Delta_{qp,\mu} \circ T(\{p\}). \quad \square$$

In this paper, for the sake of simplicity, we do not address the existence of the generalized transition matrix, since the existence problem is related to the unification of the transition matrix theory. In a forthcoming paper, we have obtained existence results, nevertheless, here we establish the existence in particular cases as one can observe in Corollary 2.9 and Theorem 3.1.

The primary goal in this section is to establish properties of the generalized topological transition matrix – including connecting orbit existence results – corresponding to those of the classical topological transition matrix. In the last two sections we present applications of the generalized topological transition matrix.

The following properties of generalized topological transition matrices are an extension of Theorem 2.1.

THEOREM 2.8. *Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi \in (\mathbf{P}, <_\lambda)}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi \in (\mathbf{P}, <_\mu)}$ be Morse decompositions, Δ_λ and Δ_μ their respective connection matrices with the flow-defined order. Moreover, assume that M_λ and M_μ are related by continuation with an admissible ordering $<$. Then the generalized topological transition matrix T satisfies the following properties:*

- (a) $T \circ \Delta_\lambda = \Delta_\mu \circ T$.
- (b) $T_{\lambda, \mu}(\{p\}) = \text{id}$ and T is upper triangular with respect to $<$.
- (c) T is an isomorphism.
- (d) $T_{\lambda, \lambda} = \text{id}$, $T_{\lambda, \nu}(\mathbf{I}) = T_{\mu, \nu} \circ T_{\lambda, \mu}(\mathbf{I})$ and $T_{\mu, \lambda}(\mathbf{I}) = T_{\lambda, \mu}^{-1}(\mathbf{I})$ are generalized topological transition matrices, for all intervals $\mathbf{I} \in \mathcal{I}$ and $p \in \mathbf{P}$, in particular $T = T(\mathbf{P})$.
- (e) Let $\omega: [0, 1] \rightarrow \Lambda$ be a path that continues M_λ to M_μ . Assume that $T_{\lambda, \mu}(p, q) \neq 0$ for all generalized topological transition matrices and $\text{GTTM}(<_m) \neq \emptyset$ for all $\omega[s, t]$, where $s, t \in [0, 1]$. Then there exist a finite sequence $0 \leq s_1 \leq \dots \leq s_n \leq 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the set of connecting orbits $C(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i))$ is nonempty.

Before proving Theorem 2.8, we present the following corollary.

COROLLARY 2.9. *Under the same hypothesis as Theorem 2.8, assume that $T_{\lambda, \mu}(p, q) \neq 0$ and $\Delta_\lambda(\mathbf{K}) = 0 = \Delta_\mu(\mathbf{K})$, where \mathbf{K} is an interval that contains p and q . Then there exist a finite sequence $0 \leq s_1 \leq \dots \leq s_n \leq 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the set of connecting orbits $C(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i))$ is nonempty.*

PROOF. Since $\Delta_\lambda(\mathbf{K}) = 0 = \Delta_\mu(\mathbf{K})$ then

$$H\Delta_\lambda(\mathbf{K}) = \bigoplus_{\pi \in \mathbf{K}} CH(M_\lambda(\pi)) \quad \text{and} \quad H\Delta_\mu(\mathbf{K}) = \bigoplus_{\pi \in \mathbf{K}} CH(M_\mu(\pi)),$$

hence $\text{GTTM}(<_m) = \{T_{\lambda,\mu}(\mathbf{K}) = \Phi_\mu^{-1} \circ F \circ \Phi_\lambda(\mathbf{K})\} \neq \emptyset$. Therefore, by item (e) of Theorem 2.8, one just needs that $T_{\lambda,\mu}(p, q) \neq 0$ in order to prove the result. \square

Note that when $\Delta_\lambda(\mathbf{K}) = 0 = \Delta_\mu(\mathbf{K})$ we cannot use the classical topological matrix to obtain Corollary 2.9, since $\Delta_\lambda(\mathbf{K}) = 0$ does not imply that there is no connection at the parameter λ . Actually one can use Corollary 2.9 whenever there is no connection at parameter λ and μ in order to prove item (e) of Theorem 2.1.

PROOF OF THEOREM 2.8 (a)–(d). (a) Since $\Delta_\lambda(\mathbf{I})$ and $\Delta_\mu(\mathbf{I})$ are boundary maps and $T_{\lambda,\mu}(\mathbf{I})$ is a chain map, we have that $T_{\lambda,\mu}(\mathbf{I}) \circ \Delta_\lambda(\mathbf{I}) = \Delta_\mu(\mathbf{I}) \circ T_{\lambda,\mu}(\mathbf{I})$ for all \mathbf{I} .

(b) Fix a base \mathfrak{B}_λ for the domain then $\mathfrak{B}_\mu = \bigoplus_{p \in \mathbf{P}} F(p)(\mathfrak{B}_\lambda)$ is a base for the codomain of the map

$$\bigoplus_{p \in \mathbf{P}} F(p): \bigoplus_{p \in \mathbf{P}} H_\lambda(p) \rightarrow \bigoplus_{p \in \mathbf{P}} H_\mu(p).$$

Therefore, $\Phi_\lambda^{-1}(\mathfrak{B}_\lambda)$ and $\Phi_\mu^{-1}(\mathfrak{B}_\mu)$ are bases for the domain and codomain of the map $\bigoplus_{p \in \mathbf{P}} T_*(\{p\})$, i.e. $T_*(\{p\}) = \text{id}$ and, since $T_*(\{p\}) = T(\{p\})$, then $T(\{p\}) = \text{id}$.

In order to prove that T is upper triangular it is enough to prove $T_{q,p} = 0$ for $q \not\prec p$. Indeed, let \mathbf{I} be an interval that has p and q at the ends, and choose the adjacent pair $(p, \mathbf{I} \setminus p)$. It follows that

$$\begin{aligned} [T(\mathbf{I})(\alpha \oplus 0)] &= T_*(\mathbf{I})[\alpha \oplus 0] = T_*(\mathbf{I}) \circ i(\alpha) = i \circ T_*(\{p\})(\alpha) \\ &= i \circ T(\{p\})(\alpha) = i(\dot{\alpha}) = [i(\dot{\alpha})] = [\dot{\alpha} \oplus 0], \end{aligned}$$

where $\alpha \in C(p)$, $\dot{\alpha} = T(\{p\})(\alpha)$ and $0 \in \bigoplus_{\pi \in \mathbf{I} \setminus p} C(\pi)$. Therefore,

$$T(\mathbf{I})(\alpha \oplus 0) - \dot{\alpha} \oplus 0 \in \text{Im } \Delta_\mu(\mathbf{I})$$

and since $\Delta'(\mathbf{I})$ is an upper triangular boundary map then there exists $\beta \in \bigoplus_{\mathbf{I}} C'(\pi)$ such as

$$T(\alpha \oplus 0) - \dot{\alpha} \oplus 0 = \Delta'(\mathbf{I})(\beta) = d \oplus 0, \quad \text{where } d \in \bigoplus_{\pi \in \mathbf{I} \setminus q} C'(\pi).$$

As $T(\alpha \oplus 0) = \dot{\alpha} \oplus \dots \oplus T_{q,p} \cdot \alpha$ then $T_{q,p} \cdot \alpha = 0$, thus $T_{q,p} = 0$.

(c) By item (b), we have that T is an upper triangular matrix with nonzero entries on the diagonal, thus T is an isomorphism.

(d) By the properties of the continuation isomorphism, specifically $F_{\lambda,\lambda} = \text{id}$, $F_{\mu,\nu} \circ F_{\lambda,\mu} = F_{\lambda,\nu}$ and $F_{\lambda,\mu} = F_{\mu,\lambda}^{-1}$, the result follows. \square

One needs a technical lemma in order to prove item (e) of Theorem 2.8.

LEMMA 2.10. *Let $M_\lambda = \{M_\lambda(\pi)\}_{\pi \in (\mathbf{P}, <_\lambda)}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi \in (\mathbf{P}, <_\mu)}$ be Morse decompositions, Δ_λ and Δ_μ their respective connection matrices with the flow-defined order. Moreover, assume that M_λ and M_μ are related by continuation with an admissible ordering $<$. If $T_{\lambda,\mu}(p, q) \neq 0$ for all generalized topological transition matrices and $\text{GTTM}(<_m) \neq \emptyset$ then there is a ucc from q to p .*

PROOF. Since $T(p, q) \neq 0$ for all $T \in \text{GTTM}(<)$, then $T(p, q) \neq 0$ for all $T \in \text{GTTM}(<_m)$, given that $<_m$ is the minimal order that continues. Therefore, by T being $<_m$ -upper triangular, we have that $p <_m q$. \square

PROOF OF THEOREM 2.8 (e). By Lemma 2.10, we have $T_{pq} \neq 0$ for all $T \in \text{GTTM}(<)$ implies that there exists a ucc from q to p .

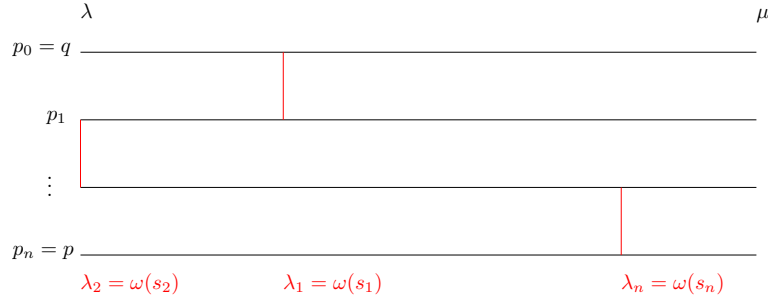


FIGURE 2. Unordered chain connections from q to p .

It remains to prove that $s_i \leq s_{i+1}$, i.e. the connections occur along the path ω . Since $\text{GTTM}(<_m) \subseteq \text{GTTM}(<)$ then it is enough to prove the result for $T_{\lambda\mu}(p, q) \neq 0$ for the order $<_m$. Indeed, we will prove by induction on the numbers of elements m that \mathbf{K} has.

Case $m = 2$. Follows directly from Lemma 2.10.

Assume the result is true for $k < m + 1$. Let $0 < s_\xi < 1$ and $\xi = \omega(s_\xi)$. Since $T_{\lambda\mu}(p, q) \neq 0$ for all $T_{\lambda\mu} \in \text{GTTM}_{\lambda\mu}(<)$, it follows that for all $T_{\xi\mu} \in \text{GTTM}_{\xi\mu}(<)$ and for all $T_{\lambda\xi} \in \text{GTTM}_{\lambda\xi}(<)$ there exists $0 \leq j \leq m$ such that

$$T_{\xi\mu}(q_0, q_j) \cdot T_{\lambda\xi}(q_j, q_m) \neq 0.$$

See Figure 3.

Since j depends on the choice of $T_{\xi\mu}$ and $T_{\lambda\xi}$, we will fix j for $T_{\xi\mu} \in \text{GTTM}_{\xi\mu}(<_{m_{\xi\mu}})$ and $T_{\lambda\xi} \in \text{GTTM}_{\lambda\xi}(<_{m_{\lambda\xi}})$, given that

$$\begin{aligned} \text{GTTM}_{\xi\mu}(<_{m_{\xi\mu}}) &\subseteq \text{GTTM}_{\xi\mu}(<_m) \subseteq \text{GTTM}_{\xi\mu}(<), \\ \text{GTTM}_{\lambda\xi}(<_{m_{\lambda\xi}}) &\subseteq \text{GTTM}_{\lambda\xi}(<), \end{aligned}$$

where $<_{m_{\xi\mu}}$ and $<_{m_{\lambda\xi}}$ are the minimal orders that continue for $\omega[s_\xi, 1]$ and for $\omega[0, s_\xi]$, respectively.

$$p = q_0 \begin{pmatrix} & & q_j \\ \text{id} & & * \\ & \ddots & \\ & & \text{id} \end{pmatrix} \begin{pmatrix} \text{id} & & \\ & \ddots & \\ & & \begin{matrix} * \\ \text{id} \end{matrix} \\ & & q_j \end{pmatrix} = \begin{pmatrix} \text{id} & & \approx \\ & \ddots & \\ & & \text{id} \end{pmatrix}$$

$T_{\xi\mu} \qquad T_{\lambda\xi} \quad q_m = q \qquad T_{\lambda\mu}$

FIGURE 3

If $j \neq 0$ and $j \neq m$ consider the submatrices in Figure 3, $(T_{\xi\mu}(q_l, q_k))_{l,k \in [0,j]}$ and $(T_{\lambda\xi}(q_l, q_k))_{l,k \in [j,m]}$. By the induction hypothesis, there exist connections between $M_\lambda(q_m)$ and $M_\xi(q_j)$ and between $M_\xi(q_j)$ and $M_\mu(q_0)$ that occur along the path ω . Using these connections, we get connections between $M_\lambda(q_m)$ and $M_\mu(q_0)$ that occur along the path ω . See Figure 4.

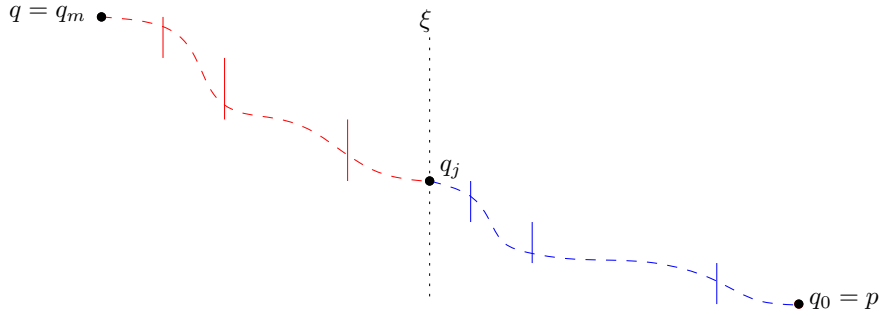


FIGURE 4. Connections between $M_\lambda(q_m)$ and $M_\mu(q_0)$ that occur along the path ω .

If $j = 0$ or $j = m$ then $T_{\lambda\xi}(p, q) \neq 0$ or $T_{\xi\mu}(p, q) \neq 0$, respectively. Suppose $T_{\xi\mu}(p, q) \neq 0$ then let $s_\xi < s_{\xi_1} < 1$ and $\xi_1 = \omega(s_{\xi_1})$. If for ξ and μ there exist $j_{\xi_1} \neq 0$ and $j_{\xi_1} \neq m$ the result follows. Nevertheless if $j_{\xi_1} = 0$ ($T_{\xi\xi_1}(p, q) \neq 0$) choose $s_\xi < s_{\xi_2} < s_{\xi_1}$ and $\xi_2 = \omega(s_{\xi_2})$ and if $j_{\xi_1} = m$ ($T_{\xi_1\mu}(p, q) \neq 0$) choose $s_{\xi_1} < s_{\xi_2} < 1$ and $\xi_2 = \omega(s_{\xi_2})$. See Figure 5.

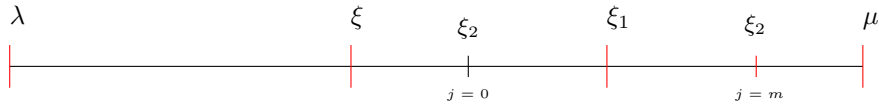


FIGURE 5

By respecting this process and assuming always that $j_{\xi_i} = 0$ or $j_{\xi_i} = m$, it follows that there exists θ such that

$$\omega(s_{\xi_i}) = \xi_i \rightarrow \theta, \quad \omega(s_{\xi_{i'}}) = \xi_{i'} \rightarrow \theta, \quad 0 \leq s_{\xi_i} < s_{\xi_{i'}} \leq 1 \quad \text{and} \quad T_{\xi_i \xi_{i'}}(p, q) \neq 0.$$

Suppose that there is no chain of connections from q to p at the parameter θ , thus $p \not\prec_{\theta} q$, where \prec_{θ} is the flow defined order at parameter θ . By Theorem 1.10, there exists a neighbourhood W of θ such that $\mathcal{M}(S)$ continues with order \prec_{θ} over W , therefore \prec_{θ} extends \prec_{m_W} , where \prec_{m_W} is the minimal order that continues over W . Choose ξ_l and $\xi_{l'}$ such that $\theta \in \omega[s_{\xi_l}, s_{\xi_{l'}}] \subset W$. It follows that $T_{\xi_l \xi_{l'}}(p, q) \neq 0$, i.e., there exists a *ucc* from q to p in $\omega[s_{\xi_l}, s_{\xi_{l'}}]$. Thus $p \prec_{m_W} q$. Since \prec_{θ} extends \prec_{m_W} we have that $p \prec_{\theta} q$ for the Morse decomposition $\mathcal{M}(S)$, which is a contradiction, given that $p \not\prec_{\theta} q$. Therefore the result follows. \square

The following Example 2.11 shows that the connections obtained from item (e) of Theorem 2.8 can occur at parameter λ or μ . This contrasts Theorem 2.1 in [17] and Theorem 3.13 in [22], where those connections cannot occur at parameters λ and μ . However, when they do occur, one sees in Example 2.11 that the classical results on transition matrices (topological, singular and algebraic) do not apply and therefore do not provide information on connections.

EXAMPLE 2.11. Consider the following family of ordinary differential equations parameterized by the variable $\theta > 0$:

$$\dot{x} = y, \quad \dot{y} = -\theta y - x \left(x - \frac{1}{3} \right) (1 - x).$$

The connection matrices for $\theta > 0$ are well known, see [9], [10] and [22]. Let μ be the parameter which has a heteroclinic connection between the Morse sets $M_{\mu}(2)$ and $M_{\mu}(3)$, and $0 < \lambda < \mu$. The order that continues is the total order and the set of connection matrices are

$$\begin{pmatrix} 0 & \approx & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \approx & \approx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, it is easy to see that $\text{id} \in \mathcal{T}_{\lambda, \mu}^U$, where $\mathcal{T}_{\lambda, \mu}^U$ is the set of algebraic transition matrices. Hence, Theorem 4.3 in [11], does not apply and one cannot retrieve dynamical information from $\mathcal{T}_{\lambda, \mu}^U$.

Now, in order to calculate the singular transition matrix, first introduce a slow drift $\dot{\theta} = \varepsilon(\lambda - \theta)(\mu - \theta)$ in the parameter space, see Figure 6. Since there are two connection matrices for the flow ordering at parameter μ , choose

$$\Delta_{\mu} = \begin{pmatrix} 0 & \approx & \approx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

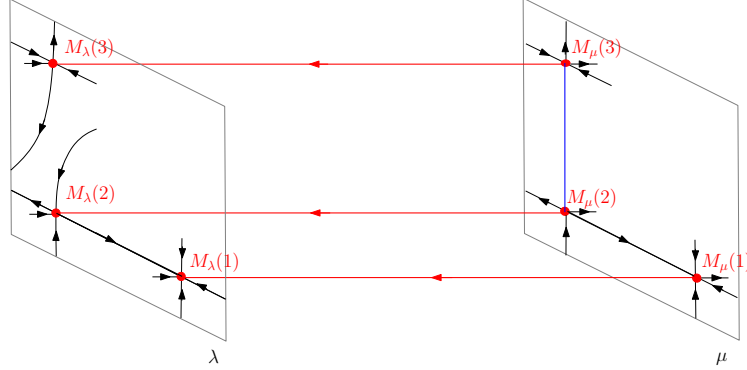


FIGURE 6. Drift flow in the parameter space.

The associated singular transition matrix is

$$\Delta = \begin{pmatrix} \Delta_\lambda & T_s \\ 0 & \Delta_\mu \end{pmatrix} = \begin{pmatrix} 0 & \approx & 0 & \approx & & & \\ 0 & 0 & 0 & & \approx & & * \\ 0 & 0 & 0 & & & \approx & \approx \\ 0 & 0 & 0 & 0 & \approx & \approx & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}.$$

Since $\Delta^2 = 0$ must be zero, we see that multiplying the top row and the last column in Δ forces $* = \Delta(M_\mu(3), M_\lambda(2)) \neq 0$. Even though the entry $T_s(p, q) \neq 0$, one cannot conclude from T_s that there exists a connection between $M_\lambda(2)$ and $M_\mu(3)$ for $\epsilon > 0$ sufficiently small. Thus, one cannot apply Theorem 3.13 in [22]. Therefore, in this example, singular transition matrices do not give dynamical information. The same thing happens for classical topological transition matrices, since they are only defined when there are no connections at parameters λ and μ .

Now, we will calculate a generalized topological transition matrix related with

$$\Delta_\lambda = \begin{pmatrix} 0 & \approx & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Delta_\mu = \begin{pmatrix} 0 & \approx & \approx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is not hard to see that

$$T_{\lambda,\mu} = \begin{pmatrix} \approx & 0 & 0 \\ 0 & \approx & \approx \\ 0 & 0 & \approx \end{pmatrix}$$

satisfies items (a)–(c) of Theorem 2.8, hence $T_{\lambda,\mu}$ is a good candidate to be a generalized topological transition matrix. Thus, it remains to prove that $T_{\lambda,\mu}$ covers the flow defined isomorphism F . Indeed, it is enough to prove that $T_{\lambda,\mu}(\mathbf{I})$

covers $F(\mathbf{I})$ for the interval $\mathbf{I} = \{2, 3\}$ ($T_{\lambda,\mu}$ covers F for others intervals, because $T_{\lambda,\mu}$ is a 0 degree chain map). For $\mathbf{I} = \{2, 3\}$ we have that $\Delta_\lambda(\mathbf{I}) = 0 = \Delta_\mu(\mathbf{I})$, therefore we can choose generators: α_ν and β_ν for $CH_1(M_\nu(\mathbf{I}))$; a_ν for $CH_1(M_\nu(3))$; and b_ν for $CH_1M_\nu(2)$, where $\nu = \lambda, \mu$. Figure 7 indicates such choices.

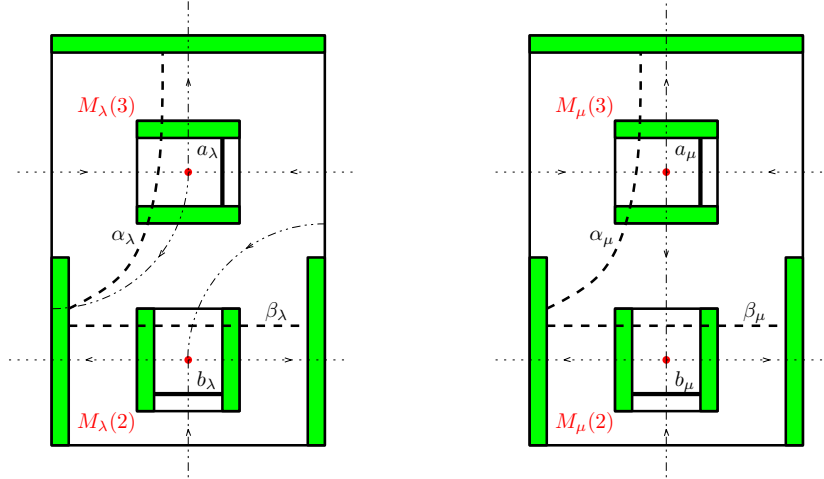


FIGURE 7. Generators of the Conley indices.

Observe that $F_{\lambda,\mu}(\mathbf{I})(\alpha_\lambda) = \alpha_\mu$, $F_{\lambda,\mu}(\mathbf{I})(\beta_\lambda) = \beta_\mu$, $\Phi_\lambda(a_\lambda) = \alpha_\lambda$, $\Phi_\lambda(b_\lambda) = \beta_\lambda$, $\Phi_\mu(a_\mu) = \alpha_\mu * \beta_\mu$ and $\Phi_\mu(b_\mu) = \beta_\mu$. Thus $T_{\lambda,\mu}(\mathbf{I}) = \Phi_\mu^{-1} \circ F_{\lambda,\mu} \circ \Phi_\lambda(\mathbf{I})$, which means that $T_{\lambda,\mu}(\mathbf{I})$ covers $F(\mathbf{I})$ and $T_{\lambda,\mu}(2, 3) \neq 0$.

Note that the total order $1 < 2 < 3$ is the minimal order that continues, therefore by Lemma 2.10 we have a connection between $M(2)$ and $M(3)$, since $T_{\lambda,\mu}(2, 3) \neq 0$. In fact, one could have used item (v) of Theorem 2.8, since it is not hard to obtain that $GTTM(<_m) \neq \emptyset$ for all $[s, t]$, where $s, t \in [\lambda, \mu]$.

3. Morse–Smale flows without periodic orbits

In this section, the generalized topological transition matrix for Morse–Smale flows without periodic orbits is presented. In other words, the Morse decomposition consists of hyperbolic rest points and whenever the stable manifold of $M(\pi)$ and the unstable manifold of $M(\pi')$ have nonempty intersection, it is transversal.

As one can see in [23], the connection matrix for Morse–Smale flows without periodic orbits is unique for the flow-defined order. It is no surprise that the generalized topological transition matrix is unique. This is verified in Theorem 3.1.

Furthermore, in [16] and [25], an alternative and easier way to compute the connection matrix in this setting is presented. Likewise, we show in Theorem 3.1

that the generalized topological transition matrix can be computed, without difficulty, from the set of the classical topological transition matrix.

THEOREM 3.1. *Let $M_\lambda = \{M_\mu(\pi)\}_{\pi \in \mathbf{P}}$ and $M_\mu = \{M_\mu(\pi)\}_{\pi \in \mathbf{P}}$ be Morse decompositions, and Δ_λ and Δ_μ be the respective connection matrices with the flow-defined order. Moreover, assume that M_λ and M_μ are related by continuation with the admissible ordering $<$ and the flow at λ and μ is Morse–Smale without periodic orbits. Then the generalized topological transition matrix T satisfies the following properties:*

- (a) $T \circ \Delta_\lambda = \Delta_\mu \circ T$.
- (b) $T_{\lambda,\mu}(\{p\}) = \text{id}$ and T is upper triangular with respect to $<$.
- (c) T is an isomorphism.
- (d) $T_{\lambda,\lambda} = \text{id}$, $T_{\lambda,\nu}(\mathbf{I}) = T_{\mu,\nu} \circ T_{\lambda,\mu}(\mathbf{I})$ and $T_{\mu,\lambda}(\mathbf{I}) = T_{\lambda,\mu}^{-1}(\mathbf{I})$, for all intervals $\mathbf{I} \in \mathcal{I}$.
- (e) Let $\omega: [0, 1] \rightarrow \Lambda$ be the path that continues M_λ and M_μ . If $T_{\lambda,\mu}(p, q) \neq 0$ then there exist a finite sequence $0 < s_1 \leq \dots \leq s_n < 1$ and a sequence $(p_i) \subseteq \mathbf{P}$ such that $p_0 = q$, $p_n = p$ and the set of connecting orbits $C(M_{\omega(s_i)}(p_{i-1}), M_{\omega(s_i)}(p_i))$ is nonempty.
- (f) $T_{\lambda,\mu}$ is unique.
- (g) The generalized topological transition matrix is a matrix in block form with submatrices being the classical topological transition matrix $T_{\text{top},i}$ of the critical points of index i

$$T_{\lambda,\mu} = \begin{pmatrix} T_{\text{top},0} & 0 & 0 & 0 \\ 0 & T_{\text{top},1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{\text{top},k} \end{pmatrix}.$$

PROOF. (g) We let Crit_i denote the set of critical points of index i , and $\text{Crit}_{i,i+1}$ denote the set of critical points that have indices i and $i+1$ along with all connecting orbits between them.

Since the order $<$ extends the flow ordering, then, without loss of generality, we can suppose that the columns of the generalized topological transition matrix T are ordered from the critical point of the lowest index to the largest index.

If $M(\mathbf{I}) \subseteq \text{Crit}_k$ then $T(\mathbf{I}) = T_{\text{top}}(\mathbf{I})$, since there is no connecting orbit between critical points with the same index at parameters λ and μ . Moreover, for $M_\lambda(p)$ and $M_\lambda(q)$ with different indices, we have that $T(p, q) = 0$, since T is a zero degree map. Therefore T must be a matrix in block form, and from

property (b) of Theorem 2.8, T must be an upper triangular matrix. Thus

$$T = \begin{pmatrix} T_{\text{top},0} & 0 & 0 & 0 \\ 0 & T_{\text{top},1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{\text{top},k} \end{pmatrix}$$

is a good candidate to be a generalized topological transition matrix.

Therefore, we will prove that:

- $\{T(\mathbf{I})\}_{\mathbf{I} \in \mathbf{I}(<)}$ is a chain map from $\mathcal{C}\Delta_\lambda$ to $\mathcal{C}\Delta_\mu$;
- T makes Diagram 1 commute for all adjacent pairs (\mathbf{I}, \mathbf{J}) .

Indeed, by Proposition 1.11, we only need to show that $T \circ \Delta_\lambda = \Delta_\mu \circ T$, i.e.

$$\begin{aligned} & \begin{pmatrix} T_{\text{top},0} & 0 & 0 & 0 \\ 0 & T_{\text{top},1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{\text{top},k} \end{pmatrix} \begin{pmatrix} 0 & \Delta_\lambda(\text{Crit}_1, \text{Crit}_0) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Delta_\lambda(\text{Crit}_k, \text{Crit}_{k-1}) \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Delta_\mu(\text{Crit}_1, \text{Crit}_0) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Delta_\mu(\text{Crit}_k, \text{Crit}_{k-1}) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_{\text{top},0} & 0 & 0 & 0 \\ 0 & T_{\text{top},1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{\text{top},k} \end{pmatrix}. \end{aligned}$$

By multiplying the matrices, one obtains

$$T_{\text{top},l-1} \circ \Delta_\lambda(\text{Crit}_l, \text{Crit}_{l-1}) = \Delta_\mu(\text{Crit}_l, \text{Crit}_{l-1}) \circ T_{\text{top},l}$$

for all $l \in \{1, \dots, k\}$. But this follows from the commutativity of the following diagram:

$$\begin{array}{ccc} H\Delta_\lambda(\text{Crit}_l) & \xrightarrow{\Delta_\lambda(\text{Crit}_l, \text{Crit}_{l-1})} & H\Delta_\lambda(\text{Crit}_{l-1}) \\ T(\text{Crit}_l) \downarrow & & \downarrow T(\text{Crit}_{l-1}) \\ H\Delta_\mu(\text{Crit}_l) & \xrightarrow{\Delta_\mu(\text{Crit}_l, \text{Crit}_{l-1})} & H\Delta_\mu(\text{Crit}_{l-1}) \end{array}$$

whereas $T(\text{Crit}_l) = T_{\text{top},l} = \Phi_\mu^{-1} \circ F(\text{Crit}_l) \circ \Phi_\lambda$.

Now we will prove that T makes Diagram 1 commute for all adjacent pairs (\mathbf{I}, \mathbf{J}) . Indeed, suppose without loss of generality, $M(\mathbf{I}) \subseteq \text{Crit}_{k-1,k}$ and $M(\mathbf{J}) \subseteq \text{Crit}_{k,k+1}$, and let $L_j = L \cap \text{Crit}_j$ where $L = \mathbf{I}$ or $L = \mathbf{J}$ and $j \in \{k-1, k, k+1\}$. First, we calculate the homologies from the long exact sequences that come from Diagram 1

$$\begin{aligned} \cdots \rightarrow H_{k+1}\Delta(\mathbf{I}) \rightarrow H_{k+1}\Delta(\mathbf{IJ}) \rightarrow H_{k+1}\Delta(\mathbf{J}) \rightarrow H_k\Delta(\mathbf{I}) \rightarrow H_k\Delta(\mathbf{IJ}) \\ \rightarrow H_k\Delta(\mathbf{J}) \rightarrow H_{k-1}\Delta(\mathbf{I}) \rightarrow H_{k-1}\Delta(\mathbf{IJ}) \rightarrow H_{k-1}\Delta(\mathbf{J}) \rightarrow \cdots \end{aligned}$$

By the definition of connection matrices, we have to make the following calculations:

$$\begin{aligned}
& (C_*\Delta(\mathbf{IJ}), \Delta(\mathbf{IJ})): \\
0 & \rightarrow \bigoplus_{\pi \in \mathbf{J}_{k+1}} CH(M(\pi)) \xrightarrow{\Delta} \bigoplus_{\pi \in \text{Crit}_k} CH(M(\pi)) \xrightarrow{\Delta} \bigoplus_{\pi \in \mathbf{I}_{k-1}} CH(M(\pi)) \rightarrow 0, \\
& (C_*\Delta(\mathbf{I}), \Delta(\mathbf{I})): \\
0 & \rightarrow 0 \rightarrow \bigoplus_{\pi \in \mathbf{I}_k} CH(M(\pi)) \xrightarrow{\Delta} \bigoplus_{\pi \in \mathbf{I}_{k-1}} CH(M(\pi)) \rightarrow 0, \\
& (C_*\Delta(\mathbf{J}), \Delta(\mathbf{J})): \\
0 & \rightarrow \bigoplus_{\pi \in \mathbf{J}_{k+1}} CH(M(\pi)) \xrightarrow{\Delta} \bigoplus_{\pi \in \mathbf{J}_k} CH(M(\pi)) \rightarrow 0 \rightarrow 0.
\end{aligned}$$

Therefore, the homologies are

$$\begin{aligned}
H_k\Delta(\mathbf{I}) &= \text{Ker } \Delta_k(\mathbf{I}), & H_{k-1}\Delta(\mathbf{I}) &= \frac{\bigoplus_{\pi \in \mathbf{I}_{k-1}} CH(M(\pi))}{\text{Im } \Delta_k(\mathbf{I})} \\
& \text{and } H_j\Delta(\mathbf{I}) = 0 & \text{for all } j \neq k, k-1,
\end{aligned}$$

$$\begin{aligned}
H_{k+1}\Delta(\mathbf{J}) &= \text{Ker } \Delta_{k+1}(\mathbf{J}), & H_k\Delta(\mathbf{J}) &= \frac{\bigoplus_{\pi \in \mathbf{J}_k} CH(M(\pi))}{\text{Im } \Delta_{k+1}(\mathbf{J})} \\
& \text{and } H_j\Delta(\mathbf{J}) = 0 & \text{for all } j \neq k+1, k,
\end{aligned}$$

$$\begin{aligned}
H_{k+1}\Delta(\mathbf{IJ}) &= \text{Ker } \Delta_{k+1}(\mathbf{IJ}), & H_k\Delta(\mathbf{IJ}) &= \frac{\text{Ker } \Delta_k}{\text{Im } \Delta_{k+1}} \\
& \text{and } H_{k-1}\Delta(\mathbf{IJ}) &= \frac{\bigoplus_{\pi \in \mathbf{I}_{k-1}} CH(M(\pi))}{\text{Im } \Delta_k} & \text{for all } j \neq k-1, k, k+1.
\end{aligned}$$

Thus, making the substitutions, one obtains Diagram 2.

Now we will prove that $\hat{T}(\mathbf{I})$ and $\hat{T}(\mathbf{J})$ are the induced maps for the map T . Indeed, consider the adjacent pair $(\mathbf{I}_{k-1}, \mathbf{I}_k)$, hence

$$\begin{array}{ccccccc}
0 & \xrightarrow{i} & H_k\Delta_\lambda(\mathbf{I}) & \xrightarrow{p} & H_k\Delta_\lambda(\mathbf{I}_k) & \xrightarrow{\Delta_\lambda} & H_{k-1}\Delta_\lambda(\mathbf{I}_{k-1}) & \xrightarrow{i} & H_{k-1}\Delta_\lambda(\mathbf{I}) & \xrightarrow{p} & 0 \\
& & \downarrow \hat{T}(\mathbf{I}) & & \downarrow \hat{T}(\mathbf{I}_k) & & \downarrow \hat{T}(\mathbf{I}_{k-1}) & & \downarrow \hat{T}(\mathbf{I}) & & \\
0 & \xrightarrow{i} & H_k\Delta_\mu(\mathbf{I}) & \xrightarrow{p} & H_k\Delta_\mu(\mathbf{I}_k) & \xrightarrow{\Delta_\mu} & H_{k-1}\Delta_\mu(\mathbf{I}_{k-1}) & \xrightarrow{i} & H_{k-1}\Delta_\mu(\mathbf{I}) & \xrightarrow{p} & 0.
\end{array}$$

Note that $T(\mathbf{I}_k)$ and $T(\mathbf{I}_{k-1})$ are the classical topological transition submatrices of T , i.e. the rightmost diagram commutes in Diagram 3.

$$\begin{array}{c}
 \begin{array}{ccccccc}
 0 \xrightarrow{i} \text{Ker } \Delta_{k+1,\lambda} \xrightarrow{p} \text{Ker } \Delta_{k,\lambda}(\mathbf{J}) \xrightarrow{\Delta^k} \text{Ker } \Delta_{k,\lambda}(\mathbf{I}) \xrightarrow{i} \frac{\text{Ker } \Delta_{k,\lambda}}{\text{Im } \Delta_{k+1,\lambda}} \xrightarrow{p} \frac{\bigoplus_{\pi \in \mathbf{J}_k} CH(M_\lambda(\pi))}{\text{Im } \Delta_{k+1,\lambda}(\mathbf{J})} \xrightarrow{\Delta^k} \frac{\bigoplus_{\pi \in \mathbf{I}_{(k-1)}} CH(M_\lambda(\pi))}{\text{Im } \Delta_{k,\lambda}(\mathbf{I})} \xrightarrow{i} \frac{\bigoplus_{\pi \in \mathbf{I}_{(k-1)}} CH(M_\lambda(\pi))}{\text{Im } \Delta_{k,\lambda}} \xrightarrow{p} 0 \\
 \hat{T}(\mathbf{J}) \quad I \quad \hat{T}(\mathbf{J}) \quad II \quad \hat{T}(\mathbf{J}) \quad II \quad \hat{T}(\mathbf{J}) \quad III \quad \hat{T}(\mathbf{J}) \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 0 \xrightarrow{i} \text{Ker } \Delta_{k+1,\mu} \xrightarrow{p} \text{Ker } \Delta_{k,\mu}(\mathbf{J}) \xrightarrow{\Delta^k} \text{Ker } \Delta_{k,\mu}(\mathbf{I}) \xrightarrow{i} \frac{\text{Ker } \Delta_{k,\mu}}{\text{Im } \Delta_{k+1,\mu}} \xrightarrow{p} \frac{\bigoplus_{\pi \in \mathbf{J}_k} CH(M_\mu(\pi))}{\text{Im } \Delta_{k+1,\mu}(\mathbf{J})} \xrightarrow{\Delta^k} \frac{\bigoplus_{\pi \in \mathbf{I}_{(k-1)}} CH(M_\mu(\pi))}{\text{Im } \Delta_{k,\mu}(\mathbf{I})} \xrightarrow{i} \frac{\bigoplus_{\pi \in \mathbf{I}_{(k-1)}} CH(M_\mu(\pi))}{\text{Im } \Delta_{k,\mu}} \xrightarrow{p} 0 \\
 \hat{T}(\mathbf{J}) \quad I \quad \hat{T}(\mathbf{J}) \quad II \quad \hat{T}(\mathbf{J}) \quad III \quad \hat{T}(\mathbf{J})
 \end{array}
 \end{array}$$

DIAGRAM 2

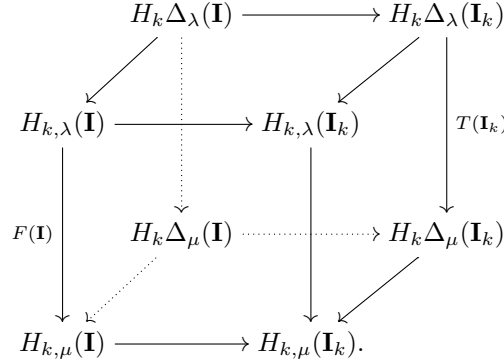


DIAGRAM 3

In order to show that the leftmost diagram in Diagram 3 commutes when we place $\widehat{T}(\mathbf{I})$, it is enough to show that the background diagram commutes, whereas the rightmost and foreground diagrams commute and p , the induced projection map $C_k \Delta_\lambda(\mathbf{I}) \rightarrow C_k \Delta_\lambda(\mathbf{I}_k)$, is the inclusion map. Indeed, as

$$H_k \Delta_\lambda(\mathbf{I}) = \text{Ker } \Delta_{k,\lambda}(\mathbf{I}) \subseteq H_k \Delta_\lambda(\mathbf{I}_k) = \bigoplus_{\mathbf{I}_k} CH(M_\lambda(\pi)),$$

it follows that p is the inclusion map and therefore

$$T(\mathbf{I}_k) \circ p = p \circ \widehat{T}(\mathbf{I}).$$

Analogously, one can show that i , the induced inclusion map

$$C_{k-1} \Delta_\lambda(\mathbf{I}_{k-1}) \rightarrow C_{k-1} \Delta_\lambda(\mathbf{I}),$$

is a projection map and

$$\widehat{T}(\mathbf{I}) \circ i = i \circ T(\mathbf{I}_{k-1}),$$

since

$$H_{k-1} \Delta_\lambda(\mathbf{I}) = \frac{\bigoplus_{\pi \in \mathbf{I}_{k-1}} CH(M_\lambda(\pi))}{\text{Im } \Delta_\lambda(\mathbf{I})} = \frac{H_{k-1} \Delta_\lambda(\mathbf{I}_{k-1})}{\text{Im } \Delta_\lambda(\mathbf{I})}.$$

In the same way, the above construction can be done for $\mathbf{J} = \mathbf{J}_k \mathbf{J}_{k+1}$ and the adjacent pair $(\mathbf{J}_k, \mathbf{J}_{k+1})$. Hence $\widehat{T}(\mathbf{I})$ and $\widehat{T}(\mathbf{J})$ are the map induced by T and make Diagram 1 commute sectionwise.

Now it remains to prove the same for $\widehat{T}(\mathbf{IJ})$, using the same idea described previously. So it is enough to prove that diagrams *I*, *II* and *III* in Diagram 2 commute. But this comes from $\Delta_\lambda(\mathbf{I})$ and $\Delta_\lambda(\mathbf{J})$ being submatrices of $\Delta_\lambda(\mathbf{IJ})$. The only diagram that deserves special attention is diagram *II*.

We will prove that diagram *II* commutes. Indeed, let $a \in \text{Ker } \Delta_{k,\lambda}(\mathbf{I})$. Then $i(a) = a + \text{Im } \Delta_{k+1,\lambda}(\mathbf{IJ}) = [a + b]$, where b is such that there exists

$c \in \bigoplus_{\pi \in \mathbf{J}_{k+1}} CH(M_\lambda(\pi))$ such that $\Delta_{k+1,\lambda}(\mathbf{I}\mathbf{J})(c) = b$. Applying $\widehat{T}(\mathbf{I}\mathbf{J})$, we have

$$\widehat{T}(\mathbf{I}\mathbf{J})([a + b]) = [T(\mathbf{I}\mathbf{J})(a + b)].$$

On the other hand,

$$i \circ \widehat{T}(\mathbf{I})(a) = \widehat{T}(\mathbf{I})(a) + \text{Im } \Delta_{k+1,\mu}(\mathbf{I}\mathbf{J}) = [T(\mathbf{I})(a) + d],$$

where $d \in \text{Im } \Delta_{k+1,\mu}(\mathbf{I}\mathbf{J})$. Therefore, we need to show

$$[T(\mathbf{I}\mathbf{J})(a + b)] = [T(\mathbf{I})(a) + d].$$

Since $T(\mathbf{I}\mathbf{J})a = T(\mathbf{I})a$, for $a \in \text{Ker } \Delta_{k,\lambda}(\mathbf{I}) \subseteq \bigoplus_{\pi \in \mathbf{I}_k} CH(M_\lambda(\pi))$, it is enough to show

$$T(\mathbf{I}\mathbf{J})a + T(\mathbf{I}\mathbf{J})b - T(\mathbf{I})a - d = T(\mathbf{I}\mathbf{J})b - d \in \text{Im } \Delta_{k,\mu}(\mathbf{I}\mathbf{J}).$$

Indeed, $d \in \text{Im } \Delta_{k,\mu}(\mathbf{I}\mathbf{J})$ so it is sufficient to prove that Diagram 4 commutes.

$$\begin{array}{ccc} \bigoplus_{\pi \in \mathbf{J}_{k+1}} CH(M_\lambda(\pi)) & \xrightarrow{\Delta_{k+1,\lambda}(\mathbf{I}\mathbf{J})} & \bigoplus_{\pi \in \mathbf{I}_k} CH(M_\lambda(\pi)) \oplus \bigoplus_{\pi \in \mathbf{J}_k} CH(M_\lambda(\pi)) \\ \downarrow T(\mathbf{J}_{k+1}) & & \downarrow T(\mathbf{I}\mathbf{J}) \\ \bigoplus_{\pi \in \mathbf{J}_{k+1}} CH(M_\mu(\pi)) & \xrightarrow{\Delta_{k+1,\mu}(\mathbf{I}\mathbf{J})} & \bigoplus_{\pi \in \mathbf{I}_k} CH(M_\mu(\pi)) \oplus \bigoplus_{\pi \in \mathbf{J}_k} CH(M_\mu(\pi)). \end{array}$$

DIAGRAM 4

Considering the adjacent pair $(\mathbf{I}_k \mathbf{J}_k, \mathbf{J}_{k+1})$, we have that the diagram commutes in the sections, since $T(\mathbf{J}_{k+1})$ and $T(\mathbf{I}_k \mathbf{J}_k)$ are precisely the classical topological transition matrices. Therefore the diagram

$$\begin{array}{ccccc} & & \bigoplus_{\mathbf{J}_{k+1}} CH(M_\lambda(\pi)) & \xrightarrow{\Delta_{k+1,\lambda}(\mathbf{I}\mathbf{J})} & \bigoplus_{\mathbf{I}_k} CH(M_\lambda(\pi)) \oplus \bigoplus_{\mathbf{J}_k} CH(M_\lambda(\pi)) & & \\ & & \downarrow & & \downarrow & & \\ H_{k+1,\lambda}(\mathbf{J}_{k+1}) & \xrightarrow{\quad} & H_{k,\lambda}(\mathbf{I}_k \mathbf{J}_k) & & & & \\ & & \downarrow & & \downarrow & & \\ F(\mathbf{J}_{k+1}) & & \bigoplus_{\mathbf{J}_{k+1}} CH(M_\mu(\pi)) & \xrightarrow{\quad} & \bigoplus_{\mathbf{I}_k} CH(M_\mu(\pi)) \oplus \bigoplus_{\mathbf{J}_k} CH(M_\mu(\pi)) & & \\ & & \downarrow & & \downarrow & & \\ H_{k+1,\mu}(\mathbf{J}_{k+1}) & \xrightarrow{\quad} & H_{k,\mu}(\mathbf{I}_k \mathbf{J}_k) & & & & \end{array}$$

in the background commutes. Observing that

$$T_k(\mathbf{I}, \mathbf{J}): \bigoplus_{\mathbf{I}_{k-1}} 0 \bigoplus_{\pi \in \mathbf{I}_k \mathbf{J}_k} CH_k(M_\lambda(\pi)) \bigoplus_{\mathbf{J}_{k+1}} 0 \xrightarrow{0 \oplus T_k(\mathbf{I}_k \mathbf{J}_k) \oplus 0} \bigoplus_{\mathbf{I}_{k-1}} 0 \bigoplus_{\pi \in \mathbf{I}_k \mathbf{J}_k} CH_k(M_\mu(\pi)) \bigoplus_{\mathbf{J}_{k+1}} 0$$

thus Diagram 4 commutes. Hence it is proved that diagram II commutes on the left. One can prove analogously that it also commutes on the right. Therefore T makes Diagram 1 commute for all adjacent pairs (\mathbf{I}, \mathbf{J}) .

Item (f) follows from the fact that the classical topological transition matrices are unique, and hence T is unique. Items (a)–(d) follow from T being a generalized topological transition matrix and from the fact that T is unique. Item (e) follows from (g) and from property (e) of Theorem 2.8. \square

4. Generalized transition matrix for the sweeping method

In this section we present an application of a generalized topological transition matrix in a continuation associated to a dynamical spectral sequence, see [5] and [12]. Our dynamical interpretation result implies the existence of connecting orbits in a fast-slow system “going from $M(q)$ to $M(p)$ ” for a nontrivial entry on T_{pq}^r associated to the spectral sequence.

Let M be an n -dimensional compact Riemannian manifold, $f: M \rightarrow \mathbb{R}$ a Morse function that is Morse–Smale, and ϕ the gradient flow of f . Choose a finite Morse decomposition $\bigcup M(p)$, $p \in \mathbf{P} = \{1, \dots, m\}$, such that there are distinct critical values c_p with $f^{-1}(c_p) \supset M(p)$. Then

$$\{F_p\}_{p=1}^m = \{f^{-1}(-\infty, c_p + \varepsilon)\}_{p=1}^m.$$

This defines an admissible ordering on M called the *filtration order*. In this case, each Morse set, $M(p)$, is a non-degenerate singularity of the gradient flow ϕ and hence the Conley index of each Morse set is the homology of a pointed k -sphere, where k is the Morse index of the singularity $M(p)$. We denote by $h_k^{(p)}$ the index k singularity in $F_p \setminus F_{p-1}$.

In the case where each $M(p)$ is a non-degenerate singularity and the stable and unstable manifolds intersect transversally, the connection matrix Δ associated to $\mathcal{D}(M)$ is unique (see [22] and [23]). It can also be defined as the differential of the graded Morse chain complex (C, Δ) , where C is generated by the singularities and graded by their indices, i.e. $C = \mathbb{Z}_2 \langle \text{Crit } f \rangle$ and Δ is determined by the maps $\Delta_k: C_k \rightarrow C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \text{Crit}_{k-1} f} n(x, y) \langle y \rangle,$$

where $n(x, y)$ is the number of connecting orbits counted mod 2 for nondegenerate singularities x and y of indices k and $k - 1$ respectively. We require that the columns of the matrix Δ are ordered such that $\Delta_k(F_p C_k) \subset F_{p-1} C_{k-1}$.

In [5] and [20] it is proved that a certain algorithm (called the sweeping method) applied to a connection matrix Δ determines a spectral sequence (E^r, d_r) of a filtered chain complex (C, Δ) . To achieve this we apply the sweeping method to the connection matrix. The sweeping method is an iterative process that, given a connection matrix, generates a collection of connection matrices $\Delta^1, \dots, \Delta^F$ and transition matrices T^1, \dots, T^F . This method singles out important nonzero entries, namely primary pivots and change of basis pivots, of the r -th diagonal of Δ^r , which are necessary to define a matrix Δ^{r+1} . At each step, Δ^{r+1} is obtained from Δ^r by a change of basis. Moreover, as r increases, the modules E_p^r change generators. In practice, the generators of the chain complex C mentioned above are very specific: singularities in the Morse case. The domain of d_r, E^r , is a certain quotient of a subgroup of C . Elements in this domain are represented by elements of C whose appropriate classes are in the kernels of all previous differentials $d_s, s < r$.

The change of basis matrices $T^r, r = 1, \dots, F$, determined by the sweeping method algorithm are called *transition matrices associated to the spectral sequence*. In [12], it was proven that such matrices satisfy the following properties.

PROPOSITION 4.1. *Each matrix T^r associated to the spectral sequence satisfies the following properties:*

- (a) $\Delta^{r+1} T^r + T^r \Delta^r = 0$;
- (b) T^r is an isomorphism;
- (c) T^r is an upper triangular matrix with respect to the filtration order.

In [12] a dynamical interpretation of the spectral sequence (E^r, d_r) is given in a setting of a fast-slow system flow,

$$\dot{x} = f(x, y), \quad \dot{y} = \varepsilon(y - 1)(y - 2),$$

where the sweeping method output of $n \times n$ connection matrices and transition matrices, $\Delta^1, T^1, \dots, T^{F-1}, \Delta^F$, reveals bifurcations that arise as a result of the nonzero entries of T^r .

In this article we consider a more general fast-slow system flow

$$\dot{x} = f(x, y), \quad \dot{y} = \varepsilon g(x, y),$$

in $M \times [0, F]$ with the following properties:

- When $\varepsilon = 0$ the parameterized system has an isolated invariant set S_y for each y which continues over the interval $[0, F]$ (slow variable), and which has a Morse decomposition $\mathcal{D}(M)_y = \{M_\lambda(p)_y \mid p = 1, \dots, n\}$

that also continues over the interval $[0, F]$. Assume also that the order from the sweeping method continues.

- For each $r = 0, 1, \dots, F$, we have that $f(x, r) = f_r$ is a gradient function which comes from a Morse function whose stable and unstable manifolds intersect transversely, and $g(M_y(p), y) \neq 0$ for $y \in (0, F)$ and $p \in \mathbf{P}$.
- For each $r = 0, 1, \dots, F$, the sweeping method connection matrices Δ^r are connection matrices of the Morse decomposition at parameter r .
- Lastly, the continuation of the Morse decomposition $\mathcal{D}(M)_y$ is such that Diagram 5 commutes. At each stage r of the sweeping method, F_k^r 's are continuation isomorphisms, $\Delta_{k+1,k}^r$ is a submatrix of the connection matrix Δ^r at parameter r , and T_k^r is a submatrix of T^r .

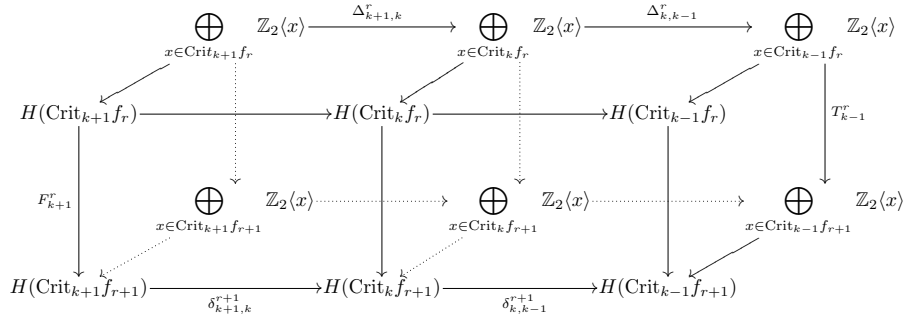


DIAGRAM 5

THEOREM 4.2. *Consider the fast-slow system defined previously. Then the sweeping method transition matrices are generalized topological transition matrices.*

Therefore, the sweeping method transition matrices inherit all properties from Theorem 3.1,

$$T_{0,1} = \begin{pmatrix} T_{\text{top},0} & 0 & 0 & 0 \\ 0 & T_{\text{top},1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{\text{top},k} \end{pmatrix}$$

as well as the connection result in Theorem 6.1 in [12]. Furthermore, in this setting, the use of topological transition matrix will allow us, in the forthcoming work, to employ the direction transition matrix theory (see [13] and [15]). As a consequence, it will be possible to obtain results that are not unidirectional (related to slow flow) as presented in Theorem 6.1 in [12].

PROOF. By Proposition 1.11, the collection of the submatrices $\{T^r(\mathbf{I})\}_{\mathbf{I} \in \mathbf{I}(<)}$ of the sweeping method transition matrices is a chain map from $\mathcal{C}\Delta_r$ to $\mathcal{C}\Delta_{r+1}$, since T^r is an upper triangular matrix and

$$T^r \circ \Delta^r = \Delta^{r+1} \circ T^r.$$

Now we will prove that T^r is a matrix in block form as $T_{r,r+1}$ the generalized topological transition matrix for Δ^r and Δ^{r+1} .

Indeed, since the boundary map Δ^r of a Morse complex is a connection matrix (see [25]) we have that for each adjacent pair of intervals (\mathbf{I}, \mathbf{J}) the following diagram is commutative:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H\Delta^r(\mathbf{I}) & \longrightarrow & H\Delta^r(\mathbf{IJ}) & \longrightarrow & H\Delta^r(\mathbf{J}) \xrightarrow{\Delta^r(\mathbf{J},\mathbf{I})} H\Delta^r(\mathbf{I}) \longrightarrow \cdots \\ & & \Phi_r(\mathbf{I}) \downarrow & & \Phi_r(\mathbf{IJ}) \downarrow & & \downarrow \Phi_r(\mathbf{J}) & \downarrow \Phi_r(\mathbf{I}) \\ \cdots & \longrightarrow & H_r(\mathbf{I}) & \longrightarrow & H_r(\mathbf{IJ}) & \longrightarrow & H_r(\mathbf{J}) \xrightarrow{\delta^r(\mathbf{J},\mathbf{I})} H_r(\mathbf{I}) \longrightarrow \cdots \end{array}$$

Set $M_r(\mathbf{I}) = \text{Crit}_{k-1}f_r$ and $M_r(\mathbf{J}) = \text{Crit}_k f_r$. One obtains $\Delta^r(\mathbf{J}, \mathbf{I}) = \Delta_{k,k-1}^r$. And since there are no connections in $M_r(\mathbf{I})$ and in $M_r(\mathbf{J})$ it follows that

$$H\Delta^r(\mathbf{I}) = \begin{cases} \bigoplus_{x \in \mathbf{I}} \mathbb{Z}_2 \langle x \rangle & \text{for } n = k-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$H\Delta^r(\mathbf{J}) = \begin{cases} \bigoplus_{x \in \mathbf{J}} \mathbb{Z}_2 \langle x \rangle, & \text{for } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore Diagram 5, in this case, is equal to Diagram 6.

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & H_k \Delta^r(\mathbf{IJ}) & \longrightarrow & \bigoplus_{x \in \mathbf{J}} \mathbb{Z}_2 \langle x \rangle \xrightarrow{\Delta_{k,k-1}^r} \bigoplus_{x \in \mathbf{I}} \mathbb{Z}_2 \langle x \rangle \\ & \swarrow & \vdots & & \downarrow & & \downarrow \\ H_k(\text{Crit}_{k-1}f_r) & \longrightarrow & H_k(\text{Crit}_{k-1,k}f_r) & \longrightarrow & H_k(\text{Crit}_k f_r) & \longrightarrow & H_{k-1}(\text{Crit}_{k-1}f_r) \\ & \downarrow & \downarrow & & \downarrow & & \downarrow \\ F_{k-1}^r(\mathbf{I}) & \downarrow & 0 & \longrightarrow & H_k \Delta^{r+1}(\mathbf{IJ}) & \longrightarrow & \bigoplus_{x \in \mathbf{J}} \mathbb{Z}_2 \langle x \rangle \xrightarrow{\Delta_{k,k-1}^{r+1}} \bigoplus_{x \in \mathbf{I}} \mathbb{Z}_2 \langle x \rangle \\ & \downarrow & \downarrow & & \downarrow & & \downarrow \\ H_k(\text{Crit}_{k-1}f_{r+1}) & \longrightarrow & H_k(\text{Crit}_{k-1,k}f_{r+1}) & \longrightarrow & H_k(\text{Crit}_k f_{r+1}) \xrightarrow{\delta^{r+1}} H_{k-1}(\text{Crit}_{k-1}f_{r+1}) \end{array}$$

DIAGRAM 6

Note that for homology dimensions different from k and $k - 1$, the sequence becomes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_n \Delta^r(\mathbf{J}) & \longrightarrow & 0 & \xrightarrow{0} & 0 \\
 \Phi_r(\mathbf{I}) \downarrow & & \Phi_r(\mathbf{J}) \downarrow & & \Phi_r(\mathbf{J}) \downarrow & & \Phi_r(\mathbf{I}) \downarrow \\
 H_n(\text{Crit}_{k-1} f_r) & \longrightarrow & H_n(\text{Crit}_{k-1,k} f_r) & \longrightarrow & H_n(\text{Crit}_k f_r) & \xrightarrow{\delta^r(\mathbf{J}, \mathbf{I})} & H_{n-1}(\text{Crit}_{k-1} f_r).
 \end{array}$$

Since $H_n \Delta^r(\mathbf{J}) = 0$ and Φ_r is an isomorphism, it follows that $H_n(\cdot) = 0$. Therefore T^r has the same block structure as $T_{r,r+1}$ and, by hypothesis, each submatrix T_k^r of T^r is actually $T_{\text{top},k}$. Thus, by item (f) and (g) of Theorem 3.1, we have that $T^r = T_{r,r+1}$. \square

Applying Proposition 2.7 to a fast-slow system as defined previously, we obtain the following corollary.

COROLLARY 4.3. *The entries from Δ^0 which are preserved, independent of continuation, are the primary pivots of the initial stages of the sweeping method.*

Note that these entries are the first step to define a new algebraic method in attempting to generalize the sweeping method, which is defined only for Morse–Smale flows without periodic orbits.

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