Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 2, 2016, 423–438
DOI: 10.12775/TMNA.2016.014

© 2016 Juliusz Schauder Centre for Nonlinear Studies
Nicolaus Copernicus University

POSITIVE SOLUTIONS FOR PARAMETRIC DIRICHLET PROBLEMS WITH INDEFINITE POTENTIAL AND SUPERDIFFUSIVE REACTION

Sergiu Aizicovici — Nikolaos S. Papageorgiou — Vasile Staicu

Abstract. We consider a parametric semilinear Dirichlet problem driven by the Laplacian plus an indefinite unbounded potential and with a reaction of superdifissive type. Using variational and truncation techniques, we show that there exists a critical parameter value $\lambda_* > 0$ such that for all $\lambda > \lambda_*$ the problem has at least two positive solutions, for $\lambda = \lambda_*$ the problem has at least one positive solution, and no positive solutions exist when $\lambda \in (0, \lambda_*)$. Also, we show that for $\lambda \geq \lambda_*$ the problem has a smallest positive solution.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric Dirichlet problem:

$$
(\mathrm{P}_\lambda) \qquad \begin{cases} -\triangle u(z) + \beta(z)u(z) = \lambda u(z)^{q-1} - f(z,\ u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0,\ \lambda > 0,\ 2 < q < 2^*, \end{cases}
$$

where

$$
2^* = \begin{cases} 2N/(N-2) & \text{if } N \geq 3, \\ +\infty & \text{if } N \in \{1, 2\}. \end{cases} \tag{1.1}
$$

2010 *Mathematics Subject Classification.* 35J20, 35J605.
Key words and phrases. Reaction of superdifussive type; maximum principle; local minimizer; mountain pass theorem; bifurcation type theorem; indefinite and unbounded potential.

Here $\beta \in L^s(\Omega)$, with $s > N$, and it may change sign. Also, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (i.e. for all $x \in \mathbb{R} z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous) which has a $(q-1)$-superlinear growth near $+\infty$. So, the reaction of (P_λ) exhibits a superdiffusive kind of behavior.

Recall that in superdiffusive logistic equations, the reaction has the form $\lambda x^{q-1} - x^{r-1}$ with $2 < q < r < 2^*$. We show that there is a critical value $\lambda_* > 0$ of the parameter such that for $\lambda > \lambda_*$ problem (P_λ) has at least two positive smooth solutions, for $\lambda = \lambda_*$ problem (P_λ) has at least one positive smooth solution, and for $\lambda \in (0, \lambda_*)$ no positive smooth solutions exist.

Positive solutions for parametric semilinear Dirichlet problems with $\beta \geq 0$ and more restrictive conditions on the reaction were obtained by Amann [2], Dancer [4], Lin [13], Ouang-Shi [15] and Rabinowitz [17]. To the best of our knowledge, no such results exist for problems with indefinite potential and general superdiffusive reaction. Recently, Gasinski–Papageorgiou [9] and Kyritsi–Papageorgiou [12] studied nonparametric semilinear problems with indefinite potential, either with double resonance (see [9]), or with superlinear reaction (see [12]). Finally, we mention the recent work of Gasinski and Papageorgiou [10] on bifurcation type results for different types of p-Laplacian equations.

Our approach is variational, based on critical point theory coupled with suitable truncation techniques.

2. Mathematical preliminaries and hypotheses

Throughout this paper, by $\|\cdot\|_p$, $1 \leq p \leq \infty$, we denote the norm of $L^p(\Omega)$, or $L^p(\Omega, \mathbb{R}^N)$ and by $\|\cdot\|$ we denote the norm of the Sobolev space $H_0^1(\Omega)$ defined by

$$\|u\| = \|Du\|_2 \quad \text{for all } u \in H_0^1(\Omega).$$

Note that if $2 < q < 2^*$ (see (1.1)), then $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, with compact embedding. Also, if $x \in \mathbb{R}$, then $x^\pm = \max\{\pm x, 0\}$. For every $u \in H_0^1(\Omega)$ we set $u^\pm(\,\cdot\,) = u(\,\cdot\,)^\pm$. We know that

$$u^\pm \in H_0^1(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-$$

(see [8]). If $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then the corresponding Nemytskiĭ map N_h is defined by

$$N_h(u)(\,\cdot\,) = h(\,\cdot\,, u(\,\cdot\,)) \quad \text{for all } u \in H_0^1(\Omega).$$

By $|\cdot|_N$ we will denote the Lebesgue measure on $\mathbb{R}^N$.

Suppose that $(X, \|\cdot\|)$ is a Banach space and X^* is its topological dual. By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair (X^*, X), and we will use the symbol "$\xrightarrow{w}$" to designate weak convergence.

We say that the Banach space X has the Kadec–Klee property if the following is true:

$$[x_n \xrightarrow{w} x \text{ and } \|x_n\| \to \|x\|] \Rightarrow [x_n \to x].$$

A Hilbert space (and more generally, a locally uniformly convex Banach space) has the Kadec–Klee property (see Gasinski and Papageorgiou [8, p. 911]).

Given $\varphi \in C^1(X)$, we say that φ satisfies the *Palais–Smale condition* (PS-*condition*, for short), if the following is true:

every sequence $\{x_n\}_{n\geq 1} \subseteq X$ *such that* $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ *is bounded and* $\varphi'(x_n) \to 0$ *in* X^* *as* $n \to \infty$ *admits a strongly convergent subsequence.*

Using this compactness-type condition, we have the following minimax theorem, known in the literature as the "mountain pass theorem":

Theorem 2.1. *If* $\varphi \in C^1(X)$ *satisfies the* PS-*condition,* $x_0, x_1 \in X$ *and* $\rho > 0$ *are such that* $\|x_1 - x_0\| > \rho$,

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} =: \eta_\rho,$$

and $c = \inf_{\gamma\in\Gamma} \inf_{t\in[0,1]}$, *where* $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}$, *then* $c \geq \eta_\rho$ *and* c *is a critical value of* φ (*i.e. there exists* $x^* \in X$ *such that* $\varphi'(x^*) = 0$ *and* $\varphi(x^*) = c$).

In the study of problem (P_λ), we will use the Sobolev space $H_0^1(\Omega)$ and the ordered Banach space $C_0^1(\overline{\Omega})$. The positive cone of the latter is

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

We consider the C^1-functional $\sigma\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \|Du\|_2^2 + \int_\Omega \beta u^2\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

We assume that $\beta \in L^s(\Omega)$ with $s > N/2$. Let $s' > 1$ denote the conjugate exponent of s, i.e. $1/s + 1/s = 1$. We have

$$2s' = 2\frac{s}{s-1} < 2^* \tag{2.1}$$

(see (1.1)). Then, by virtue of the Sobolev embedding theorem, we have $H_0^1(\Omega) \hookrightarrow L^{2s'}(\Omega)$ and the embedding is compact. Using Hölder's inequality, we have

$$\left|\int_\Omega \beta u^2\,dz\right| \leq \|\beta\|_s \|u\|_{2s'}^2. \tag{2.2}$$

Since $2 < 2s' < 2^*$ (see (2.1)), we have $H_0^1(\Omega) \hookrightarrow L^{2s'}(\Omega) \hookrightarrow L^2(\Omega)$, and, as we already mentioned, the first embedding is compact. Invoking Ehrling's inequality (see, for example, Papageorgiou and Kyritsi [16, p. 698]), given $\xi j > 0$, we can find $C(\xi j) > 0$ such that

$$\|u\|_{2s'}^2 \leq \varepsilon \|u\|^2 + C(\varepsilon)\|u\|_2^2 \quad \text{for all } u \in H_0^1(\Omega). \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$\|Du\|_2^2 - \int_\Omega \beta u^2 dz \leq \|Du\|_2^2 + \varepsilon \|\beta\|_s \|u\|^2 + C(\xi j)\|\beta\|_s \|u\|_2^2,$$

hence $(1 - \varepsilon\|\beta\|_s)\|u\|^2 \leq \sigma(u) + C(\varepsilon)\|\beta\|_s\|u\|_2^2$. Choosing $\varepsilon \in (0, 1/\|\beta\|_s)$, we have

$$\|u\|^2 \leq C_1(\sigma(u) + \widehat{C}\|u\|_2^2) \quad \text{for some } C_1, \widehat{C} > 0 \text{ and all } u \in H_0^1(\Omega). \tag{2.4}$$

Consider the continuous bilinear form $\alpha\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\alpha(u,\ y) = C_1\bigg[\langle A(u), y\rangle + \int_\Omega \beta u y\, dz\bigg] \quad \text{for all } u, y \in H_0^1(\Omega),$$

where $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is defined by

$$\langle A(u), y\rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N} \quad \text{for all } u, y \in H_0^1(\Omega).$$

From (2.4) we have

$$\alpha(u, u) + C_1\widehat{C}\|u\|_2^2 \geq \|u\|^2 \quad \text{for all } u \in H_0^1(\Omega). \tag{2.5}$$

From (2.5) and Corollary $7D$ of Showalter (see [18, p. 78]), it follows that the linear differential operator $u \mapsto -\triangle u + \beta u$, $u \in H_0^1(\Omega)$, has a spectrum, consisting of a sequence of distinct eigenvalues $\{\widehat{\lambda}_k\}_{k\geq 1}$ such that

$$-C_1\widehat{C} < \widehat{\lambda}_1 < \ldots < \widehat{\lambda}_k \to \infty \quad \text{as } k \to \infty.$$

We know that $\widehat{\lambda}_1$ is simple and admits the following variational characterization:

$$\widehat{\lambda}_1 = \inf\bigg\{\frac{\sigma(u)}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\bigg\} \tag{2.6}$$

(see also Mugnai and Papageorgiou [14]). Note that if $\beta \geq 0$, then $\widehat{\lambda}_1 > 0$. More generally, if λ_1 is the first eigenvalue of $(-\triangle, H_0^1(\Omega))$ and $\beta \in L^s(\Omega)$ satisfies

$$\beta^-(z) := \max\{-\beta(z), 0\} \leq \overline{\lambda}_1 \quad \text{a.e. on } \Omega,\ \beta^- \neq \overline{\lambda}_1,$$

then

$$\sigma(u) \geq \|Du\|_2^2 - \int_\Omega \beta^- u^2 dz \geq \xi_0\|u\|^2 \quad \text{for some } \xi_0 > 0,\ \text{all } u \in H_0^1(\Omega)$$

(see Gasinski and Papageorgiou [9, Lemma 2.1]), hence $\widehat{\lambda}_1 > 0$.

The infimum in (2.6) is achieved on the eigenspace of $\widehat{\lambda}_1$. Let $\widehat{u}_1$ be the L^2-normalized (i.e. $\|\widehat{u}_1\|_2 = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. It is clear from (2.6) that we may assume that $\widehat{u}_1 \geq 0$.

If $s > N$, then the regularity theory for Dirichlet problems (see Struwe [19, pp. 218–219]) and the maximum principle of Vazquez [20] imply $\widehat{u}_1 \in \operatorname{int} C_+$.

We will also use a "weighted" version of the previous eigenvalue problem. Namely, let $\xi \in L^\infty(\Omega)_+, \xi \neq 0$ and consider the following eigenvalue problem:

$$-\triangle u(z) + \beta(z)u(z) = \lambda\xi(z)u(z) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0.$$

As before, we have an increasing sequence of eigenvalues denoted by $\widehat{\lambda}_k(\xi), k \geq 1$, and $\widehat{\lambda}_k(\xi) \to \infty$ as $k \to \infty$. Moreover, the unique continuation property (see Garofalo and Lin [7]) implies that:

$$\text{if } \xi(z) \leq \xi'(z) \text{ a.e. in } \Omega \text{ and } \xi \neq \xi', \text{ then } \widehat{\lambda}_k(\xi') < \widehat{\lambda}_k(\xi) \text{ for all } k \geq 1.$$

The hypotheses on the potential function $\beta(\,\cdot\,)$ are the following:

H(β) $\beta \in L^s(\Omega)$ with $s > N$, $\beta^+ = \max\{\beta,\ 0\} \in L^\infty(\Omega)$.

The hypotheses on the perturbation $f(z,x)$ are the following:

H(f) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(z,0) = 0$ almost everywhere in Ω and

(a) $|f(z,x)| \leq a(z) + Cx^{r-1}$ for almost all $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)_+$, $C > 0$, $2 < r < 2^*$;

(b) $\lim\limits_{x\to\infty} f(z,x)/x^{q-1} = +\infty$ uniformly for almost all $z \in \Omega$;

(c) $f(z,x) \geq -\eta(z)x$ for almost all $z \in \Omega$, all $x \geq 0$ with $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_1$ almost everywhere in Ω, $\eta \neq \widehat{\lambda}_1$;

(d) for almost all $z \in \Omega$, $x \to f(z,x)/x$ is nondecreasing on $(0,\infty)$;

(e) there exists $\tau > 2$ such that for every $\rho > 0$, one can find $\gamma_\rho > 0$ with the property that for almost all $z \in \Omega$, the map

$$x \mapsto \gamma_\rho(x^{\tau-1} + x) - f(z,x)$$

is nondecreasing on $[0,\rho]$.

REMARK 2.2. Since we are interested in positive solutions and all of the above hypotheses concern only the nonnegative half-axis $[0,+\infty)$, without any loss of generality, we may (and will) assume that $f(z,x) = 0$ for almost all $z \in \Omega$, all $x \leq 0$.

As we illustrate in the examples that follow, hypotheses H(f) dictate a reaction of superdiffusive type.

EXAMPLES 2.3. The following functions satisfy hypotheses H(f) (for the sake of simplicity, we drop the z-dependence):

$$f_1(x) = x^{\tau-1} + \eta x \quad \text{for a.a. } x \geq 0,$$

where $\tau > q$, $\eta > 0$ with $-\eta < \widehat{\lambda}_1$, if $\widehat{\lambda}_1 \leq 0$;

$$f_2(x) = \begin{cases} \eta x & \text{if } x \in [0,1], \\ \eta x + x^{q-1}\ln(x) & \text{if } x > 1, \end{cases}$$

where $\eta > 0$ with $-\eta < \widehat{\lambda}_1$, if $\widehat{\lambda}_1 \leq 0$.

By a positive solution of (P_λ), we mean a function $u \in H_0^1(\Omega) \setminus \{0\}$ such that $u(z) \geq 0$ almost everywhere in Ω, which is a weak solution of (P_λ).

From the regularity theory of Dirichlet problems (see Struwe [19, pp. 218–219]), we have $u \in C_+ \setminus \{0\}$ and

$$-\triangle u(z) + \beta(z)u(z) = \lambda u(z)^{q-1} - f(z, u(z)) \quad \text{a.e. in } \Omega.$$

Let $\rho = \|u\|_\infty$ and let $\gamma_\rho > 0$ be as postulated by hypothesis $\mathrm{H}(f)$(e). Then

$$\begin{aligned} -\triangle u(z) &+ (\beta(z) + \gamma_\rho)u(z) + \gamma_\rho u(z)^{\tau-1} \\ &= \lambda u(z)^{q-1} + \gamma_\rho u(z) + \gamma_\rho u(z)^{\tau-1} - f(z,\ u(z)) \geq 0 \end{aligned}$$

almost everywhere in Ω, hence

$$\triangle u(z) \leq (\|\beta^+\|_\infty + \gamma_\rho(1 + \rho^{\tau-2}))u(z) \quad \text{a.e. in } \Omega,$$

therefore $u \in \operatorname{int} C_+$ (see Vazquez [20]). So, we see that every positive solution of (P_λ) belongs to $\operatorname{int} C_+$.

3. A bifurcation-type theorem

In this section, we study the dependence on the parameter $\lambda > 0$ of the positive solutions of (P_λ) and eventually obtain a bifurcation-type theorem, describing this dependence. Let

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\}.$$

First we show that this set is nonempty and upward directed.

Proposition 3.1. *If hypotheses* $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold, then* $\mathcal{L} \neq \emptyset$, *and if* $\lambda \in \mathcal{L}$ *with* $\eta > \lambda$, *then* $\eta \in \mathcal{L}$.

Proof. By virtue of hypotheses $\mathrm{H}(f)$(a), (b), given any $\xi > 0$, we can find $C_2 = C_2(\xi) > 0$ such that

$$F(z,x) \geq \frac{\xi}{q}(x^+)^q - C_2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{3.1}$$

where $F(z,x) = \int_0^x f(z,s)\,ds$. Let $g_\lambda(z,x) = \lambda(x^+)^{q-1} - f(z,x) + \widehat{C}x^+$ for all $(z,x) \in \Omega \times \mathbb{R}$ (see (2.4)). This is a Carathéodory function. We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)\,ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda \colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{C}}{2}\|u\|_2^2 - \int_\Omega G_\lambda(z, u(z))\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

We have

$$\begin{aligned}\text{(3.2)}\qquad \widehat{\varphi}_\lambda(u) &\geq \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q \\ &\quad + \frac{\xi}{q}\|u^+\|_q^q - \frac{\widehat{c}}{2}\|u^+\|_2^2 - C_2|\Omega| \quad \text{(see (3.1))} \\ &= \frac{1}{2}\sigma(u^+) + \frac{\xi-\lambda}{q}\|u^+\|_q^q + \frac{1}{2}(\sigma(u^-) + \widehat{C}\|u^-\|_2^2) - C_2|\Omega|_N.\end{aligned}$$

Since $\xi > 0$ is arbitrary, we choose $\xi > \lambda$. Then, by (3.2), (2.4) and since $q > 2$, we obtain

$$\begin{aligned}\text{(3.3)}\quad \widehat{\varphi}_\lambda(u) &\geq \frac{1}{2}\sigma(u^+) + C_3\|u^+\|_2^q + \frac{1}{2C_1}\|u^-\|^2 - C_2|\Omega|_N \quad \text{for some } C_3 > 0 \\ &\geq \frac{1}{2C_1}\|u\|^2 + C_3\|u^+\|_2^q - C_4\|u^+\|_2^2 - C_2|\Omega|_N \quad \text{for some } C_4 > 0.\end{aligned}$$

Because $q > 2$, from (3.3) we infer that $\widehat{\varphi}_\lambda$ is coercive. Exploiting the compact embedding of $H_0^1(\Omega)$ into $L^r(\Omega)$ and $L^q(\Omega)$, we can easily show that $\widehat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in H_0^1(\Omega)$ such that

$$\text{(3.4)}\qquad \widehat{\varphi}_\lambda(u_0) = \inf\{\widehat{\varphi}_\lambda(u) : u \in H_0^1(\Omega)\} =: m_\lambda.$$

Let $\overline{u} \in \operatorname{int} C_+$. Then

$$\widehat{\varphi}_\lambda(\overline{u}) = \frac{1}{2}\sigma(\overline{u}) - \frac{\lambda}{q}\|\overline{u}\|_q^q + \int_\Omega F(z, \overline{u}(z))\,dz$$

and so, it is clear that for $\lambda > 0$ large we have $\widehat{\varphi}_\lambda(\overline{u}) < 0$. Hence

$$\widehat{\varphi}_\lambda(u_0) = m_\lambda < 0 = \widehat{\varphi}_\lambda(0) \quad \text{for } \lambda > 0 \text{ large}$$

i.e. $u_0 \neq 0$. From (3.4) we derive $\widehat{\varphi}_\lambda'(u_0) = 0$, hence

$$\text{(3.5)}\qquad A(u_0) + (\beta + \widehat{C})u_0 = N_{g_\lambda}(u_0).$$

On (3.5) we act with $-u_0^- \in H_0^1(\Omega)$ and obtain

$$\|Du_0^-\|_2^2 + \int_\Omega \beta(u_0^-)^2\,dz + \widehat{C}\|u_0^-\|_2^2 = 0$$

hence $\|u_0^-\|_2^2/C_1 \leq 0$ (see (2.4)), i.e. $u_0 \geq 0$, $u_0 \neq 0$. Therefore (3.5) becomes

$$A(u_0) + \beta u_0 = \lambda u_0^{q-1} - N_f(u_0),$$

hence

$$-\triangle u_0(z) + \beta(z)u_0(z) = \lambda u_0(z)^{q-1} - f(z, u(z)) \quad \text{a.e. in } \Omega,\ u_0|_{\partial\Omega} = 0,$$

therefore $u_0 \in \operatorname{int} C_+$ is a positive solution of (P_λ) for $\lambda > 0$ large. This proves that $\mathcal{L} \neq \emptyset$. Now, let $\lambda \in \mathcal{L}$ and $\eta > \lambda$. Since $\lambda \in \mathcal{L}$ we can find $u_\lambda \in \operatorname{int} C_+$, a solution of problem (P_λ). Let $\theta \in (0,1)$ be such that

$$\text{(3.6)}\qquad \lambda = \theta^{q-2}\eta$$

(recall that $q > 2$). One has

$$-\triangle(\theta u_\lambda)(z) + \beta(z)(\theta u_\lambda)(z) = \theta\lambda u_\lambda(z)^{q-1} - \theta f(z, u_\lambda(z)) \leq \eta(\theta u_\lambda)^{q-1}(z) - f(z, \theta u_\lambda(z)) \tag{3.7}$$

almost everywhere in Ω (see H(f)(d)).

We set $\underline{u} = \theta u_\lambda \in \operatorname{int} C_+$ and consider the following truncation-perturbation of the reaction of problem (P_η):

$$h_\eta(z,\ x) = \begin{cases} \eta\underline{u}(z)^{q-1} - f(z,\ \underline{u}(z)) + \widehat{C}u(z) & \text{if } x \leq \underline{u}(z), \\ \eta x^{q-1} - f(z,\ x) + \widehat{C}x & \text{if } \underline{u}(z) < x. \end{cases} \tag{3.8}$$

This is a Carathéodory function. We set $H_\eta(z,x) = \int_0^x h_\eta(z,\ s)ds$ and consider the C^1-functional ψ_η: $H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\eta(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{C}}{2}\|u\|_2^2 - \int_\Omega H_\eta(z,\ u(z))\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

Using (3.8), we have

$$\begin{aligned} \psi_\eta(u) &\geq \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u\|_2^2 - \int_{\{\underline{u}<u\}} \left(\frac{\eta}{q}u^q - F(z,u) + \frac{\widehat{c}}{2}u^2\right) dz - C_5 \\ &\qquad\qquad \text{for some } C_5 > 0 \\ &\geq \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u^-\|_2^2 - \frac{\eta}{q}\|u^+\|_q^q + \int_\Omega F(z,u)\,dz - C_6 \\ &\qquad\qquad \text{for some } C_6 > 0 \\ &\geq \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u^-\|_2^2 - \frac{\eta}{q}\|u^+\|_q^q + \frac{\xi}{q}\|u^+\|_q^q - C_7 \\ &\qquad\qquad \text{for some } C_7 > 0 \text{ (see(3.1))} \\ &= \frac{1}{2}\sigma(u^+) + \frac{\xi-\eta}{q}\|u^+\|_q^q + \frac{1}{2}\sigma(u^-) + \frac{\widehat{c}}{2}\|u^-\|_2^2 - C_7 \\ &\geq \frac{1}{2C_1}\|u\|^2 + \frac{\xi-\eta}{q}\|u^+\|_q^q - \frac{\widehat{c}}{2}\|u^+\|_2^2 - C_7 \qquad \text{(see (2.4)).} \end{aligned}$$

Since $\xi > 0$ is arbitrary, we choose $\xi > \eta$ and infer

$$\psi_\eta(u) \geq \frac{1}{2C_1}\|u\|^2 + C_8\|u^+\|_2^q - \frac{\widehat{C}}{2}\|u^+\|_2^2 - C_7 \quad \text{for some } C_8 > 0.$$

Because $q > 2$, it follows that ψ_η is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\eta \in H_0^1(\Omega)$ such that

$$\psi_\eta(u_\eta) = \inf\{\psi_\eta(u) : u \in H_0^1(\Omega)\}.$$

Then $\psi'_\eta(u_\eta) = 0$, hence

$$A(u_\eta) + (\beta + \widehat{C})u_\eta = N_{h_\eta}(u_\eta). \tag{3.9}$$

On (3.9) we act with $(\underline{u}-u_\eta)^+ \in H_0^1(\Omega)$ and use (3.8) to obtain

$$\begin{aligned}\langle A(u_\eta),(\underline{u}-u_\eta)^+\rangle &+ \int_\Omega (\beta+\widehat{C})u_\eta(\underline{u}-u_\eta)^+\,dz \\ &= \int_\Omega (\eta\underline{u}^{q-1} - f(z,\underline{u}) + \widehat{C}\underline{u})(\underline{u}-u_\eta)^+\,dz \\ &\geq \langle A(\underline{u}),(\underline{u}-u_\eta)^+\rangle + \int_\Omega (\beta+\widehat{C})\underline{u}(\underline{u}-u_\eta)^+\,dz\end{aligned}$$

(see (3.7)), hence

$$\langle A(\underline{u})-A(u_\eta),\ (\underline{u}-u_\eta)^+\rangle + \int_\Omega (\beta+\widehat{C})[(\underline{u}-u_\eta)^+]^2\,dz \leq 0,$$

therefore

$$\|D(\underline{u}-u_\eta)^+\|_2^2 + \int_\Omega \beta[(\underline{u}-u_\eta)^+]^2\,dz + \widehat{C}\|(\underline{u}-u_\eta)^+\|_2^2 \leq 0.$$

This implies $\|(\underline{u}-u_\eta)^+\|^2/C_1 \leq 0$ (see (2.4)), hence $\underline{u} \leq u_\eta$. So, (3.9) becomes

$$A(u_\eta) + \beta u_\eta = \eta u_\eta^{q-1} - N_f(u_\eta)$$

(see (3.8)) and we conclude that $u_\eta \in \operatorname{int} C_+$ solves (P_η), i.e. $\eta \in \mathcal{L}$. □

Now let

$$\lambda_* = \inf \mathcal{L}. \tag{3.10}$$

Proposition 3.2. *If hypotheses* H(β) *and* H(f) *hold, then* $\lambda_* > 0$.

Proof. First assume that $\widehat{\lambda}_1 > 0$. By virtue of hypotheses H(f)(a)–(c), we can find $\lambda_0 > 0$ small such that

$$\lambda x^{q-1} < \widehat{\lambda}_1 x + f(z,x) \quad \text{for a.a. } z\in\Omega,\ \text{all } x\geq 0 \text{ and all } \lambda\in(0,\lambda_0).$$

Suppose that for $\lambda \in (0,\lambda_0)$, we have $\lambda \in \mathcal{L}$. Then, there exists $u_\lambda \in \operatorname{int} C_+$, a positive solution of problem (P_λ), such that

$$-\triangle u_\lambda(z) + \beta(z)u_\lambda(z) = \lambda u_\lambda(z)^{q-1} - f(z,\ u_\lambda(z)) < \widehat{\lambda}_1 u_\lambda(z) \quad \text{a.e. in } \Omega.$$

Then $\sigma(u_\lambda) < \widehat{\lambda}_1\|u_\lambda\|_2^2$ which contradicts (2.6), hence $\lambda_* \geq \lambda_0 > 0$.

Next assume that $\widehat{\lambda}_1 \leq 0$. Suppose that $\lambda_* = 0$. We can find $\{\lambda_n\}_{n>1} \subset \mathcal{L}$ such that $\lambda_n > \lambda_{n+1}$, $\lambda_n \downarrow 0$ as $n\to\infty$. For $n \geq 1$, let $u_n = u_{\lambda_n} \in \operatorname{int} C_+$ be a positive solution of problem (P_{λ_n}). We have

$$A(u_n) + \beta u_n = \lambda_n u_n^{q-1} - N_f(u_n) \quad \text{for all } n\geq 1, \tag{3.11}$$

hence

$$\sigma(u_n) = \lambda_n\|u_n\|_q^q - \int_\Omega f(z,u_n)u_n\,dz. \tag{3.12}$$

By hypothesis H(f)(b), given any $\xi > 0$, we can find $M = M(\xi) \geq 1$, such that

$$f(z,x)x \geq \xi x^q \quad \text{for a.a. } z\in\Omega,\ \text{all } x\geq M. \tag{3.13}$$

On the other hand, hypothesis H(f)(c) implies that

$$f(z,x)x \geq -\eta(z)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,M]. \tag{3.14}$$

Returning to (3.12), we have

$$\begin{aligned}\sigma(u_n) &= \lambda_n \|u_n\|_q^q - \int_{\{u_n \geq M\}} f(z,u_n)u_n\,dz - \int_{\{0<u_n<M\}} f(z,\ u_n)u_n\,dz \\ &\leq \lambda_n\|u_n\|_q^q - \xi\int_{\{u_n\geq M\}} u_n^q\,dz + \int_{\{0<u_n<M\}} \eta u_n^2\,dz \quad \text{(see (3.13), (3.14))} \\ &\leq (\lambda_n + \|\eta\|_\infty - \xi)\int_{\{u_n\geq M\}} u_n^q\,dz + \lambda_n\int_{\{0<u_n<M\}} u_n^q\,dz + \int_\Omega \eta u_n^2\,dz\end{aligned}$$

(recall that $q > 2$, $M \geq 1$), hence

$$\sigma(u_n) - \int_\Omega \eta u_n^2\,dz \leq (\lambda_n + \|\eta\|_\infty - \xi)\int_{\{u_n\geq M\}} u_n^q\,dz + \lambda_n\int_{\{0<u_n<M\}} u_n^q\,dz. \tag{3.15}$$

Recall that $\xi > 0$ is arbitrary. So, choosing $\xi > \lambda_1 + \|\eta\|_\infty \geq \lambda_n + \|\eta\|_\infty$ for all $n \geq 1$, from (3.15) and Lemma 2.1 of [9] it follows that there exists $C_9 = C_9(\xi) > 0$ such that $\|u_n\|^2 \leq \lambda_n C_9$ for all $n \geq 1$, hence

$$u_n \to 0 \quad \text{in } H_0^1(\Omega). \tag{3.16}$$

Let $y_n = u_n/\|u_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{2s'}(\Omega) \text{ as } n \to \infty. \tag{3.17}$$

From (3.11), we have

$$A(y_n) + \beta y_n = \lambda_n u_n^{q-2} y_n - \frac{N_f(u_n)}{\|u_n\|} \quad \text{for all } n \geq 1. \tag{3.18}$$

Note that $\{N_f(u_n)/\|u_n\|\}_{n\geq 1} \subset L^{r'}(\Omega)$ is bounded (see hypotheses H(f)(a) and (d)). Hence acting in (3.18) with $y_n - y \in H_0^1(\Omega)$, passing to the limit as $n \to \infty$ and using (3.17), we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

hence $\|Dy_n\|_2 \to \|Dy\|_2$ and by the Kadec–Klee property of the Hilbert space $H_0^1(\Omega)$, we infer that

$$y_n \to y \quad \text{in } H_0^1(\Omega), \quad \text{hence} \quad \|y\| = 1. \tag{3.19}$$

We may assume that

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} \beta \quad \text{in } L^{r'}(\Omega) \quad \text{and} \quad \beta = -\widehat{\eta}y \quad \text{with } \widehat{\eta} \leq \eta \tag{3.20}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 31). So, if in (3.18) we pass to the limit as $n \to \infty$ and use (3.16), (3.19) and (3.20), we obtain

$$A(y) + \beta y = \widehat{\eta} y, \quad y \neq 0,$$

hence

$$\sigma(y) - \int_\Omega \widehat{\eta} y^2 \, dz = 0,$$

therefore $C_{10}\|y\|^2 \leq 0$ for some $C_{10} > 0$ (see Lemma 2.1 of [9]). It follows that $y = 0$, a contradiction (see (3.19)). □

PROPOSITION 3.3. *If hypotheses* H(β) *and* H(f) *hold and* $\lambda > \lambda_*$, *then problem* (P_λ) *has at least two positive smooth solutions* $u_0, \widehat{u} \in \operatorname{int} C_+$.

PROOF. Let $\lambda' \in (\lambda_*, \lambda) \cap \mathcal{L}$ and let $u_{\lambda'} \in \operatorname{int} C_+$ be a positive solution of problem ($\mathrm{P}_{\lambda'}$). As in the proof of Proposition 3.1, let $\theta \in (0, 1)$ be such that $\lambda' = \theta^{q-2}\lambda$ and set $\underline{u} = \theta u_{\lambda'} \in \operatorname{int} C_+$. We introduce the following truncation-perturbation of the reaction in problem (P_λ):

$$h_\lambda(z, x) = \begin{cases} \lambda \underline{u}(z)^{q-1} - f(z, \underline{u}(z)) + \widehat{C} u(z) & \text{if } x \leq \underline{u}(z), \\ \lambda x^{q-1} - f(z, x) + \widehat{C} x & \text{if } \underline{u}(z) < x. \end{cases} \tag{3.21}$$

This is a Carathéodory function. We set $H_\lambda(z, x) = \int_0^x h_\lambda(z, s) \, ds$ and consider the C^1-functional $\psi_\lambda \colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{C}}{2}\|u\|^2 - \int_\Omega H_\lambda(z, u(z)) \, dz, \quad \text{for all } u \in H_0^1(\Omega).$$

Reasoning as in the proof of Proposition 3.1, we can find $u_0 \in \operatorname{int} C_+$, with $\underline{u} \leq u_0$, such that

$$\psi_\lambda(u_0) = \inf\{\psi_\lambda(u) : u \in H_0^1(\Omega)\} \tag{3.22}$$

and u_0 is a solution of problem (P_λ). As in the proof of Proposition 3.1, $\widehat{\varphi}_\lambda \colon H_0^1(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{C}}{2}\|u\|_2^2 - \int_\Omega G_\lambda(z, u(z)) \, dz \quad \text{for all } u \in H_0^1(\Omega),$$

where $G_\lambda(z, x) = \int_0^x g_\lambda(z, s) \, ds$ and

$$g_\lambda(z, x) = \lambda (x^+)^{q-1} - f(z, x) + \widehat{C} x^+ \quad \text{for all } (z, x) \in \Omega \times \mathbb{R}.$$

Let $[\underline{u}) := \{u \in H_0^1(\Omega) : \underline{u}(z) \leq u(z) \text{ a.e. in } \Omega\}$. From (3.21) it follows that

$$\psi_\lambda|_{[\underline{u})} = \widehat{\varphi}_\lambda|_{[\underline{u})} - C_{11} \quad \text{with } C_{11} \in \mathbb{R}. \tag{3.23}$$

Let $\rho = \|u_0\|_\infty$ and let $\gamma_\rho > 0$ and $\tau > 2$ be as postulated by hypothesis H(f)(e). Then

$$\begin{aligned}
&-\triangle u_0(z) + \beta(z)u_0(z) + \gamma_\rho(u_0(z)^{\tau-1} + u_0(z)) \\
&\quad = \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \gamma_\rho(u_0(z)^{\tau-1} + u_0(z)) \\
&\quad \geq \lambda \underline{u}(z)^{q-1} - f(z, \underline{u}(z)) + \gamma_\rho(\underline{u}(z)^{\tau-1} + \underline{u}(z)) \quad (\text{since } \underline{u} \leq u_0, \text{ see H}(f)(\text{e})) \\
&\quad \geq -\triangle \underline{u}(z) + \beta(z)\underline{u}(z) + \gamma_\rho(\underline{u}(z)^{\tau-1} + \underline{u}(z)) \quad \text{a.e. in } \Omega,
\end{aligned}$$

(see (3.7) with η replaced by λ, and λ by λ'). Hence

$$\begin{aligned}
\triangle(u_0 - \underline{u})(z) &\leq (\beta(z) + \gamma_\rho)(u_0(z) - \underline{u}(z)) + \gamma_\rho(u_0(z)^{\tau-1} - \underline{u}(z)^{\tau-1}) \\
&\leq (\|\beta^+\|_\infty + \gamma_\rho + C_{12})(u_0(z) - \underline{u}(z))
\end{aligned}$$

almost everywhere in Ω, for some $C_{12} > 0$, therefore

$$u_0 - \underline{u} \in \operatorname{int} C_+ \tag{3.24}$$

(see Vazquez [20]). From (3.22)–(3.24) it follows that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\varphi}_\lambda$. From Brezis and Nirenberg [3], we infer that u_0 is a local $H_0^1(\Omega)$-minimizer of $\widehat{\varphi}_\lambda$. Next, for all $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\widehat{\varphi}_\lambda(u) &\geq \frac{1}{2}\sigma(u) + \frac{\widehat{c}}{2}\|u^-\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q - \int_\Omega \eta(u^+)^2\,dz \\
&\qquad\qquad\qquad\qquad \text{(see H}(f)(\text{c})) \\
&\geq \frac{1}{2}\sigma(u^+) - \frac{1}{2}\int_\Omega \eta(u^+)^2\,dz + \frac{1}{2}\sigma(u^-) + \frac{\widehat{c}}{2}\|u^-\|_2^2 - C_{13}\|u\|^q \\
&\qquad\qquad\qquad\qquad \text{for some } C_{13} > 0, \\
&\geq \frac{C_{14}}{2}\|u^+\|^2 + \frac{1}{2C_1}\|u^-\|^2 - C_{13}\|u\|^q \quad \text{for some } C_{14} > 0 \\
&\qquad\qquad\qquad\qquad \text{(see [9, Lemma 2.1])} \\
&\qquad\qquad\qquad\qquad \text{and (2.4))} \\
&\geq C_{15}\|u\|^2 - C_{13}\|u\|^q \qquad \text{for some } C_{15} > 0.
\end{aligned} \tag{3.25}$$

Since $q > 2$, from (3.25) it follows that $u = 0$ is a local minimizer of $\widehat{\varphi}_\lambda$. Without any loss of generality, we may assume that $0 = \widehat{\varphi}_\lambda(0) \leq \widehat{\varphi}_\lambda(u_0)$ (the reasoning is similar if the opposite inequality is true). Since u_0 is a local minimizer of $\widehat{\varphi}_\lambda$, reasoning as in [1] (see the proof of Proposition 29), we can find $\rho \in (0,1)$ small such that

$$0 = \widehat{\varphi}_\lambda(0) \leq \widehat{\varphi}_\lambda(u_0) < \inf\{\widehat{\varphi}_\lambda(u) : \|u - u_0\| = \rho\} = \widehat{\eta}_\lambda. \tag{3.26}$$

Recall that $\widehat{\varphi}_\lambda$ is coercive (see the proof of Proposition 3.1). Hence it satisfies the PS-condition. This fact and (3.26) enable us to use Theorem 2.1 (the mountain pass theorem) and obtain $\widehat{u} \in H_0^1(\Omega)$ such that

$$0 = \widehat{\varphi}_\lambda(0) \leq \widehat{\varphi}_\lambda(u_0) < \widehat{\eta}_\lambda \leq \widehat{\varphi}_\lambda(\widehat{u}) \tag{3.27}$$

(see (3.26)) and

$$\widehat{\varphi}'_\lambda(\widehat{u}) = 0. \tag{3.28}$$

From (3.27) we see that $\widehat{u} \notin \{0, u_0\}$. From (3.28) it follows that $\widehat{u} \in \operatorname{int} C_+$ solves problem (P_λ). $\square$

Next we see what happens for $\lambda = \lambda^*$ (the "*critical case*").

Proposition 3.4. *If hypotheses* $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold, then* $\lambda_* \in \mathcal{L}$ *and so,* $\mathcal{L} = [\lambda_*, +\infty)$.

Proof. Let $\{\lambda_n\}_{n\geq 1} \subset \mathcal{L}$ be such that $\lambda_n \downarrow \lambda_*$ as $n \to \infty$ (cf. (3.10)). For $n \geq$ let $u_n = u_{\lambda_n} \in \operatorname{int} C_+$ be a positive solution of problem (P_{λ_n}). We have

$$A(u_n) + \beta u_n = \lambda_n u_n^{q-1} - N_f(u_n) \quad \text{for all } n \geq 1. \tag{3.29}$$

By virtue of hypotheses $\mathrm{H}(f)$(a), (b), given any $\xi > 0$, we can find $C_{16} = C_{16}(\xi) > 0$ such that

$$f(z,x)x \geq \xi x^{q-1} - C_{16} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.30}$$

On (3.29) we act with $u_n \in \operatorname{int} C_+$ and obtain

$$\sigma(u_n) = \lambda_n \|u_n\|_q^q - \int_\Omega f(z,u_n)u_n\,dz \leq \lambda_n \|u_n\|_q^q - \xi \|u_n\|_q^q + C_{16}|\Omega|_N,$$

hence

$$\sigma(u_n) + (\xi - \lambda_n)\|u_n\|_q^q \leq C_{16}|\Omega|_N. \tag{3.31}$$

Choosing $\xi > \sup_{n\geq 1} \lambda_n$ big, and recalling that $q > 2$, from (2.4) and (3.31) we infer that $\{u_n\}_{n\geq 1} \subset H_0^1(\Omega)$ is bounded. Therefore we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^{2s'}(\Omega) \text{ and in } L^r(\Omega) \text{ as } n \to \infty.$$

So, passing to the limit as $n \to \infty$ in (3.29), we obtain

$$A(u_*) + \beta u_* = \lambda_* u_*^{q-1} - N_f(u_*),$$

hence $u_* \in C_+$ is a solution of (P_{λ_*}). We need to show that $u_* \neq 0$ (then we will have $u_* \in \operatorname{int} C_+$). We argue by contradiction. So, suppose that $u_* = 0$. We set $y_n = u_n/\|u_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$, So, passing to a suitable subsequence, if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(\Omega) \text{ and } y_n \to y \text{ in } L^{2s'}(\Omega) \text{ and in } L^r(\Omega) \text{ as } n \to \infty. \tag{3.32}$$

From (3.29) we have

$$A(y_n) + \beta y_n = \lambda_n u_n^{q-2} y_n - \frac{N_f(u_n)}{\|u_n\|} \quad \text{for all } n \geq 1. \tag{3.33}$$

By virtue of hypothesis $\mathrm{H}(f)$(a), we can find $C_{17} > 0$ such that

$$0 \leq f(z,x) \leq C_{17}(1 + x^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0,$$

and we conclude that $\{N_f(u_n)/\|u_n\|\}_{n\geq 1} \subset L^{r'}(\Omega)$ is bounded. We may assume that

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} -y\xi \quad \text{in } L^{r'}(\Omega) \text{ with } \xi \in L^\infty(\Omega)_+,\ \xi \leq \eta \tag{3.34}$$

(see [1]). On (3.33) we act with $y_n - y \in H_0^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.32) and (3.34), as before (see the proof of Proposition 3.1). By the Kadec–Klee property of Hilbert spaces we infer that

$$y_n \to y \quad \text{in } H_0^1(\Omega), \quad \text{hence} \quad \|y\| = 1. \tag{3.35}$$

So, if we pass to the limit as $n \to \infty$ in (3.33) and use (3.35), then

$$A(y) + \beta y = \xi y \quad \text{with } y \geq 0,\ \|y\| = 1. \tag{3.36}$$

But since $\xi \leq \eta \leq \widehat{\lambda}_1$, $\eta \neq \widehat{\lambda}_1$, we have $\widehat{\lambda}_1(\xi) > \widehat{\lambda}_1(\widehat{\lambda}_1) = 1$, and so it follows that y may not be an eigenfunction of (3.36); consequently, $y \equiv 0$, a contradiction. This proves that $u_* \neq 0$, and so $u_* \in \operatorname{int} C_+$ is a positive solution of (P_{λ_*}), and we conclude that $\lambda_* \in \mathcal{L}$. $\square$

Proposition 3.5. *If hypotheses* $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold and* $\lambda \geq \lambda_*$, *then problem* (P_λ) *has a smallest positive solution* $\overline{u}_\lambda \in \operatorname{int} C_+$.

Proof. Let $\lambda \geq \lambda_*$ and let $S(\lambda)$ be the set of positive solutions of (P_λ). From Propositions 3.2 and 3.3 we know that $S(\lambda) \neq \emptyset$ and $S(\lambda) \subset \operatorname{int} C_+$. Let C be a chain (i.e. a totally ordered subset) of $S(\lambda)$. Invoking Dunford and Schwartz [5, p. 336], we can find $\{u_n\}_{n\geq 1} \subset C$ such that

$$\inf_c = \inf_{n\geq 1} u_n.$$

Moreover, by virtue of Lemma 1.1.5 of [11, p. 15], we can choose $\{u_n\}_{n\geq 1}$ to be decreasing. Then

$$A(u_n) + \beta u_n = \lambda u_n^{q-1} - N_f(u_n) \quad \text{and} \quad 0 \leq u_n \leq u_1 \quad \text{for all } n \geq 1,$$

hence $\{u_n\}_{n\geq 1} \subset H_0^1(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \overline{u}_\lambda \quad \text{in } H_0^1(\Omega), \quad \text{and} \quad u_n \to \overline{u}_\lambda \quad \text{in } L^{2s'}(\Omega) \text{ and in } L^r(\Omega) \text{ as } n \to \infty.$$

Assuming that $\overline{u}_\lambda = 0$ and using $y_n = u_n/\|u_n\|$, $n \geq 1$, as in the proof of Proposition 3.3, we reach a contradiction. So, $\overline{u}_\lambda \neq 0$ and $\overline{u}_\lambda \in S(\lambda)$. Then $\overline{u}_\lambda = \inf C \in S(\lambda)$ and since C is an arbitrary chain, we can apply the Kuratowski–Zorn lemma and find $\overline{u}_\lambda \in S(\lambda)$, a minimal element. From Lemma 4.3 of Filippakis, Kristaly and Papageorgiou [6] it follows that $S(\lambda)$ is downward directed (i.e. if $u, u' \in S(\lambda)$, one can find $y \in S(\lambda)$ such that $y \leq \min\{u, u'\}$). Therefore, we conclude that $\overline{u}_\lambda \in \operatorname{int} C_+$ is the smallest positive solution of (P_λ). $\square$

Summarizing the above results for problem (P_λ), we conclude that the following bifurcation-type theorem holds true:

Theorem 3.6. *If hypotheses* $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold, then there exists* $\lambda_* > 0$ *such that*:

(a) *for* $\lambda > \lambda_*$, *problem* (P_λ) *has at least two positive smooth solutions* $u_0, \widehat{u} \in \operatorname{int} C_+$;
(b) *for* $\lambda = \lambda_*$, *problem* (P_λ) *has at least one positive solution* $u_* \in \operatorname{int} C_+$;
(c) *for* $\lambda \in (0, \lambda_*)$, *problem* (P_λ) *has no positive solution.*

Moreover, problem (P_λ) *has a smallest positive solution* $\overline{u}_\lambda \in \operatorname{int} C_+$, *for every* $\lambda \geq \lambda_*$.

Acknowledgements. This work was supported in part by the Portuguese Foundation for Science and Technology (FCT–Fundação para a Ciência e a Tecnologia), through CIDMA – Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013 and, for the third author, through the Sabbatical Fellowship SFRH/BSAB/113647/2015 during his sabbatical leave, while he was a Visiting Professor at the Department of Information Engineering, Computer Science and Mathematics (DISIM) of the University of L'Aquila (Italy). The hospitality and partial support of DISIM are gratefully acknowledged.

References

[1] S. Aizicovici, N.S. Papageorgiou and V. Staicu, *Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints*, Mem. Amer. Math. Soc. **196** (915), 2008.

[2] H. Amann, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Funct. Anal. **11** (1972), 346–384.

[3] H. Brézis and L. Nirenberg, *H^1 versus C^1 local minimizers*, C.R. Math. Acad. Sci. Paris **317** (1993), 465–472.

[4] E.N. Dancer, *On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large*, Proc. London Math. Soc. **53** (1986), 429–452.

[5] N. Dunford and J.T. Schwartz, *Linear Operators*, Part. I, Interscience, New York, 1958.

[6] M. Filippakis, A. Kristaly and N.S. Papageorgiou, *Existence of five nonzero solutions with exact sign for a p-Laplacian equation*, Discrete Contin. Dyn. Systems, Ser. A **24** (2009), 405–440.

[7] N. Garofalo and F.H. Lin, *Unique continuation for elliptic operators*: *A geometric-variational approach*, Comm. Pure Appl. Math. **40** (1987), 347–366.

[8] *L. Gasinski and N.S. Papageorgiou*, *Nonlinear Analysis*, Chapman & Hall/CRC Press, Boca Raton, 2006.

[9] ———, *Dirichlet problems with double resonance and an indef- inite potential*, Nonlinear Anal. **75** 92012), 4560–4595.

[10] ———, *Bifurcation-type results for nonlinear parametric elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A. **142** (2012), 595–623.

[11] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Non-linear Differential Equations*, Marcel Dekker, New York, 1994.

[12] S. Kyritsi and N.S. Papageorgiou, *Multiple solutions for superlinear Dirichlet problems with an indefinite potential*, Ann. Mat. Pura Appl. **192** (2013), 297–315.

[13] S. Lin, *On the number of positive solutions for nonlinear elliptic equations when a parameter is large*, Nonlinear Anal. **16** (1991), 263–297.

[14] D. Mugnai and N.S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super Pisa Cl. Sci. Vol. XI, Fasc. 4 (2012), 729–788.

[15] T. Ouang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problems*, J. Differential Equations **146** (1998), 121–156.

[16] N.S. Papageorgiou and S. Kyritsi Yiallourou, *Handbook of Applied Analysis*, Springer, New York, 2009.

[17] P. Rabinowitz, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **23** (1973), 172–185.

[18] R. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, London, 1977.

[19] M. Struwe, *Variational Methods*, Springer Verlag, Berlin, 1990.

[20] J. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

Manuscript received September 30, 2013
accepted March 17, 2014

Sergiu Aizicovici
Department of Mathematics
Ohio University
Athens, OH 45701, USA
E-mail address: aizicovs@ohio.edu

Nikolaos S. Papageorgiou
Department of Mathematics
National Technical University
Zografou Campus
Athens 15780, GREECE
E-mail address: npapg@math.ntua.gr

Vasile Staicu
CIDMA – Center for Research and Development
in Mathematics and Applications
Department of Mathematics
University of Aveiro
Aveiro, PORTUGAL
E-mail address: vasile@ua.pt