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BOUNDEDNESS OF LARGE-TIME SOLUTIONS TO A CHEMOTAXIS MODEL WITH NONLOCAL AND SEMILINEAR FLUX

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Abstract. A semilinear version of parabolic-elliptic Keller–Segel system with the *critical* nonlocal diffusion is considered in one space dimension. We show boundedness of weak solutions under very general conditions on our semilinearity. It can degenerate, but has to provide a stronger dissipation for large values of a solution than in the critical linear case or we need to assume certain (explicit) data smallness. Moreover, when one considers a logistic term with a parameter r, we obtain our results even for diffusions slightly weaker than the critical linear one and for arbitrarily large initial datum, provided $r > 1$. For a mild logistic dampening, we can improve the smallness condition on the initial datum up to $\sim 1/(1-r)$.

1. Introduction

In this paper we study the following model:

$$\partial_t u = \partial_x(-\mu(u)Hu + u\partial_x v) + ru(1-u), \quad x \in \mathbb{T},\ t \in \mathbb{R}^+, \tag{1.1}$$

$$\partial_x^2 v = u - \langle u \rangle, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+, \tag{1.2}$$

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where $u = u(x,t)$, $v = v(x,t)$, H stands for the (periodic) Hilbert transform, i.e.

$$\widehat{Hu}(\xi) = -i\,\frac{\xi}{|\xi|}\,\widehat{u}(\xi).$$

$\mathbb{T} = [-\pi,\pi]$, $r \geq 0$ and μ is a certain function (semilinearity), precised in what follows. Before formulating our results let us explain our motivations to study system (1.1)–(1.2).

1.1. Motivation. (a) *Mathematical biology.* One of the basic systems studied in the context of chemotaxis is the parabolic–elliptic Keller–Segel system (also known as the Smoluchowski–Poisson system)

$$\partial_t u = \nabla \cdot (\mu \nabla u - u \nabla \phi), \quad x \in \mathbb{T},\ t \in \mathbb{R}^+, \tag{1.3}$$

where $d \geq 1$ denotes the spatial dimension, $\mathbb{T}^d = [-\pi,\pi]^d$, $\mu > 0$ is a constant and ϕ is recovered from u through some operator, i.e. $\phi(x,t) = T(u(x,t))$. In many cases ϕ satisfies the Poisson equation

$$-\Delta\phi = u - \langle u \rangle, \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+. \tag{1.4}$$

In this notation, u represents the concentration of cells, $\langle u \rangle$ its space average and ϕ gives us the concentration of a chemical substance that attracts cells. It is biologically justified to enrich equation (1.3) with the logistic term, obtaining

$$\partial_t u = \nabla \cdot (\mu \nabla u - u \nabla \phi) + ru(1-u), \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.5}$$

where $r \geq 0$. Model (1.5)–(1.4) is related to the parabolic-elliptic simplification of the cell kinetics model M8 in [28], that describes a bacterial pattern formation or cell movement and growth during angiogenesis.

Another application of model (1.5)–(1.4) occurs in tumor growth. In particular, this model is related to the three-component urokinase plasminogen invasion model (see [29]). There is a huge literature on the mathematical study of a numerous versions of (1.5)–(1.4) in the context of mathematical biology, see [5], [7], [9], [10], [16], [25], [30] and the references therein.

(b) *Natural sciences.* Let us take in (1.5)–(1.4), $v := -\phi$. The resulting system

$$\partial_t u = \nabla \cdot (\mu \nabla u + u \nabla v) + ru(1-u), \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.6}$$

$$\Delta v = u - \langle u \rangle, \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.7}$$

in the case $r = 0$ is important in mathematical cosmology and gravitation theory. It is very similar in spirit to the Zel'dovich approximation used in cosmology to study the formation of large-scale structure in the primordial universe, see also [1], [4]. It is also connected with the Chandrasekhar equation for the gravitational equilibrium of polytropic stars, statistical mechanics and the Debye system for electrolytes, see [6].

Remark 1.1. In what follows, we consider a system with the sign "+" in front of terms $u\nabla v$ and Δv, compare motivation (b), remembering that letting $\phi := -v$ we get the equations studied in mathematical biology (see motivation (a)).

1.2. Central problem. The focal point in the studies of solutions to (1.6)–(1.7) is the matter of distinguishing between the blowup and global-in-time regimes in correlation with the dimension d, initial data and parameters of the system. Roughly speaking, it turns out that with $\mu = 1$, $r = 0$ there is a 8π criticality of the initial mass $\|u_0\|_{L^1}$ in two dimensions. Below this threshold one can have the global existence of bounded solutions and above it there is a finite time blowup (see for instance [11]). In the one-dimensional case, the diffusion $\nabla \cdot (\nabla u)$ is strong enough to give the global existence. On the other hand, for $d > 2$ it is too weak (let us remark here that the logistic term $ru(1-u)$ generally helps the global existence, compare [35]). In this context it is mathematically interesting to find, for a fixed dimension d, a "critical" diffusive operator that sits on the borderline of the blowup and global-in-time regimes. There are at least two approaches to this problem, both justified also from the point of view of applications. One is to consider semilinear diffusion $\nabla \cdot (\mu(u)\nabla u)$, see for instance [3], [8], [23], [13], [22], [32]. Another one is to replace the standard diffusion with the fractional one. We focus on this case. Let us consider

$$\partial_t u = -\mu\Lambda^\alpha u + \nabla \cdot (u\nabla v) + ru(1-u), \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.8}$$

$$\Delta v = u - \langle u \rangle, \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}^+, \tag{1.9}$$

where $\mu > 0$ is a constant and the operator Λ^α is defined using the Fourier transform

$$\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi).$$

It turns out that with $r = 0$ and $d = 1$ there are global-in-time solutions for $\alpha > 1$ and blowups for $\alpha < 1$, compare [12] and [26] (see also [1]). The case $\alpha = 1$ seems critical and, to the best of our knowledge, the sharpest result up to now is the global boundedness for small data. In particular, it is shown in [12] that there exists a constant K, such that $\|u_0\|_{L^1} \leq K$ implies the global existence of solutions. Later on, in [1], the authors proved that $\|u_0\|_{L^1} \leq 1/(2\pi)$ implies global existence and the convergence towards the homogeneous steady state. In this context we refer also to [31].

In this paper we propose a slight semilinear strengthening of the diffusion in (1.8) that provides bounded solutions for any $r \geq 0$. We also study the regularization due the logistic dampening for the diffusions equal to that of (1.8) or slightly weaker, compare example (2.9).

The case of (1.1)–(1.2) with $\alpha = 1$, $r = 0$ and $\mu(s) = s + \nu$ appears in [27], where the authors address the local/global existence and the qualitative behavior of the solutions. Similar equations have been studied in [2], [17], [19]–[21] in the context of fluid dynamics. In particular, the equation

$$\partial_t u = \partial_x(-[u+\nu]Hu) \tag{1.10}$$

has been proposed as a one-dimensional model of the 2D Vortex Sheet problem or the 2D surface quasi-geostrophic equation. Notice that (1.1) reduces to (1.10) when $v \equiv 0$, $r = 0$ and $\mu(x) = x + \nu$.

1.3. Basic notation and plan of the paper. We write $H^s(\mathbb{T})$ for the usual L^2-based Sobolev spaces with norm

$$\|f\|_{H^s}^2 := \|f\|_{L^2}^2 + \|f\|_{\dot{H}^s}^2, \qquad \|f\|_{\dot{H}^s} := \|\Lambda^s f\|_{L^2},$$

and

$$\langle u \rangle := \frac{1}{2\pi}\int_{\mathbb{T}} u(x)\,dx.$$

For a given initial data u_0, we introduce the following notation:

$$\mathcal{N}_1 := \max\{2\pi, \|u_0\|_{L^1}\}. \tag{1.11}$$

Notice that for periodic functions, the half-laplacian in one dimension has the following kernel representation:

$$\Lambda f(x) = \frac{\text{p.v.}}{2\pi}\int_{\mathbb{T}} \frac{f(x)-f(y)}{\sin^2((x-y)/2)}\,dy.$$

The remainder of this paper is organized as follows: In Section 2 we present the statement of our results. In Section 3 we prove Theorem 2.4. In Section 4 we prove Theorem 2.6. Finally, in Section 6 we present our proof of Theorem 2.9.

2. Statement of results

Given the initial data $u_0(x) \geq 0$, we have the following definition of a weak solution to system (1.1)–(1.2):

Definition 2.1. Choose $u_0 \in L^2(\mathbb{T})$. Fix arbitrary $T \in (0,\infty)$. The couple

$$(u,v) \in L^\infty(0,T;L^2(\mathbb{T})) \times L^\infty(0,T;H^1(\mathbb{T}))$$

is a solution of (1.1)–(1.2) if and only if

$$\int_0^T \int_{\mathbb{T}} -\partial_t\phi\, u + \partial_x\phi\,(-\mu(u)Hu + u\partial_x v) - \phi\, ru(1-u)\,dx\,dt \\ - \int_{\mathbb{T}} \phi(x,0)u_0\,dx = 0,$$

$$\int_0^T \int_{\mathbb{T}} \partial_x\varphi\,\partial_x v + \varphi\,(u - \langle u\rangle)\,dx\,dt = 0,$$

for every test function $\phi(x,t), \varphi(x,t) \in C^\infty((-1,T)\times\mathbb{T})$ with a compact support in time and periodic in space.

Definition 2.2. If a solution (u, v) verifies Definition 2.1 for any $T < \infty$, we call it a *large-time weak solution*.

We will use the following entropy (or free energy) functional:

$$\mathcal{F}(u(t)) = \int_{\mathbb{T}} u(t)\log(u(t)) - u(t) + 1. \tag{2.1}$$

2.1. Case of linearly degenerating μ and any $r \geq 0$. The results presented in this section do not use extra information in estimates that follows from the logistic term $ru(1-u)$. Hence they hold for any value of $r \geq 0$, including small ones. The semilinearity μ of our diffusion will be generated here by a function γ as follows

$$\mu(s) := \gamma(s)s. \tag{2.2}$$

Let us introduce also

$$\Gamma(s) := \int_0^s \gamma(y)\,dy.$$

We work within

Assumption 2.3. The semilinearity μ is differentiable and its derivative μ' is bounded for bounded arguments, i.e. there exists a finite function C such that $\mu'(s) \leq C(s)$ for any $s \in [0, \infty)$. Moreover, γ of (2.2) satisfies, for any $y \in [0, \infty)$,

$$\gamma(y) \geq \delta > 0 \tag{2.3}$$

for a fixed $\delta > 0$ and there exists $0 \leq y_0 < \infty$ such that

$$\gamma(y) \geq 1 \quad \text{for } y \geq y_0. \tag{2.4}$$

The fact that μ is *linearly degenerating* is understood in the sense of condition (2.3), as it allows for $\mu(s) = \delta s$ for small s.

Theorem 2.4. *Let $0 \leq u_0 \in L^\infty$ be the initial data for* (1.1)–(1.2) *under Assumption* 2.3. *Then there exists at least one large-time weak solution to* (1.1)–(1.2) (*in the sense of Definitions* 2.1, 2.2). *Furthermore, this solution enjoys additionally the following regularity:*

$$u \in L^\infty(0,T;L^\infty(\mathbb{T})) \cap L^2(0,T;H^{1/2}(\mathbb{T})) \quad \textit{for all } T < \infty,$$

where the L^∞ bound is T-independent.

2.2. Results using the logistic dampening. Now we formulate a result that allows for the the critical linear nonlocal diffusion (*i.e.* $\mu \equiv c$) at the cost of using a relation between the lower bound on μ, the initial mass $\langle u_0 \rangle$ and r. Moreover, for strictly positive data it generalizes Theorem 2.4 over any $\mu(s)$ that is positive for $s > 0$. Particularly, we do not need to assume here the sublinear profile of degeneration of μ.

Now we assume the following hypothesis on the semilinearity.

Assumption 2.5. The semilinearity μ is differentiable and its derivative μ' is bounded for bounded arguments, i.e. there exists a finite function C such that $\mu'(s) \leq C(s)$ for any $s \in [0, \infty)$. Moreover, μ is positive for positive arguments, i.e. if $\mu(x_0) = 0$ then $x_0 = 0$, and there exists $\delta \geq 0$ such that $\mu(s) \geq \delta$.

Observe that above we allow $\delta = 0$ (then the condition of positivity for positive arguments prevails). In the following result $\delta \geq 0$ comes from Assumption 2.5 and r is the parameter of the logistic term.

Theorem 2.6. *Let* $0 \leq u_0 \in L^\infty$. *If, in addition to Assumption* 2.5 *we have*

$$r + \delta(4\pi^2 \max\{\langle u_0 \rangle, 1\})^{-1} > 1, \tag{2.5}$$

and either

$$\delta > 0 \tag{2.6}$$

or

$$\operatorname*{ess\,min}_x u_0 > 0, \tag{2.7}$$

then there exists at least one large-time weak solution to (1.1)–(1.2) (*in the sense of Definitions* 2.1, 2.2). *This solution enjoys additionally the following regularity:*

$$u \in L^\infty(0, T; L^\infty(\mathbb{T})) \cap L^2(0, T; H^{1/2}(\mathbb{T})) \quad \text{for all } T < \infty,$$

where the L^∞ *bound is* T*-independent.*

In the case $r = 0$ we can provide a simpler condition, according to

Corollary 2.7. *In the case* $r = 0$ *Theorem* 2.6 *is valid with condition* (2.5) *replaced with*

$$\frac{\delta}{4\pi^2 \langle u_0 \rangle} > 1. \tag{2.8}$$

2.3. Remarks. The weakest semilinear diffusion allowed by Theorem 2.4 is $\mu(s) = \delta(s)s$ with: $\delta(s) \geq 1$ for large s and being arbitrary positive constant otherwise. This is the previously mentioned *semilinear strengthening* of the critical nonlocal linear diffusion $-\mu\Lambda u$ with constant μ.

On the other hand, in Theorem 2.6 this critical diffusion is admissible with any $\mu \equiv c$ for arbitrary $c > 0$, provided the logistic parameter $r \geq 1$. What is more, for $r > 1$ and strictly positive data we can allow for

$$\mu(s) \begin{cases} = 0 & \text{for } s = 0, \\ > 0 & \text{for } s > 0, \end{cases} \tag{2.9}$$

hence we do not assume any precise profile of degeneracy at 0 of our semilinearity μ and it can be slightly weaker than the linear $\mu \equiv c > 0$. From the proof of Theorem 2.6 one sees that even a diffusion that vanishes outside a certain interval of arguments is allowed, compare Remark 4.1. Finally, for $r \in [0,1)$ we need in Theorem 2.6 to mitigate the weaker logistic dampening with larger diffusion, namely such that

$$\delta > (1-r)(4\pi^2 \max\{\langle u_0 \rangle, 1\}).$$

Observe that the bigger the initial mean value, the stronger diffusions we need. See also Corollary 2.7.

Let us now compare Theorems 2.4, 2.6 with known results.

(a) In order to get the system considered in [27] we take $r = 0$ and $\gamma(x) = 1 + \nu/x$. This falls under our assumption (2.4). We see now that our Theorem 2.4 recovers the result of Theorem 5.2 in [27] and sharpens it with respect to the admissible initial data. Namely, we have removed the $H^{1/2}$ smoothness requirement and, more importantly, the smallness assumption $\|u_0\|_{L^1} \leq 2\nu/3$ of [27].

(b) We allow in Theorem 2.4 for much more general semilinear diffusions μ than in [27] and for the logistic term.

(c) When restricted to linear case, *i.e.* $\mu \equiv c$, Corollary 2.7 is inline with results in [1], [12]. Moreover, our condition (2.8) is explicit and says that the threshold mass for a (debatable) blowup is at least $\delta/(2\pi)$.

2.4. Results for a nonlocal porous medium type equation. If we take $v \equiv 0$ and $r = 0$ in (1.1) we get

$$\partial_t u = \partial_x(-\mu(u) H u), \quad x \in \mathbb{T},\ t \in \mathbb{R}^+. \tag{2.10}$$

We have the following result:

Theorem 2.8. *Let* $0 \leq u_0 \in L^\infty$ *be the initial data for* (2.10) *with* $\mu(s)$ *following Assumption* 2.3*. Then there exists at least one large-time weak solution* u *of* (2.10)*. Furthermore, this solution is*

$$u \in L^\infty(0,T; L^\infty(\mathbb{T})) \cap L^2(0,T; H^{1/2}(\mathbb{T})) \quad \text{for all } T < \infty,$$

where the L^∞ *bound is* T*-independent.*

The proof of this theorem is similar to the proof of Theorem 2.4, so we omit it.

Finally, we provide also the following result on asymptotics of solutions to a special case of (2.10).

Theorem 2.9. *Let the initial data for* (2.10) *with* $\mu(x) = x$ *be* $u_0 \in L^\infty$ *such that* $\operatorname{ess\,min}_x u_0 > 0$ *with* $\mu(x) = x$ *and assume that* $\langle u_0 \rangle = 1$*. Then the large-time solution* $u(x,t)$ *of* (2.10) *tends to the homogeneous steady state* $u_\infty \equiv 1$ *and satisfies*

$$\mathcal{F}(u(t)) \leq C(u_0) e^{-2(\operatorname{ess\,inf}_x u_0)\, t}.$$

In particular this theorem covers the case $\lambda = 0$, $s = 1/2$ in [18] for periodic and positive initial data.

3. Proof of Theorem 2.4

3.1. Approximate problems. Let us consider a family of Friedrichs mollifiers $\mathcal{J}_\varepsilon$. We define the regularized initial data

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) = \mathcal{J}_\varepsilon * u_0(x) \geq 0$$

and consider the approximate problems

$$\partial_t u^\varepsilon = \partial_x(-\mu(u^\varepsilon)Hu^\varepsilon + u^\varepsilon \partial_x v^\varepsilon) + ru^\varepsilon(1-u^\varepsilon) + \varepsilon\partial_x^2 u^\varepsilon, \tag{3.1}$$

$$\partial_x^2 v^\varepsilon = u^\varepsilon - \langle u^\varepsilon \rangle. \tag{3.2}$$

Each of these problems has a local-in-time smooth solution with the maximal time of existence T_ε. This can be shown via a fixed point argument (basically, Picard's Theorem in Banach spaces). For the time being we will work within this local time of existence.

3.2. L^1 bound. Integrating equation (3.1) and using Jensen's inequality, we have

$$\|u^\varepsilon(t)\|_{L^1} \leq \mathcal{N}_1 = \max\{\|u_0\|_{L^1}, 2\pi\}. \tag{3.3}$$

3.3. Pointwise bounds. Let us denote the point where $\min_x u^\varepsilon(t)$ is attained as $\underline{x}_t$. Similarly, we write $\overline{x}_t$ for the point where the maximum is reached. In other words,

$$\min_{x\in\mathbb{T}} u^\varepsilon(t) = u^\varepsilon(\underline{x}_t, t), \qquad \max_{x\in\mathbb{T}} u^\varepsilon(t) = u^\varepsilon(\overline{x}_t, t). \tag{3.4}$$

Then, using the same arguments as in [1], [2], [27], we prove

$$\min_{x\in\mathbb{T}} u^\varepsilon(t) \geq \min_{x\in\mathbb{T}} u_0^\varepsilon e^{\int_0^t -\gamma(u^\varepsilon(\underline{x}_s))\Lambda u^\varepsilon(\underline{x}_s) + u^\varepsilon(\underline{x}_s) - \langle u^\varepsilon(s)\rangle + r(1-u^\varepsilon(\underline{x}_s))\,ds} \geq 0 \tag{3.5}$$

and, using $\Lambda u^\varepsilon(\overline{x}_t, t) \geq u^\varepsilon(\overline{x}_t, t) - \langle u^\varepsilon(t)\rangle$, for $X(t) := \|u^\varepsilon(t)\|_{L^\infty}$, we get the following ODI:

$$\dot{X} \leq X\big((X - \langle u^\varepsilon\rangle)(1-\gamma(X)) + r(1-X)\big) =: XI. \tag{3.6}$$

Before proceeding with analysis of (3.6), let us explain why X is differentiable (almost-everywhere). Take two $t \leq s$ and to fix ideas, let us assume that $X(t) \geq X(s)$, then

$$|X(t) - X(s)| = X(t) - X(s) = u^\varepsilon(\overline{x}_t, t) - u^\varepsilon(\overline{x}_s, s) \leq u^\varepsilon(\overline{x}_t, t) - u^\varepsilon(\overline{x}_t, s),$$

where the inequality follows from the definition of $X(s)$. Consequently, smoothness of u^ε gives Lipschitz continuity of X. The Rademacher Theorem provides now almost everywhere differentiability of X.

Let us come back to considering (3.6). First, consider the case $r > 0$. Recalling Assumption 2.3 (in particular condition (2.4)), we can ensure the existence of $s_0 \in \mathbb{R}^+$ such that $\gamma(s) \geq 1$ if $s \geq s_0$. We can always choose s_0 such that $s_0 \geq 1$ and $X(0) \leq u_0 < s_0$. Then, we have the alternative:

(i) either $X(t) \leq s_0$ for all times, or

(ii) there exists $t_0 > 0$ such $X(t_0) = s_0$ and X crosses s_0 for the first time at t_0.

Let us focus on the case (ii). Here, in view of the above choices of s_0 and continuity of X, we see that there exists $\delta > 0$ such that for $\tau \in [t_0, t_0 + \delta]$ we have $1 - \gamma(X(\tau)) \leq 0$ and $r(1 - X(\tau)) \leq 0$. This in tandem with $X(t) \geq \langle u^\varepsilon(t)\rangle$ implies that in (3.6), $I(\tau) \leq 0$.

Applying integration to our ODI (3.6), we get

$$X(t) \leq X(t_0)e^{\int_{t_0}^t I(\tau)\,d\tau} = s_0 e^{\int_{t_0}^t I(\tau)\,d\tau} \quad \text{for } t \in [t_0, t_0 + \delta], \tag{3.7}$$

but then $X(t) \leq s_0$ for $t \in [t_0, t_0 + \delta]$, because the exponential function has nonpositive exponent I. Hence we have falsified (ii). As a consequence, we have (i), i.e. $X(t) \leq s_0$ for every time.

In the case $r = 0$, we have $\langle u^\varepsilon(t)\rangle = \langle u_0\rangle$ and an analogous argument follows. Consequently we have the bound

$$\|u^\varepsilon(t)\|_{L^\infty} \leq s_0(\|u_0\|_{L^\infty}, r, \gamma). \tag{3.8}$$

3.4. $\dot{H}^1$ bound. For the time being, we have worked within the local time of existence T_ε. Now we fix any $T \in (0, \infty)$ and prove that $T_\varepsilon \geq T$ for any ε. We test (3.1) against $-\partial_x^2 u^\varepsilon$. We get

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|_{\dot{H}^1}^2 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

with

$$I_1 = \int_{\mathbb{T}} \mu(u^\varepsilon)\Lambda u^\varepsilon \partial_x^2 u^\varepsilon\,dx, \qquad I_2 = \int_{\mathbb{T}} (\mu'(u^\varepsilon))\partial_x u^\varepsilon H u^\varepsilon \partial_x^2 u^\varepsilon\,dx,$$

$$I_3 = -\int_{\mathbb{T}} u^\varepsilon(u^\varepsilon - \langle u^\varepsilon(t)\rangle)\partial_x^2 u^\varepsilon\,dx, \qquad I_4 = -\int_{\mathbb{T}} \partial_x u^\varepsilon \partial_x v^\varepsilon \partial_x^2 u^\varepsilon\,dx,$$

$$I_5 = -\int_{\mathbb{T}} r u^\varepsilon(1 - u^\varepsilon)\partial_x^2 u^\varepsilon dx, \qquad I_6 = -\varepsilon\|\partial_x^2 u^\varepsilon\|_{L^2}^2.$$

As we have the bound (3.8), using Assumption 2.3m we get

$$|\mu(u^\varepsilon)| + |\mu'(u^\varepsilon)| \leq \mathcal{C}(s_0).$$

Recall the following property of the Hilbert transform: $\|Hf\|_{L^p} \leq c_p\|f\|_{L^p}$, for $1 < p < \infty$.

As a consequence, we find the bounds:

$$
\begin{aligned}
I_1 &\le \mathcal{C}(s_0)\|u^\varepsilon\|_{\dot H^1}\|\partial_x^2 u^\varepsilon\|_{L^2},\\
I_2 &\le \mathcal{C}(s_0)\|Hu^\varepsilon\|_{L^4}\|\partial_x u^\varepsilon\|_{L^4}\|\partial_x^2 u^\varepsilon\|_{L^2} \le \mathcal{C}(s_0)c_4\|u^\varepsilon\|_{L^4}3\|u^\varepsilon\|_{L^\infty}^{0.5}\|\partial_x^2 u^\varepsilon\|_{L^2}^{1.5},\\
I_3 &\le \|\partial_x^2 u^\varepsilon\|_{L^2}\|u^\varepsilon\|_{L^2}[\|u^\varepsilon\|_{L^\infty} + \langle u^\varepsilon(t)\rangle],\\
I_4 &\le \frac{1}{2}\|u^\varepsilon\|_{\dot H^1}^2[\|u^\varepsilon\|_{L^\infty} + \langle u^\varepsilon(t)\rangle],\\
I_5 &\le r\|\partial_x^2 u^\varepsilon\|_{L^2}\|u^\varepsilon\|_{L^2}[\|u^\varepsilon\|_{L^\infty} + 1].
\end{aligned}
$$

Using Young's inequality and the ε-dissipative term I_6, we have

$$\frac{d}{dt}\|u^\varepsilon\|_{\dot H^1}^2 \le C_1(\varepsilon) + C_2(\varepsilon)\|u^\varepsilon\|_{\dot H^1}^2, \tag{3.9}$$

where C_1 and C_2 depend also on $\mathcal{N}_1$, s_0, r and γ.

Let us assume now that for a given ε it holds $T_\varepsilon < T$. Using Gronwall's and Poincaré's inequalities in (3.9), we obtain existence of an approximate solution in H^1 up to T_ε. It is also bounded in view of (3.8). Hence it is smooth by bootstrapping in (3.1). Therefore it can be continued beyond T_ε, which consequently cannot be the maximal time of existence. We have $T_\varepsilon \ge T$ and (3.3), (3.8) hold on $[0, T]$.

3.5. Uniform estimates. At this stage, we have ε-uniform estimates for $u^\varepsilon(x,t)$ in $L_t^\infty L_x^p$ with $1 \le p \le \infty$ on $[0,T]$ (see (3.8)). Recalling (2.2), we have

$$\int_{\mathbb{T}} \partial_t u^\varepsilon \log(u^\varepsilon)\,dx = I_7 + I_8 + I_9 + I_{10},$$

with

$$
\begin{aligned}
I_7 &= \int_{\mathbb{T}} \partial_x u^\varepsilon \gamma(u^\varepsilon) H u^\varepsilon dx = \int_{\mathbb{T}} \partial_x \Gamma(u^\varepsilon) H u^\varepsilon\,dx = -\int_{\mathbb{T}} \Gamma(u^\varepsilon)\Lambda u^\varepsilon\,dx,\\
I_8 &= -\int_{\mathbb{T}} \partial_x u^\varepsilon \partial_x v^\varepsilon\,dx = \int_{\mathbb{T}} u^\varepsilon(u^\varepsilon - \langle u^\varepsilon\rangle)\,dx,\\
I_9 &= \int_{\mathbb{T}} r u^\varepsilon(1-u^\varepsilon)\log(u^\varepsilon)\,dx \le 0,\\
I_{10} &= -4\varepsilon\int_{\mathbb{T}} |\partial_x(\sqrt{u^\varepsilon})|^2\,dx.
\end{aligned}
$$

Consequently, the evolution of the entropy functional (2.1) is given by

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) + \int_{\mathbb{T}} \Gamma(u^\varepsilon)\Lambda u^\varepsilon\,dx + 4\varepsilon\int_{\mathbb{T}} |\partial_x(\sqrt{u^\varepsilon})|^2\,dx&\\
= \|u^\varepsilon - \langle u^\varepsilon\rangle\|_{L^2}^2 + \int_{\mathbb{T}} r u^\varepsilon(1-u^\varepsilon)\log(u^\varepsilon)\,dx.&
\end{aligned}
$$

Symmetrizing the integral I_7, we have

$$\begin{aligned}-I_7 &= \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{u^\varepsilon(x)-u^\varepsilon(y)}{\sin^2((x-y)/2)}(\Gamma(u^\varepsilon(x))-\Gamma(u^\varepsilon(y)))\,dx\,dy\\ &= \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\int_0^1 \frac{(u^\varepsilon(x)-u^\varepsilon(y))^2}{\sin^2((x-y)/2)}\gamma(su^\varepsilon(x)+(1-s)u^\varepsilon(y))\,ds\,dx\,dy\\ &\geq \frac{\delta}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(u^\varepsilon(x)-u^\varepsilon(y))^2}{\sin^2((x-y)/2)}\,dx\,dy = \delta\int_{\mathbb{T}} \Lambda u^\varepsilon u^\varepsilon\,dx = \delta\|u^\varepsilon\|^2_{\dot{H}^{0.5}},\end{aligned}$$

where we have used (2.3) of Assumption 2.3. Integrating in time, we get the uniform bound

$$\mathcal{F}(u^\varepsilon(t)) + \delta\int_0^t \|u^\varepsilon(s)\|^2_{\dot{H}^{0.5}}\,ds \leq C_3(\gamma,\mathcal{N}_1,s_0)t + \mathcal{F}(u_0^\varepsilon),$$

so

$$\delta\int_0^t \|u^\varepsilon(s)\|^2_{\dot{H}^{0.5}}\,ds \leq C_4(\gamma,\mathcal{N}_1,s_0)(t+1). \tag{3.10}$$

For the derivative $\partial_t u^\varepsilon$, using the duality pairing, we obtain

$$\|\partial_t u^\varepsilon\|_{H^{-1.5}} \leq \varepsilon\|u^\varepsilon\|_{H^{1/2}} + C(\gamma,s_0), \tag{3.11}$$

where we have used that $\|u^\varepsilon\|_{L^2}$ controls $\|\partial_x v^\varepsilon\|_{L^\infty}$.

Let us sum up the obtained uniform bounds. Given a finite, arbitrary $T \in (0,\infty)$, we have now the ε-uniform bounds for

$$u^\varepsilon \text{ in } L^\infty(0,T;L^\infty(\mathbb{T})) \cap L^2(0,T;H^{1/2}(\mathbb{T})),$$

in view of (3.8) and (3.10) as well as for $\partial_t u^\varepsilon$ in $L^2(0,T;H^{-1.5}(\mathbb{T}))$ in view of (3.11), provided $\varepsilon \leq 1$.

For any $T<\infty$, applying the sequential $*$-weak compactness of spaces with a separable predual space, we have the existence of a limit function u in appropriate spaces. Equipped with this u we define v using (1.2).

3.6. Compactness. Applying Aubin–Lions's Lemma, we have the strong convergences (up to a subsequence $\varepsilon_n \to 0$ as $n \to \infty$)

$$\lim_{n\to\infty}\int_0^T \|u^{\varepsilon_n}(s)-u(s)\|^2_{L^2}\,ds = 0.$$

Notice that

$$\lim_{n\to\infty}\int_0^T |\langle u^{\varepsilon_n}\rangle - \langle u\rangle|^2\,ds = 0.$$

Consequently, using $\|\partial_x v^{\varepsilon_n} - \partial_x v\|_{L^2} \leq \|u^{\varepsilon_n} - u + \langle u\rangle - \langle u^{\varepsilon_n}\rangle\|_{L^2}$, we get

$$\lim_{n\to\infty}\int_0^T \|\partial_x v^{\varepsilon_n} - \partial_x v\|^2_{L^2}\,ds = 0. \tag{3.12}$$

3.7. Passing to the limit. Here we show that (3.1) is (1.1) in the limit $n \to \infty$, in the sense of Definition 2.1. We have in view of Assumption 2.3 and (3.8)

$$|\mu(u^{\varepsilon_n}(x,t)) - \mu(u(x,t))| \leq \mathcal{C}(\gamma, \mathcal{N}_\infty)|u^{\varepsilon_n}(x,t) - u(x,t)|,$$

and, consequently,

$$\begin{aligned}\bigg|\int_0^T \int_{\mathbb{T}} \partial_x \phi[\mu(u^{\varepsilon_n}) - \mu(u)] H u^{\varepsilon_n}\, dx\, ds\bigg| \\ \leq \|\partial_x \phi\|_{L^\infty_t L^\infty_x} \|\mu(u^{\varepsilon_n}) - \mu(u)\|_{L^2_t L^2_x} \|H u^{\varepsilon_n}\|_{L^2_t L^2_x} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Similarly we show

$$\bigg|\int_0^T \int_{\mathbb{T}} \partial_x \phi \mu(u)[H u^{\varepsilon_n} - H u]\, dx\, ds\bigg| \xrightarrow{n\to\infty} 0.$$

Using (3.12) we get

$$\bigg|\int_0^T \int_{\mathbb{T}} \partial_x \phi [u^{\varepsilon_n} - u] \partial_x v^{\varepsilon_n}\, dx\, ds\bigg| \xrightarrow{n\to\infty} 0,$$
$$\bigg|\int_0^T \int_{\mathbb{T}} \partial_x \phi u [\partial_x v^{\varepsilon_n} - \partial_x v]\, dx\, ds\bigg| \xrightarrow{n\to\infty} 0.$$

We can pass to the limit in the logistic term with the same ideas. Dealing with the term with laplacian is straightforward since it is linear. The only term left is the one corresponding to the initial data. This term can be handled using the properties of mollifiers. We have proved that (u, v) is a solution of (1.1)–(1.2) according to Definition 2.1 and enjoys regularity properties from the statement of Theorem 2.4.

4. Proof of Theorem 2.6

Several steps are similar to those in the proof of Theorem 2.4, so we omit them. We consider the same approximate problems and we get the same $L^\infty_t L^1_x$ bound (3.3). Furthermore, these approximate problems have the large-time solution in H^1. Consequently, we focus on the uniform estimates up to a fixed (but otherwise arbitrary) $0 < T < \infty$.

4.1. Pointwise bounds. We use the same notation (3.4) as before and we get that positivity is preserved. In particular, using

$$-\Lambda u^\varepsilon(\underline{x}_t, t) \geq -(u^\varepsilon(\underline{x}_t, t) - \langle u^\varepsilon(t)\rangle),$$

we can sharpen our bound. In this case, we have the ODE

$$\begin{aligned}\frac{d}{dt}\min_{x\in\mathbb{T}} u^\varepsilon(t) &= \partial_t u^\varepsilon(\underline{x}_t, t)\\ &\geq (1-\gamma(u^\varepsilon(\underline{x}_t)))u^\varepsilon(\underline{x}_t)(u^\varepsilon(\underline{x}_t) - \langle u^\varepsilon(t)\rangle) + ru^\varepsilon(\underline{x}_t)(1-u^\varepsilon(\underline{x}_t))\\ &\geq -\mu(u^\varepsilon(\underline{x}_t))(u^\varepsilon(\underline{x}_t) - \langle u^\varepsilon(t)\rangle) + u^\varepsilon(\underline{x}_t)(1-\langle u^\varepsilon(t)\rangle)\\ &\geq -u^\varepsilon(\underline{x}_t)\max\{1,\langle u_0\rangle\},\end{aligned}$$

so $\min_{x\in\mathbb{T}} u^\varepsilon(t) \geq \operatorname*{ess\,inf}_{x\in\mathbb{T}} u_0 e^{-\max\{1,\langle u_0\rangle\}t} \geq 0$ for all $0 \leq t \leq T < \infty$. In particular, if the initial data is strictly positive,

$$\min_{0\leq t\leq T}\min_{x\in\mathbb{T}} u^\varepsilon(t) \geq \operatorname*{ess\,inf}_{x\in\mathbb{T}} u_0 e^{-\max\{1,\langle u_0\rangle\}T} > 0. \tag{4.1}$$

We have to deal with the bound for $\|u^\varepsilon(t)\|_{L^\infty}$. To this end let us recall the inequality (see Lemma 1 in [27])

$$\Lambda u^\varepsilon(\overline{x}_t) \geq \frac{u^\varepsilon(\overline{x}_t)^2}{4\pi^2\langle u^\varepsilon(t)\rangle} \geq \frac{u^\varepsilon(\overline{x}_t)^2}{4\pi^2\max\{\langle u_0\rangle, 1\}}, \tag{4.2}$$

that is valid provided $u^\varepsilon(\overline{x}_t) \geq 4\langle u^\varepsilon(t)\rangle$. The ODI for $X(t) := u^\varepsilon(\overline{x}_t)$ reads

$$\begin{aligned}\dot{X} &\leq -\mu(X)\Lambda X + X(X - \langle u^\varepsilon(t)\rangle) + rX(1-X))\\ &\leq X\left(\left(1 - r - \mu(X)\frac{\Lambda X}{X^2}\right)X + r\right) = XI'.\end{aligned}$$

We proceed as before via a blowup alternative. Recall Assumption 2.5. In its context, choose $s_0 \in \mathbb{R}^+$ so large that $\mu(s) \geq \delta$ for $s \geq s_0$ and that

$$s_0 \geq \frac{2}{\pi}\mathcal{N}_1, \qquad s_0 \geq \frac{-2r}{1 - r - \delta(4\pi^2\max\{\langle u_0\rangle, 1\})^{-1}}. \tag{4.3}$$

$\mathcal{N}_1$ is given by (1.11). The second choice in (4.3) is possible thanks to $r + \delta(4\pi^2\max\{\langle u_0\rangle, 1\})^{-1} > 1$ assumed for our theorem. We have the alternative:

(i) either $X(t) \leq s_0$ for all times, or
(ii) there exists $t_0 > 0$ such $X(t_0) = s_0$ and X crosses s_0 for the first time at t_0.

It the latter case, we can use (4.2) thanks to choice (4.3) and (3.3). Hence we get from our ODI that $\dot{X} \leq -rX$, which excludes the case (ii) analogously to the argument involving (3.7).

4.2. Uniform estimates. We define

$$\mathbb{M}(s) := \int_0^s \mu(y)\,dy. \tag{4.4}$$

We test equation (3.1) against u^ε and use (4.4). Hence

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |u^\varepsilon(t)|^2 = \int_{\mathbb{T}} \partial_x(\mathbb{M}(u^\varepsilon))Hu^\varepsilon - \frac{1}{2}\partial_x(|u^\varepsilon|^2)\partial_x v^\varepsilon + r|u^\varepsilon(t)|^2 - r|u^\varepsilon(t)|^3.$$

After integration by parts and use of $v^\varepsilon_{xx} = u^\varepsilon - \langle u^\varepsilon\rangle$ it yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|u^\varepsilon(t)|^2 + \int_{\mathbb{T}}\mathbb{M}(u^\varepsilon)\Lambda u^\varepsilon + \left(r-\frac{1}{2}\right)|u^\varepsilon(t)|^3 = \left(r - \frac{\langle u^\varepsilon(t)\rangle}{2}\right)\int_{\mathbb{T}}|u^\varepsilon(t)|^2.$$

Hence we obtain

$$\frac{d}{dt}\|u^\varepsilon(t)\|^2_{L^2} + 2\int_{\mathbb{T}}\mathbb{M}(u^\varepsilon)\Lambda u^\varepsilon \leq \left(2r + \frac{\mathcal{N}_1}{2\pi}\right)\|u^\varepsilon(t)\|^2_{L^2} + \left(\frac{1}{2} - r\right)\|u^\varepsilon(t)|\|^3_{L^3},$$

where we have used bound (3.3) to control $\langle u^\varepsilon(t)\rangle$. For $r \geq 1/2$ the last term above provides extra dissipation, but for any $r \geq 0$ we can use our pointwise bounds to write

$$\frac{d}{dt}\|u^\varepsilon(t)\|^2_{L^2} + 2\int_{\mathbb{T}}\mathbb{M}(u^\varepsilon)\Lambda u^\varepsilon \leq \frac{1}{2}s_0^3 + \left(2r + \frac{\mathcal{N}_1}{2\pi}\right)s_0^2. \tag{4.5}$$

In order to extract $H^{1/2}$ information from (4.5) we symmetrize its second term analogously to I_7 of Section 3, getting

$$2\int_{\mathbb{T}}\mathbb{M}(u^\varepsilon)\Lambda u^\varepsilon = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\int_0^1 \frac{(u^\varepsilon(x) - u^\varepsilon(y))^2}{\sin^2((x-y)/2)}\mu(su^\varepsilon(x) + (1-s)u^\varepsilon(y))\,ds\,dx\,dy.$$

First, let us consider the case when μ verifies (2.7). Hence we have (4.1). To simplify notation, we define

$$0 < s_1(T) = \operatorname*{ess\,inf}_{x\in\mathbb{T}} u_0 e^{-\max\{1,\langle u_0\rangle\}T}, \tag{4.6}$$

and $\underline{\mu}_T = \min_{s_1\leq s\leq s_0}\mu(s)$, where s_0 is the L^∞ bound from the previous subsection. We have

$$\begin{aligned} 2\int_{\mathbb{T}}\mathbb{M}(u^\varepsilon)\Lambda u^\varepsilon &\geq \frac{\underline{\mu}_T}{2\pi}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{(u^\varepsilon(x)-u^\varepsilon(y))^2}{\sin^2((x-y)/2)}\,dx\,dy\,ds \\ &= \frac{\underline{\mu}_T}{2\pi}\int_{\mathbb{T}}\Lambda u^\varepsilon u^\varepsilon\,dx = \frac{\underline{\mu}_T}{2\pi}\|u^\varepsilon\|^2_{\dot{H}^{0.5}}. \end{aligned}$$

Observe that $\underline{\mu}_T > 0$ as μ is positive for positive arguments.

If instead of (2.7) we assume (2.6), we have for $\delta > 0$

$$2\int_{\mathbb{T}}\mathbb{M}(u^\varepsilon)\Lambda u^\varepsilon \geq \frac{\delta}{2\pi}\|u^\varepsilon\|^2_{\dot{H}^{0.5}}.$$

In any case we obtain via (4.5) the ε-uniform bound for u^ε in $L^2(0,T;H^{1/2}(\mathbb{T}))$ for every fixed $0 < T < \infty$.

All in all, for every fixed $0 < T < \infty$, we have the uniform bounds

$$u^\varepsilon \in L^\infty(0,T;L^\infty(\mathbb{T})) \cap L^2(0,T;H^{1/2}(\mathbb{T})).$$

With these uniform estimates, we can follow along the lines of the proof of Theorem 2.4 and conclude the result.

REMARK 4.1. In fact, for Theorem 2.6 we need to impose assumptions on our semilinearity $\mu(s)$ only within the interval of existence of u. More precisely, this interval belongs to

$$I := \Big[\operatorname*{ess\,inf}_{x\in\mathbb{T}} u_0 e^{-\max\{1,\langle u_0\rangle\}T}, s_0\Big],$$

where s_0, δ_0 is a pair that satisfies

$$s_0 \geq \frac{2}{\pi}\mathcal{N}_1, \qquad s_0 \geq \frac{-2r}{1-r-\delta_0(4\pi^2\max\{\langle u_0\rangle, 1\})^{-1}}, \qquad \mu(s_0) > \delta_0 \geq 0.$$

For the lower end of I see (4.6) and for the upper one – (4.3). In particular, μ can vanish on I^c. As the above condition on s_0 is implicit for $\delta_0 > 0$, we have used in Theorem 2.6 a more traceable assumption. For $\delta_0 = 0$ we need $r > 1$, but then

$$I := \left[\operatorname*{ess\,inf}_{x\in\mathbb{T}} u_0 e^{-\max\{1,\langle u_0\rangle\}T}, \frac{2r}{r-1}\right]$$

and in fact 2 above can be replaced with any $k > 1$.

5. Proof of Corollary 2.7

For $r = 0$ the conservation of mass in (3.1) gives $\|u^\varepsilon(t)\|_{L^1} \leq \|u_0\|_{L^1}$ instead of (3.3). Consequently, (4.2) reads now $\Lambda u^\varepsilon(\overline{x}_t) \geq u^\varepsilon(\overline{x}_t)^2/4\pi^2\langle u_0\rangle$. Hence, to follow the lines of proof of Theorem 2.6, it suffices to assume (2.8) instead of (2.5). This gives Corollary 2.7.

6. Proof of Theorem 2.9

We consider the vanishing viscosity approximation of (2.10) with $\gamma(x) \equiv 1$

$$\partial_t u^\varepsilon = -\partial_x(u^\varepsilon H u^\varepsilon) + \varepsilon\partial_x^2 u^\varepsilon, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+.$$

Notice that the solution to this equation verifies

$$\min_x u^\varepsilon(x,t) \geq \operatorname*{ess\,min}_x u_0 > 0, \qquad \max_x u^\varepsilon(x,t) \leq \max_x u_0.$$

By a direct computation, we have

$$\frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) + \mathcal{I}(u^\varepsilon) + \varepsilon\left\|\frac{\partial_x u^\varepsilon}{\sqrt{u^\varepsilon}}\right\|_{L^2}^2 = 0,$$

where $\mathcal{F}$ is the entropy given by (2.1) and Fisher's information $\mathcal{I}$ is $\mathcal{I}(u^\varepsilon) = \|\Lambda^{0.5}u^\varepsilon\|_{L^2}^2$, and verifies

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(u^\varepsilon(t)) + 2\varepsilon\|\partial_x\Lambda^{0.5}u^\varepsilon\|_{L^2}^2 &= -\|\sqrt{u^\varepsilon}\Lambda u^\varepsilon\|_{L^2}^2 - \|\sqrt{u^\varepsilon}\partial_x u^\varepsilon\|_{L^2}^2 \\ &\leq -2\min_x u_0^\varepsilon\,\mathcal{I}(u^\varepsilon),\end{aligned}$$

where the middle term follows from the Tricomi relation

$$H(H\partial_x u\partial_x u) = \frac{1}{2}((H\partial_x u)^2 - (\partial_x u)^2),$$

compare with [27]. In particular, by the Gronwall inequality, we conclude that u^ε tends to the homogeneous state $\langle u_0^\varepsilon\rangle$ exponentially fast (recall that by assumption $\operatorname{ess\,min}_x u_0 > 0$). We also have

$$\begin{aligned}(6.1)\quad \frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) + \varepsilon\bigg\|\frac{\partial_x u^\varepsilon}{\sqrt{u^\varepsilon}}\bigg\|_{L^2}^2 &= -\mathcal{I}(u^\varepsilon(t)) \\ &\geq \frac{1}{2\min\limits_x u_0^\varepsilon}\bigg[\frac{d}{dt}\mathcal{I}(u^\varepsilon(t)) + 2\varepsilon\|\partial_x\Lambda^{0.5}u^\varepsilon\|_{L^2}^2\bigg]\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) + \frac{\varepsilon}{\min\limits_x u_0^\varepsilon}\|\partial_x u^\varepsilon\|_{L^2}^2 &\geq \frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) + \varepsilon\bigg\|\frac{\partial_x u^\varepsilon}{\sqrt{u^\varepsilon}}\bigg\|_{L^2}^2 \\ &\geq \frac{1}{2\min\limits_x u_0^\varepsilon}\frac{d}{dt}\mathcal{I}(u^\varepsilon(t)) + \frac{\varepsilon}{\min\limits_x u_0^\varepsilon}\|\partial_x\Lambda^{0.5}u^\varepsilon\|_{L^2}^2.\end{aligned}$$

Due to the Poincaré inequality, we get

$$\frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) \geq \frac{1}{2\min\limits_x u_0^\varepsilon}\frac{d}{dt}\mathcal{I}(u^\varepsilon(t)).$$

Equivalently,

$$\begin{aligned}\int_t^\infty \frac{d}{dt}(-\mathcal{F}(u^\varepsilon(t))) &\leq \int_t^\infty \frac{-1}{2\min\limits_x u_0^\varepsilon}\frac{d}{dt}\mathcal{I}(u^\varepsilon(t)), \\ -\mathcal{F}(u^\varepsilon(\infty)) + \mathcal{F}(u^\varepsilon(t)) &\leq \frac{-1}{2\min\limits_x u_0^\varepsilon}\mathcal{I}(u^\varepsilon(\infty)) + \frac{1}{2\min\limits_x u_0^\varepsilon}\mathcal{I}(u^\varepsilon(t)).\end{aligned}$$

As $\langle u_0^\varepsilon\rangle = \langle u_0\rangle = 1$, we have $\mathcal{F}(u^\varepsilon(\infty)) = 0$, and we obtain

$$\mathcal{F}(u^\varepsilon(t)) \leq \frac{1}{2\min\limits_x u_0^\varepsilon}\mathcal{I}(u^\varepsilon(t)).$$

Using (6.1),

$$-\mathcal{F}(u^\varepsilon(t)) \geq \frac{1}{2\min\limits_x u_0^\varepsilon}[-\mathcal{I}(u^\varepsilon(t))] = \frac{1}{2\min\limits_x u_0^\varepsilon}\bigg[\frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) + \varepsilon\bigg\|\frac{\partial_x u^\varepsilon}{\sqrt{u^\varepsilon}}\bigg\|_{L^2}^2\bigg]$$

we conclude that

$$-2\min_x u_0^\varepsilon\mathcal{F}(u^\varepsilon(t)) \geq \frac{d}{dt}\mathcal{F}(u^\varepsilon(t)) \quad\text{and}\quad \mathcal{F}(u^\varepsilon(t)) \leq \mathcal{F}(u_0^\varepsilon)e^{-2(\operatorname{ess\,min}_x u_0)\,t}.$$

Hence, via $\mathcal{F}(u_0^\varepsilon) \leq C(u_0)$ we have a uniform bound. Theorem 3.20 in [24] used for $a = b = 0$ implies that the functional

$$f \mapsto \int_{\mathbb{T}} f\log(f) - f + 1\,dx$$

is weakly lower semicontinuous in L^2. Hence

$$\mathcal{F}(u(t)) \leq \liminf_{\varepsilon\to 0}\mathcal{F}(u^\varepsilon(t)) \leq \mathcal{F}(u_0^\varepsilon)e^{-2(\operatorname{ess\,min}\limits_x u_0)\,t}.$$

7. Concluding remarks

A preliminary computation (see the proofs of Theorem 2.4 and 2.8) suggests that solutions of (1.8), (1.9) in the case of subcritical diffusion $\Lambda^\alpha u$ with $\alpha \in (0,1)$ should be bounded in $L^\infty_t L^\infty_x$, provided logistic dampening constant satisfies $r \geq 1$. For details on this and beyond, see [14].

In this paper, we have only marginally touched the matter of time-asymptotics of the considered systems, which is an interesting matter to study.

In the context of the critical diffusion Λu and lack of the logistic term, the conjecture in [12] (see also [1]) says that there should be a threshold mass that divides the global existence/blowup regimes. We are rather inclined against this hypothesis, along lack of the one-dimensional critical diffusion (i.e. with threshold mass phenomenon) for the Smoluchowski–Poisson system with semilinear diffusion, compare [22]. The authors show there that there is no one-dimensional analogue to the multidimensional phenomenon of the critical diffusion in the setting of semilinear, but not fractional diffusion. More precisely, for $d = 1$ the system

$$\partial_t u = \partial_x((1+u)^{(d-2)/d}\partial_x u + u\partial_x v), \qquad \partial_x^2 v = u - \langle u \rangle,$$

has bounded solutions for any initial mass. For lack of blowup in the critical case, see our recent [15].

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