MULTIPLE SOLUTIONS
FOR AN IMPULSIVE BOUNDARY VALUE PROBLEM
ON THE HALF-LINE VIA MORSE THEORY

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Abstract. In this paper, Morse theory is used to establish the existence of multiple solutions for an impulsive boundary value problem posed on the half-line.

1. Introduction

The aim of this paper is to study the following impulsive boundary value problem:

$$(1.1) \quad \begin{cases} -(p(t)u'(t))' = q(t)f(t, u(t)), & t \neq t_j, \ j \in \{1, 2, \ldots\}, \ t > 0, \\ u(0) = u(+\infty) = 0, \\ \Delta(p(t_j)u'(t_j)) = h(t_j)I_j(u(t_j)), & j \in \{1, 2, \ldots\}, \end{cases}$$

where $f \in C^1([0, +\infty) \times \mathbb{R}, \mathbb{R})$, $1/p \in L^1((0, +\infty), (0, +\infty))$, $q \in L^1((0, +\infty), \mathbb{R}^+)$, $q > 0$ almost everywhere, and such that

$$M_1 = \int_0^{+\infty} \left(\int_t^{+\infty} \frac{ds}{p(s)} \right) dt < \infty \quad \text{and} \quad M_2 = \int_0^{+\infty} q(t) \left(\int_t^{+\infty} \frac{ds}{p(s)} \right) dt < \infty.$$

$I_j \in C^1(\mathbb{R}, \mathbb{R})$, $j = \{1, 2, \ldots\}$ are the impulse functions and $t_0 = 0 < t_1 < \ldots < t_j < \ldots < t_m \to +\infty$, as $m \to +\infty$, are infinite impulse points. We let

$$\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-),$$

where $u'(t_j^+) = \lim\limits_{t \overset{>}{\to} t_j} u'(t)$ and $u'(t_j^-) = \lim\limits_{t \overset{<}{\to} t_j} u'(t)$ stand for the right and the left limits of u' at t_j, respectively. Finally, $h \colon \mathbb{R}^+ \to \mathbb{R}$ is a function such that $\sum\limits_{j=1}^{\infty} |h(t_j)| < \infty$.

Many problems modeling perturbed phenomena in nonlinear dynamics which are subject to jump discontinuities in velocity can be represented by impulsive boundary value problems (see, e.g., [3] and the references therein). In the last couple of years, various mathematical results based on topological methods (fixed point theorems, Leray–Schauder degree, ...) have been obtained in connection with such problems (see, e.g., [15]). Variational approaches have also been shown to be efficient tools in discussing the question of the solvability; we quote the minimization principle and the mountain pass theorem by Ambrosetti and Rabinowitz (as developed in [1], [2], [18]). Among works from the recent literature, we refer to [10] and also to the paper [8] which has discussed a problem with linear differential operator, a nonlocal condition at the origin, and a Neumann condition at positive infinity for a problem posed on the half-line. In [4], some existence results of a single solution are obtained for problem (1.1). The aim of this work is to investigate the existence of multiple solutions to problem (1.1). We first prove the existence of three distinct solutions, one of which is trivial, under sub-linear growth conditions upon the nonlinearity f. Our second existence result provides existence of infinitely many solutions under conditions including super-linear nonlinearities. Each existence result is illustrated by means of an example of application. The proofs of our main existence results are based on Morse theory. For this purpose, some basic notions and important results are recalled hereafter. For more details, we refer the reader to [5]–[7], [13], [14], [17].

Let $\mathcal{H}$ be a Hilbert space and $J \in C^1(\mathcal{H}, \mathbb{R})$ a functional. For a topological pair (A, B), we denote by $H_k(A, B)$ the k-th singular relative homology group with coefficients in a ring $\mathbb{F}$ with characteristic zero (see [14]) and by $\beta_k = \dim H_k(A, B)$ the k-th Betti number. In algebraic topology, the k-th Betti number denotes the rank of the k-th homology group. Intuitively, the first Betti number of a space counts the maximum number of cuts that can be made without dividing the space into two pieces. Each Betti number is a natural number or $+\infty$. They are topological invariants. Finally $\beta_k(a, b) = \dim H_k(J^b, J^a)$ is the k-th Betti number with respect to the interval (a, b).

Let p be an isolated critical point of J, i.e., $J'(p) = 0$ and let U be a neighbourhood of p such that J has only p as a critical point in U. The critical groups

of J at p are defined by

$$C_k(J, p) = H_k(J^c \cap U, J^c \cap U \setminus \{p\}), \quad \text{for all } k \in \mathbb{N},$$

where $J^c = \{u \in \mathcal{H}, J(u) \leq c\}$ is the c-sublevel set and $c = J(p) \in \mathbb{R}$. Let $K^c = \{u \in \mathcal{H}, J'(u) = 0, J(u) = c\}$ be the set of critical points at level c. In the case where $J''(p)$ exists with p a critical point of J, the Morse index of p is defined as the supremum of dimensions of the vector subspaces of $\mathcal{H}$ on which $J''(p)$ is negative definite (it can be equal to ∞) (see [7, Definition 5.1.12]). We say that p is nondegenerate if the Hessian $J''(p)$ is invertible.

DEFINITION 1.1. Let $J \in C^1(\mathcal{H}, \mathbb{R})$ and $c \in \mathbb{R}$. The functional J satisfies the Palais–Smale condition at level c (shortly PS_c) if any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that

(1.2) $$J(u_n) \to c, \quad J'(u_n) \to 0$$

has a convergent subsequence. J satisfies the Palais–Smale condition (PS in brief) if it satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.

DEFINITION 1.2. Let $-\infty < a < b < +\infty$ be regular values and assume that J has only isolated critical values $c_1 < c_2 < \ldots$ in (a, b) with a finite number of critical points at each level and satisfies PS_c, for all $c \in [a, b]$. Then the Morse type numbers of J with respect to the interval (a, b) are defined by

$$M_k(a, b) = \sum_i \dim H_k(J^{a_{i+1}}, J^{a_i}) \quad (k \in \mathbb{N}),$$

where $a = a_1 < c_1 < \ldots < c_l < a_l = b$. They are independent of the a_i by the second deformation lemma and are related to the critical groups by the formula

$$M_k(a, b) = \sum_{i=1}^{l} \sum_{u \in K^{c_i}} \dim C_k(J, u) \quad (k \in \mathbb{N}).$$

The following result will be crucial in proving the first existence result of the paper.

LEMMA 1.3 ([7, Theorem 5.1.35]). *Assume that $J \in C^2(\mathcal{H}, \mathbb{R})$ is bounded from below, satisfies the PS condition, and has a nondegenerate, non-minimum critical point p with finite Morse index. Then J has at least three distinct critical points.*

Our second existence result is based on the following lemma.

LEMMA 1.4 ([6, Theorem 4.3]). *Assume that $J \in C^1(\mathcal{H}, \mathbb{R})$ satisfies the PS condition and let $a < b$ be two regular values of J. Suppose that J has at most finitely many critical points on $J^{-1}[a, b]$ and the dimension of the critical group for every critical point is finite. Then the following inequality holds:*

$$M_k - M_{k-1} + \ldots + (-1)^k M_0 \geq \beta_k - \beta_{k-1} + \ldots + (-1)^k \beta_0, \quad k = 0, 1, \ldots$$

In addition

$$\sum_{k=0}^{\infty} (-1)^k M_k = \sum_{k=0}^{\infty} (-1)^k \beta_k,$$

whenever the left-hand series of the equality converges.

However, in order to compute M_k and β_k, some classical properties of the Leray–Schauder degree are required.

LEMMA 1.5 ([7, Theorem 5.1.32]). *Assume that $J \in C^2(\mathcal{H}, \mathbb{R})$ satisfies the PS condition, $J' = I - A$, where I is the identity operator, and $A \colon \mathcal{H} \to \mathcal{H}$ is compact. If p is an isolated critical point of J, then there exists a neighbourhood U of p such that p is the unique critical point of J in $\overline{U}$ and*

$$\deg(I - A, U, \theta) = \sum_{k=0}^{\infty} (-1)^k \dim C_k(J, p),$$

where $\deg(I - A, U, \theta)$ stands for the Leray–Schauder topological degree with respect to the zero element $\theta \in \mathcal{H}$.

REMARK 1.6. (a) The proof of Lemma 1.3 appeals to [7, Corollary 4.8.4] which only needs J to be of C^1-class and also uses the additivity of Euler-Poincaré characteristic which holds whenever the Morse index is defined, i.e. when J'' exists at p (see the proof in [7, p. 343]). The condition that $J''(p)$ exists is then sufficient for application of Lemma 1.3.

(b) Lemma 1.5 equally holds when merely $J''(p)$ exists, a requirement for the Morse index to be defined (see the proof in [7, p. 341]).

The following result is known as Borsuk's theorem:

LEMMA 1.7 [9, Theorem 9.4]. *Let X be a Banach space and $\Omega \subset X$ a symmetric bounded open set including θ, the origin. If $K \colon \overline{\Omega} \to X$ is an odd compact map with $\theta \notin (I - K)(\partial\Omega)$, then the degree $\deg(I - K, \Omega, \theta)$ is odd.*

The following technical lemma will be useful; it is classical in operator theory (see [19, Lemma 2]).

LEMMA 1.8. *Suppose that X is a reflexive Banach space with dual space X^* and $T \in \mathcal{L}(X, X^*)$ with $T^* = T$, where $\mathcal{L}(X, X^*)$ is the set of all continuous linear operators from X to X^* and T^* is the dual operator of T. Assume that there exists a constant $C > 0$ such that $\|Tx\| \geq C\|x\|$, for all $x \in X$. Then T^{-1} exists and $T^{-1} \in \mathcal{L}(X^*, X)$.*

2. Main existence results

Consider the Hilbert space:

$$H_{0,p}^1 = \{u \in AC([0, +\infty), \mathbb{R}) : u(0) = u(+\infty) = 0, \ \sqrt{p}u' \in L^2(0, +\infty)\}$$

with the scalar product

$$(u, v) = \int_0^{+\infty} p(t)u'(t)v'(t)\, dt + \int_0^{+\infty} u(t)v(t)\, dt$$

and the corresponding norm (see Lemma 2.1):

$$||u||_{H_{0,p}^1} = \sqrt{\int_0^{+\infty} p(t)u'(t)^2\, dt + \int_0^{+\infty} u(t)^2\, dt}.$$

Let the space $C_0 = \left\{ u \in C([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} u(t) = 0 \right\}$ be endowed with the norm $||u||_\infty = \sup_{t \geq 0} |u(t)|$. Concerning these spaces, we have

LEMMA 2.1 ([4]). *On $H_{0,p}^1$, the quantity $||u|| = \sqrt{\int_0^{+\infty} p(t)u'(t)^2\, dt}$ defines a norm which is equivalent to the $H_{0,p}^1$-norm.*

LEMMA 2.2 ([4]).
(a) *$H_{0,p}^1$ is continuously embedded in C_0, more precisely for every $u \in H_{0,p}^1$, one has $||u||_\infty \leq d||u||$, where $d = \left(\int_0^{+\infty} 1/p(s)\, ds \right)^{1/2}$.*
(b) *The embedding $H_{0,p}^1 \hookrightarrow C_0$ is compact.*

Consider the Hilbert space

$$L_q^2 = \{ u : (0, +\infty) \to \mathbb{R} \text{ measurable such that } \sqrt{q}u \in L^2(0, +\infty) \}$$

equipped with the norm

$$||u||_{L_q^2} = \sqrt{\int_0^{+\infty} q(t)u^2(t)\, dt}.$$

Since $||u||_{L_q^2}^2 \leq ||u||_\infty^2 ||q||_{L^1}$, we have

PROPOSITION 2.3. *C_0 is continuously embedded in L_q^2.*

COROLLARY 2.4. *$H_{0,p}^1$ is compactly embedded in L_q^2.*

We are now interested in the first eigenvalue $\lambda_1(q)$ of the linear problem:

(2.1)
$$\begin{cases} -(p(t)u'(t))' = \lambda q(t)u(t), & t > 0, \\ u(0) = u(+\infty) = 0, \end{cases}$$

namely

$$\lambda_1(q) = \inf_{u \in H_{0,p}^1 \setminus \{0\}} \frac{||u||^2}{||u||_{L_q^2}^2}.$$

REMARK 2.5. Notice that no impulse functions are involved in the linear problem (2.1). The reason is that only the characterization and the positivity of $\lambda_1(q)$ are needed (see the proof of Theorem 2.9, Step 1). Likewise the characterization of the associated eigenfunction φ_1 will be used in the proof of Theorem 2.9, Step 3.

We have

LEMMA 2.6. $\lambda_1(q)$ *is positive and is achieved for some positive function* φ_1 *in* $H^1_{0,p} \setminus \{0\}$.

PROOF. For $u \in H^1_{0,p}$, let $I_1(u) = \|u\|^2$, $I_2(u) = \|u\|^2_{L^2_q}$, and define the quotient functional $Q \colon H^1_{0,p} \setminus \{0\} \to \mathbb{R}$ by

$$Q(u) = \frac{I_1(u)}{I_2(u)}.$$

Then $\lambda_1(q) = \inf\limits_{u \in H^1_{0,p} \setminus \{0\}} Q(u)$.

Step 1. Using the Poincaré inequality, we have that $\lambda_1(q) \geq 1/M_2 > 0$. Indeed

$$|u(t)|^2 = \left| \int_t^{+\infty} u'(s)\, ds \right|^2 = \left| \int_t^{+\infty} \sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}\, ds \right|^2$$

$$\leq \left(\int_t^{+\infty} p(s)u'^2(s)\, ds \right)\left(\int_t^{+\infty} \frac{ds}{p(s)} \right)$$

$$\leq \left(\int_0^{+\infty} p(s)u'^2(s)\, ds \right)\left(\int_t^{+\infty} \frac{ds}{p(s)} \right).$$

Hence

$$q(t)u(t)^2 \leq \left(\int_0^{+\infty} p(s)u'^2(s)\, ds \right)\left(q(t) \int_t^{+\infty} \frac{ds}{p(s)} \right),$$

i.e. $\|u\|^2_{L^2_q} \leq M_2 \|u\|^2$, which implies $\|u\|^2/\|u\|^2_{L^2_q} \geq 1/M_2 > 0$. Passing to the infimum yields

$$\lambda_1(q) = \inf\limits_{u \in H^1_{0,p} \setminus \{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2_q}} \geq \frac{1}{M_2} > 0.$$

Step 2. Let (u_n) be a minimizing sequence. Since $(|u_n|)$ is a minimizing sequence for Q, we may suppose that $u_n(t) \geq 0$, for $t \in [0, +\infty)$. Moreover, the functional Q satisfies $Q(\alpha u) = Q(u)$, for every $\alpha \in \mathbb{R}$; by setting $\widetilde{u_n} = u_n/\|u\|_{L^2_q}$, for every n, we can assume that $\|u_n\|_{L^2_q} = 1$. Then $(I_1(u_n))$ must be bounded independently of n. In fact,

$$\lim\limits_{n \to +\infty} Q(u_n) = \inf\limits_{u \in H^1_{0,p} \setminus \{0\}} Q(u) = \lambda_1(q).$$

So the sequence $(Q(u_n))$ is bounded. From this and since $Q(u_n) = \|u_n\|^2$, we deduce that (u_n) is bounded in $H^1_{0,p}$. By Lemma 2.2(b) and the reflexivity and separability of $H^1_{0,p}$, there exists a subsequence (u_{n_k}) of (u_n) such that, as $k \to +\infty$,

$$u_{n_k} \rightharpoonup \overline{u} \quad \text{in }, H^1_{0,p}, \qquad u_{n_k} \to \overline{u} \quad \text{in } C_0,$$

so $u_{n_k}(t) \to \overline{u}(t)$, for all $t \in [0, +\infty)$. By Corollary 2.4, (u_{n_k}) converges in norm to $\overline{u}$ in L^2_q; thus $\|\overline{u}\|_{L^2_q} = 1$ and $\overline{u}(t) \geq 0$, for $t \in [0, +\infty)$. Finally, the weak

lower semi-continuity of the norm guarantees that

$$Q(\overline{u}) = I_1(\overline{u}) \leq \liminf_k I_1(u_{n_k}) = \liminf_k Q(u_{n_k}) = \lambda_1(q),$$

so $\overline{u} \in H^1_{0,p} \setminus \{0\}$ and $Q(\overline{u}) = \lambda_1(q)$. $\square$

Now, we define what we mean by a solution of problem (1.1).

DEFINITION 2.7. We say that u is a weak solution of problem (1.1) if

$$\int_0^{+\infty} p(t)u'(t)v'(t)\,dt + \sum_{j=1}^{\infty} h(t_j)I_j(u(t_j))v(t_j) = \int_0^{+\infty} q(t)f(t,u(t))v(t)\,dt,$$

for all $v \in H^1_{0,p}$, the limits $p(t_j^+)u'(t_j^+)$ and $p(t_j^-)u'(t_j^-)$, $j = 1, 2, \ldots$ exist and the impulsive conditions in problem (1.1) hold.

REMARK 2.8. If u were in $H^1_{0,p}$ with $u'' \in L^2$, then u would be of C^1-class and so no impulsive conditions exist for u'. If $u \in H^1_{0,p}$, then u is absolutely continuous and $\sqrt{p}u' \in L^2$. In this case $\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-) = 0$ does not necessarily hold and the derivative $\sqrt{p}u'$ may present some discontinuities, leading to the impulsive effects.

In [16] (see also [8], [20] and references therein), the definition of a classical solution is also introduced with $u \in H^2(t_j, t_{j+1})$, which is natural, and u satisfies the first equation of the considered impulsive boundary value problem, namely $-u''(t) + \lambda u(t) = \sigma(t)$, for almost every $t \in [0, T]$. Unfortunately this cannot hold almost everywhere on $[0, T]$ for otherwise $u \in H^1$ and $\sigma \in L^2(0, T)$ imply $u \in H^2(0, T)$ and as a consequence u' has no discontinuity at the impulsive point t_j, as the authors of [16] have previously noticed. In fact, the equation should be understood only in the distributional sense, i.e.

$$-u''(t) + \lambda u(t) = \sigma(t) + \sum_{j=1}^{n} \Delta u'(t_j)\delta_j, \quad t \in [0, T],$$

where δ_j is the Dirac mass distribution defined by

$$\left\langle \sum_{j=1}^{n} \Delta u'(t_j)\delta_j, \varphi \right\rangle = \sum_{j=1}^{n} \Delta u'(t_j)\varphi(t_j), \quad \text{for all } \varphi \in \mathcal{D}(0, T);$$

here $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{D}, \mathcal{D}'}$ stands for the distribution pairing. Of course, we have that $-u''(t) + \lambda u(t) = \sigma(t)$, for all $t \in (t_j, t_{j+1})$ and, for $j \in \{1, \ldots, n\}$, $t \neq t_j$ merely means $t \in (t_j, t_{j+1})$, $j \in \{1, \ldots, n\}$. This justifies the above definition of weak solution.

We are now ready to state our first existence result.

THEOREM 2.9. Assume that the following conditions hold:
(H$_1$) $f(t,0) = 0$, for $t \in [0, +\infty)$,

(H_2) *there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha < \lambda_1(q)$ such that*

$$|f(t,u)| \leq \alpha|u| + \beta, \quad \text{for all } (t,u) \in [0,+\infty) \times \mathbb{R},$$

(H_3) *there exists $\varepsilon_0 > 0$ such that $|f'_u(t,0)| \leq \lambda_1(q) - \varepsilon_0$, for all positive t, where $f'_u(t,u)$ denotes the first-order derivative of f in the second argument u,*

(H_4) *there exist $\delta > 0$, $C_1 \in \mathbb{R}$, and $\eta \in [0,1)$ such that*

$$f(t,u) \geq C_1 u^\eta, \quad \text{for all } t \geq 0, \ u \in [0,\delta],$$

(H_5) $I'_j(0) = 0$, $j \in \{1,2,\ldots\}$,

(H_6) *there exist $k > 0$ and $\sigma \in (\eta, 1)$ such that $|I_j(s)| \leq k|s|^\sigma$, for all $s \in \mathbb{R}$, $j \in \{1,2,\ldots\}$.*

Then problem (1.1) has at least three distinct weak solutions in $H^1_{0,p}$, one of them is the trivial solution.

The functional $J \colon H^1_{0,p} \to \mathbb{R}$ associated with problem (1.1) is defined by

$$(2.2) \qquad J(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{\infty} h(t_j) \int_0^{u(t_j)} I_j(s)\,ds - \int_0^{+\infty} q(t)F(t,u(t))\,dt,$$

where $F(t,u) = \int_0^u f(t,v)\,dv$. Let the operator A be defined by

$$(2.3) \qquad Au(t) = \int_0^{+\infty} G(t,s)q(s)f(s,u(s))\,ds - \sum_{j=1}^{+\infty} G(t,t_j)h(t_j)I_j(u(t_j)),$$

where G is the Green's function of the problem $-(p(t)u'(t))' = u(0) = u(+\infty) = 0$, namely

$$G(t,s) = \frac{1}{\|1/p\|_{L^1}} \begin{cases} \varphi_1(t)\varphi_2(s), & t \leq s, \\ \varphi_1(s)\varphi_2(t), & s \leq t, \end{cases}$$

with fundamental system of solutions

$$\varphi_1(t) = \int_0^t \frac{ds}{p(s)} \quad \text{and} \quad \varphi_2(t) = \int_t^{+\infty} \frac{ds}{p(s)}.$$

It is easy to note that, by (H_2) and (H_6), the functional J and the perator A are well defined. Next, we study the compactness of A.

LEMMA 2.10. *Under assumptions (H_2) and (H_6), we have:*

(a) *A self-maps the Sobolev space $H^1_{0,p}$.*

(b) *A is compact.*

PROOF. (a) Given $u \in H^1_{0,p}$, we show that $Au \in H^1_{0,p}$; the proof will be performed in three steps:

Step 1. $Au \in AC([0,+\infty))$.

(i) The continuity of Au follows from that of the Green's function, assumptions (H_2), (H_6) and the Lebesgue dominated convergence theorem.

(ii) $(Au)' \in L^1(0, +\infty)$. We have

$$(2.4) \quad |(Au)'(t)| = \left| \int_0^{+\infty} G_t'(t, s) q(s) f(s, u(s)) \, ds - \sum_{j=1}^{+\infty} G_t'(t, t_j) h(t_j) I_j(u(t_j)) \right|$$

$$\leq \int_0^{+\infty} |G_t'(t, s)| q(s) |f(s, u(s))| \, ds$$

$$+ \sum_{j=1}^{+\infty} |G_t'(t, t_j)| |h(t_j)| |I_j(u(t_j))|$$

$$\leq \frac{1}{p(t)} \int_0^{+\infty} q(s) |f(s, u(s))| \, ds + \frac{1}{p(t)} \sum_{j=1}^{+\infty} |h(t_j)| |I_j(u(t_j))|.$$

This together with (H$_2$) and (H$_6$) guarantee that $\int_0^{+\infty} |(Au)'(t)| \, dt < \infty$.

Step 2. $G(0, s) = 0$, for all $s \geq 0$ and $\lim\limits_{t \to +\infty} G(t, s) = 0$, for all $s \geq 0$; hence $Au(0) = Au(+\infty) = 0$.

Step 3. $\sqrt{p}(Au)' \in L^2(0, +\infty)$. From the inequality (2.4), we obtain that

$$p(t)|(Au)'(t)|^2 \leq \frac{1}{p(t)} \left(\int_0^{+\infty} q(s) |f(s, u(s))| \, ds \right)^2 + \frac{1}{p(t)} \left(\sum_{j=1}^{+\infty} |h(t_j)| |I_j u(t_j)| \right)^2$$

$$+ \frac{2}{p(t)} \left(\int_0^{+\infty} q(s) |f(s, u(s))| \, ds \right) \left(\sum_{j=1}^{\infty} |h(t_j)| |I_j(u(t_j))| \right),$$

and then (H$_2$) and (H$_6$) imply that $\int_0^{+\infty} p(t) |(Au)'(t)|^2 dt < +\infty$.

(b) Let (u_n) be a bounded sequence in the reflexive separable space $H^1_{0,p}$. Then there exists a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u$ in $H^1_{0,p}$, as $k \to +\infty$. We will prove that the sequence (Au_{n_k}) is convergent. We have the estimates:

$$\|Au_{n_k} - Au\| = \sup_{\|v\| \leq 1} |(Au_{n_k} - Au, v)|$$

$$= \sup_{\|v\| \leq 1} \left| \int_0^{\infty} p(t)(Au_{n_k} - Au)'(t) v'(t) \, dt \right|$$

$$= \sup_{\|v\| \leq 1} \left| \int_0^{+\infty} -(p(t)(Au_{n_k} - Au)'(t))' v(t) \, dt \right.$$

$$+ \left. \sum_{j=1}^{+\infty} h(t_j) [I_j(u_{n_k}(t_j)) - I_j(u(t_j))] v(t_j) \right|$$

$$= \sup_{\|v\| \leq 1} \left| \int_0^{+\infty} [-(p(t)(Au_{n_k})'(t))' - (-p(t)(Au)'(t))'] v(t) \, dt \right.$$

$$+ \left. \sum_{j=1}^{+\infty} h(t_j) [I_j(u_{n_k}(t_j)) - I_j(u(t_j))] v(t_j) \right|$$

$$= \sup_{||v|| \leq 1} \left| \int_0^{+\infty} q(t)[f(t, u_{n_k}(t)) - f(t, u(t))]v(t)\, dt \right.$$

$$\left. + \sum_{j=1}^{+\infty} h(t_j)[I_j(u_{n_k}(t_j)) - I_j(u(t_j))]v(t_j) \right|$$

$$\leq \sup_{||v|| \leq 1} \left[\int_0^{+\infty} ||v||_\infty q(t)|f(t, u_{n_k}(t)) - f(t, u(t))|\, dt \right.$$

$$\left. + \sum_{j=1}^{+\infty} ||v||_\infty |h(t_j)||I_j(u_{n_k}(t_j)) - I_j(u(t_j))| \right]$$

$$\leq d \sup_{||v|| \leq 1} \left[\int_0^{+\infty} ||v|| q(t)|f(t, u_{n_k}(t)) - f(t, u(t))|\, dt \right.$$

$$\left. + \sum_{j=1}^{+\infty} ||v|||h(t_j)||I_j(u_{n_k}(t_j)) - I_j(u(t_j))| \right]$$

$$\leq d \int_0^{\infty} q(t)|f(t, u_{n_k}(t)) - f(t, u(t))|\, dt$$

$$+ d \sum_{j=1}^{+\infty} |h(t_j)||I_j(u_{n_k}(t_j)) - I_j(u(t_j))|.$$

Using Lemma 2.2(b), (H_2), (H_6), and the Lebesgue dominated convergence theorem, we obtain that $||Au_{n_k} - Au|| \to 0$, as $k \to +\infty$. The same estimates show that $||Au_n - Au|| \to 0$ whenever $||u_n - u|| \to 0$, as $n \to +\infty$. The compactness of A is then proved. In the same way, one can show that A is continuous. $\qquad \square$

The properties of the first and second derivatives of J are now studied.

Lemma 2.11. *We have*:

(a) *Under assumptions* (H_2) *and* (H_6), *J is continuously differentiable and satisfies*

$$(J'(u), v) = \int_0^{+\infty} p(t)u'(t)v'(t)\, dt$$

$$+ \sum_{j=1}^{\infty} h(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} q(t)f(t, u(t))v(t)\, dt,$$

for all $u, v \in H_{0,p}^1$ and $J' = I - A$.

(b) *Under assumptions* (H_3) *and* (H_5), *$J''(\theta)$ exists, is continuous, and, for all $v, w \in H_{0,p}^1$, satisfies*

$$(J''(\theta)v, w) = \int_0^{+\infty} p(t)v'(t)w'(t)\, dt$$

$$+ \sum_{j=1}^{\infty} h(t_j) I_j'(0) v(t_j) w(t_j) - \int_0^{+\infty} q(t) f_u'(t,0) v(t) w(t) \, dt.$$

PROOF. (a) We only check that J' is well defined and $J' = I - A$. From the Cauchy–Schwarz inequality, (H$_2$), and (H$_6$), we obtain for all $u, v \in H_{0,p}^1$ that:

$$|(J'(u), v)| \le \left(1 + kd^{\sigma+1} \sum_{j=1}^{+\infty} |h(t_j)| + (d^2\alpha + \beta)||q||_{L^1} \right) ||u|| \, ||v|| < \infty.$$

Moreover, it can be easily checked that (see Remark 2.8)

$$-(p(t)(Au)'(t))' = q(t) f(t, u(t)) + \sum_{j=1}^{\infty} \delta_{t=t_j} h(t_j) I_j(u(t_j)), \quad t \ge 0.$$

Hence $-(p(t)(Au)'(t))' = q(t) f(t, u(t))$, $t \ne t_j$, $j \in \{1, 2, \ldots\}$ and as a consequence

$$(J'(u), v) = (u, v)_{H_{0,p}^1} + \sum_{j=1}^{\infty} h(t_j) I_j(u(t_j)) v(t_j) - \int_0^{+\infty} -(p(t)(Au)'(t))' v(t) \, dt$$

$$= (u, v)_{H_{0,p}^1} + \sum_{j=1}^{\infty} h(t_j) I_j(u(t_j)) v(t_j)$$

$$- \int_0^{+\infty} p(t)(Au)'(t) v'(t) \, dt - \sum_{j=1}^{\infty} h(t_j) I_j(u(t_j)) v(t_j)$$

$$= (u, v)_{H_{0,p}^1} - (Au, v)_{H_{0,p}^1} = (u - Au, v)_{H_{0,p}^1},$$

as claimed.

(b) Under assumptions (H$_3$) and (H$_5$), $(J''(\theta) v, w)$ is well defined for all $v, w \in H_{0,p}^1(0, +\infty)$. Indeed, using (H$_3$), (H$_5$), and again the Cauchy–Schwarz inequality, we obtain for all $v, w \in H_{0,p}^1$ that

$$|(J''(\theta) v, w)| = \left| \int_0^{+\infty} p(t) v'(t) w'(t) \, dt + \sum_{j=1}^{\infty} h(t_j) I_j'(0) v(t_j) w(t_j) \right.$$

$$\left. - \int_0^{+\infty} q(t) f_u'(t, 0) v(t) w(t) \, dt \right|$$

$$\le |(v, w)| + \int_0^{+\infty} q(t) |f_u'(t, 0)| |v(t)| \, |w(t)| \, dt$$

$$\le (1 + d^2 ||q||_{L^1} (\lambda_1(q) - \varepsilon_0)) ||v|| \, ||w|| < +\infty.$$

Finally, the continuity of $J''(\theta)$ is easily proved using the Lebesgue dominated convergence theorem. □

Lemma 2.11 tells us that critical points of J, i.e. weak solutions of problem (1.1), are precisely fixed points of A.

We are now in position to prove Theorem 2.9.

PROOF OF THEOREM 2.9. We have to prove that J has at least three distinct critical points in $H_{0,p}^1$. To this end, it is sufficient to show that J satisfies the hypotheses of Lemma 1.3 (see also Remark 1.6). The proof is split into five steps.

Step 1. There exists a constant $C > 0$ such that

$$||J''(\theta)v|| \geq C||v||^2, \quad \text{for all } v \in H_{0,p}^1.$$

We proceed as in [11, Lemma 3.2]). From the Cauchy–Schwarz inequality, we have

$$||J''(\theta)v|||||v|| \geq (J''(\theta)v, v), \quad \text{for all } v \in H_{0,p}^1.$$

Using (H_3) and (H_5), we get

$$(2.5) \qquad (J''(\theta)v, v) = ||v||^2 - \int_0^{+\infty} q(t) f_u'(t, 0)(v(t))^2 \, dt$$

$$\geq ||v||^2 - \int_0^{+\infty} (\lambda_1(q) - \varepsilon_0) q(t)(v(t))^2 \, dt$$

$$\geq ||v||^2 - (\lambda_1(q) - \varepsilon_0) \int_0^{+\infty} q(t)(v(t))^2 \, dt$$

$$= ||v||^2 - \frac{\lambda_1(q) - \varepsilon_0}{\lambda_1(q)} \lambda_1(q) ||v||_{L_q^2}^2$$

$$\geq \left(1 - \frac{\lambda_1(q) - \varepsilon_0}{\lambda_1(q)}\right) ||v||^2 = \frac{\varepsilon_0}{\lambda_1(q)} ||v||^2,$$

for all $v \in H_{0,p}^1$, proving our claim.

Step 2. J is bounded from below on $H_{0,p}^1$. Indeed, (H_2) implies that

$$(2.6) \qquad F(t, u) \leq \frac{\alpha}{2} u^2 + \beta |u|, \quad \text{for all } (t, u) \in [0, +\infty) \times \mathbb{R},$$

where $F(t, u) = \int_0^u f(t, v) \, dv$. Then from (2.6), (H_6), and Lemmas 2.2, 2.4, we have the estimates, for all $u \in H_{0,p}^1$,

$$(2.7) \quad J(u) = \frac{1}{2} ||u||^2 + \sum_{j=1}^\infty h(t_j) \int_0^{u(t_j)} I_j(s) ds - \int_0^{+\infty} q(t) F(t, u(t)) \, dt$$

$$\geq \frac{1}{2} ||u||^2 - \frac{k \sum_{j=1}^\infty |h(t_j)|}{\sigma + 1} |u(t_j)|^{\sigma+1}$$

$$- \frac{\alpha}{2} \int_0^{+\infty} q(t) |u(t)|^2 \, dt - \beta \int_0^{+\infty} q(t) |u(t)| \, dt$$

$$\geq \frac{1}{2} ||u||^2 - \frac{k \sum_{j=1}^\infty |h(t_j)|}{\sigma + 1} ||u||_\infty^{\sigma+1}$$

$$- \frac{\alpha}{2\lambda_1(q)} \lambda_1(q) ||u||_{L_q^2}^2 - \beta ||q||_{L^1} ||u||_\infty$$

$$\geq \left(\frac{1}{2} - \frac{\alpha}{2\lambda_1(q)}\right)||u||^2 - d\beta||q||_{L^1}||u|| - \frac{kd^{\sigma+1}\sum_{j=1}^{\infty}|h(t_j)|}{\sigma+1}||u||^{\sigma+1},$$

and our claim follows.

Step 3. θ is not a global minimum. Let φ_1 the eigenfunction associated with the eigenvalue $\lambda_1(q)$ and let ξ be such that $0 < \xi \leq \delta/||\varphi_1||_\infty$. From (H$_4$) and (H$_6$), we derive the estimates:

$$J(\xi\varphi_1) = \frac{1}{2}\xi^2||\varphi_1||^2 + \sum_{j=1}^{+\infty}h(t_j)\int_0^{\xi\varphi_1(t_j)}I_j(s)\,ds - \int_0^{+\infty}q(t)F(t, \xi\varphi_1(t))\,dt$$

$$\leq \frac{1}{2}\xi^2||\varphi_1||^2 + \xi^{\sigma+1}\frac{k\sum_{j=1}^{+\infty}|h(t_j)|}{\sigma+1}(\varphi_1(t_j))^{\sigma+1} - \frac{1}{\eta+1}\int_0^{+\infty}C_1q(t)(\xi\varphi_1(t))^{\eta+1}\,dt$$

$$\leq \frac{1}{2}\xi^2||\varphi_1||^2 + \xi^{\sigma+1}\frac{k\sum_{j=1}^{+\infty}|h(t_j)|}{\sigma+1}||\varphi_1||_\infty^{\sigma+1} - \frac{\xi^{\eta+1}C_1}{\eta+1}\int_0^{+\infty}q(t)(\varphi_1(t))^{\eta+1}\,dt$$

$$\leq \frac{1}{2}||\varphi_1||^2\xi^2 + \frac{kd^{\sigma+1}\sum_{j=1}^{+\infty}|h(t_j)|}{\sigma+1}||\varphi_1||^{\sigma+1}\xi^{\sigma+1} - \frac{C_1}{\eta+1}||\varphi_1||_{L_q^{\eta+1}}^{\eta+1}\xi^{\eta+1},$$

where

$$||\varphi_1||_{L_q^{\eta+1}} = \left(\int_0^{\infty}q(t)|\varphi_1(t)|^{\eta+1}\,dt\right)^{1/(\eta+1)} \leq \widetilde{k}||\varphi_1||_{L_q^2} < +\infty,$$

for some positive constant $\widetilde{k}$. Indeed, all norms are equivalent in the one-dimensional space spanned by φ_1 (see [12, p. 86]). Now, since $0 \leq \eta < 1$ and $\eta < \sigma$, then for sufficiently small ξ, the leading term of the right-hand polynomial is $\xi^{\eta+1}$. Hence there is some ξ_0 such that $J(\xi\varphi_1) < 0 = J(\theta)$, for $0 < \xi \leq \xi_0$.

Step 4. J satisfies the PS condition. Suppose that $(u_n) \subset H_{0,p}^1$ and there exists $M > 0$ such that $|J(u_n)| \leq M$ and $J'(u_n) = u_n - Au_n \to \theta$ in $H_{0,p}^1$, as $n \to +\infty$. From (2.7), we know that (u_n) is bounded in $H_{0,p}^1$. By the compactness of $A\colon H_{0,p}^1 \to H_{0,p}^1$ (Lemma 2.10), there exists a subsequence (Au_{n_k}) such that $A(u_{n_k}) \to w$, as $k \to +\infty$. Then $||u_{n_k} - w|| \leq ||u_{n_k} - Au_{n_k}|| + ||Au_{n_k} - w||$ and since $u_{n_k} - Au_{n_k} \to \theta$ in $H_{0,p}^1$, as $k \to +\infty$, we obtain that (u_n) has a convergent subsequence (u_{n_k}) such that $u_{n_k} \to w$, as $k \to +\infty$. Therefore J satisfies the PS condition on $H_{0,p}^1$.

Step 5. (a) θ is a critical point. From (H$_1$) and (H$_5$), θ is a critical point of J and $J(\theta) = 0$.

(b) θ is nondegenerate. From Step 1 and Lemma 1.8, $J''(\theta)$ is invertible, so θ is nondegenerate.

(c) The Morse index is finite. From (2.5), $J''(\theta)$ is positive defined throughout the space $H^1_{0,p}$; then the Morse index of J at θ is equal to 0; so it is finite and $j = 0 < +\infty$.

Conclusion. To sum up, we have proved that θ is a nondegenerate, non global minimum critical point and its Morse index is finite. By Lemma 1.3, we conclude that J has at least three distinct critical points. Therefore problem (1.1) has at least three distinct weak solutions in $H^1_{0,p}$.   □

EXAMPLE 2.12. The following impulsive boundary value problem has at least three distinct weak solutions in $H^1_{0,p}$, including the trivial solution:

$$(2.8) \quad \begin{cases} -(e^t u'(t))' = \lambda_1 e^{-t} \dfrac{u(t) + \sin u(t)}{4}, & t \neq t_j, \ j \in \{1, 2, \ldots\}, \ t > 0, \\ u(0) = u(+\infty) = 0, \\ \Delta(e^{t_j} u'(t_j)) = \dfrac{1}{1 + t_j^2} \dfrac{u(t_j)^2}{1 + u(t_j)^2}, & j \in \{1, 2, \ldots\}, \end{cases}$$

where λ_1 is the first eigenvalue of the linear problem:

$$(2.9) \quad \begin{cases} -(e^t u'(t))' = \lambda e^{-t} u(t), & t > 0, \\ u(0) = u(+\infty) = 0. \end{cases}$$

Indeed, we have:

(H_1) $f(t, 0) = 0$, for $t \in [0, +\infty)$,
(H_2) for $\alpha = \lambda_1/2$ and $\beta = 0$, we have $|f(t, u)| \leq \alpha|u| + \beta$,
(H_3) there exists $\varepsilon_0 \in (0, \lambda_1/2)$ such that $f'_u(t, 0) = \lambda_1/2 \leq \lambda_1 - \varepsilon_0$,
(H_4) for $C_1 = \lambda_1/16$, $\eta = 1/3$, we have $f(t, u) \geq C_1 u^\eta$, whenever $u > 0$ is small enough,
(H_5) $I'_j(u) = 2u/(1 + u^2)^2$ and so $I'_j(0) = 0$,
(H_6) for $k = 1$, $\sigma = 1/2$, we have $|I_j(u)| \leq k|u|^\sigma$.

Regarding λ_1, the first eigenvalue of the linear problem (2.9), it has been computed for different values of a parameter R for a problem posed on an interval $(0, R)$; it approaches 271.76159 for $R = 60$ and then stabilized. The obtained values are depicted in the graph and table (see Figure 1).

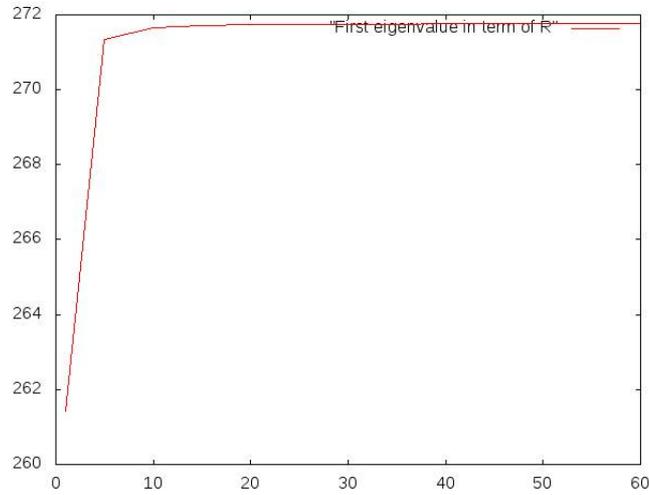Our second existence result of multiple solutions is now presented when h is positive.

THEOREM 2.13. *Assume that the following conditions hold:*

(L_1) *there exist $\nu > 2$ and $C_3 > 0$ such that*

$$\nu F(t, u) \leq u f(t, u) + \nu C_3, \quad \text{for all } t \geq 0, \ u \in \mathbb{R},$$

(L_2) *there exist $C_4, C_5 \in \mathbb{R}$ ($C_4 > 0$) such that*

$$F(t, u) \geq C_4 |u|^\nu - C_5, \quad \text{for all } t \geq 0, \ u \in \mathbb{R},$$

R	$\lambda_1(R)$	R	$\lambda_1(R)$
1	261.43004	35	271.75596
5	271.34599	40	271.75796
10	271.65983	45	271.75933
15	271.71798	50	271.76031
20	271.73833	55	271.76104
25	271.74775	60	271.76159
30	271.75287		

FIGURE 1. Problem (2.9): λ_1 in terms of R.

(L_3) $f(t, u)$ is odd in u, i.e., $f(t, -u) = -f(t, u)$, for all $t \geq 0$, $u \in \mathbb{R}$,

(L_4) there exists a positive continuous function g with

$$|f(t, u)| \leq g(u), \quad \text{for all } t \geq 0, \ u \in \mathbb{R},$$

(L_5) $I_j(\cdot)$ is odd $(j \in \{1, 2, \ldots\})$,

(L_6) there exist $\mu < \nu$ such that

$$\mu \int_0^u I_j(s) \, ds \geq u I_j(u) > 0, \quad \text{for all } u \in \mathbb{R} \setminus \{0\}, \ j \in \{1, 2, \ldots\}.$$

Then problem (1.1) has infinitely many weak solutions in $H^1_{0,p}$.

PROOF. We have to prove that J has infinitely many critical points in $H^1_{0,p}$. The proof is split into two steps.

Step 1. J satisfies the PS condition. Indeed, suppose that $(u_n) \subset H^1_{0,p}$ and there exists $M > 0$ such that $|J(u_n)| \leq M$ and $J'(u_n) = u_n - Au_n \to \theta$ in $H^1_{0,p}$,

as $n \to +\infty$. Using (L_1) and (L_6), we get the estimates:

$$\nu J(u_n) - (J'(u_n), u_n)$$

$$= \frac{\nu}{2} ||u_n||^2 + \nu \sum_{j=1}^{\infty} h(t_j) \int_0^{u_n(t_j)} I_j(s) \, ds - \int_0^{+\infty} \nu q(t) F(t, u(t)) \, dt$$

$$- ||u_n||^2 - \sum_{j=1}^{\infty} h(t_j) I_j(u_n(t_j)) u_n(t_j) + \int_0^{+\infty} q(t) f(t, u_n(t)) u_n(t) \, dt$$

$$\geq \left(\frac{\nu}{2} - 1\right) ||u_n||^2 + \nu \sum_{j=1}^{+\infty} h(t_j) \int_0^{u_n(t_j)} I_j(s) \, ds$$

$$- \int_0^{+\infty} [q(t) f(t, u_n(t)) u_n(t) + \nu C_3 q(t)] \, dt$$

$$- \sum_{j=1}^{\infty} h(t_j) I_j(u_n(t_j)) u_n(t_j) + \int_0^{+\infty} q(t) u_n(t) f(t, u_n(t)) \, dt$$

$$\geq \left(\frac{\nu}{2} - 1\right) ||u_n||^2 + \nu \sum_{j=1}^{\infty} h(t_j) \int_0^{u_n(t_j)} I_j(s) \, ds$$

$$- \mu \sum_{j=1}^{\infty} h(t_j) \int_0^{u_n(t_j)} I_j(s) ds - \nu C_3 ||q||_{L^1}$$

$$\geq \left(\frac{\nu}{2} - 1\right) ||u_n||^2 + \nu \sum_{j=1}^{\infty} h(t_j) \int_0^{u_n(t_j)} I_j(s) \, ds$$

$$- \nu \sum_{j=1}^{\infty} h(t_j) \int_0^{u_n(t_j)} I_j(s) \, ds - \nu C_3 ||q||_{L^1}$$

$$= \left(\frac{\nu}{2} - 1\right) ||u_n||^2 - \nu C_3 ||q||_{L^1}.$$

Hence

$$J(u_n) \geq \left(\frac{\nu}{2} - 1\right) ||u_n||^2 + \frac{1}{\nu} (J'(u_n), u_n) - C_3 ||q||_{L^1}$$

$$\geq \left(\frac{\nu}{2} - 1\right) ||u_n||^2 - \frac{1}{\nu} ||J'(u_n)|| ||u_n|| - C_3 ||q||_{L^1}.$$

Since $J'(u_n) \to 0$, as $n \to +\infty$, there exists $n_0 \in \mathbb{N}$ such that

$$M \geq J(u_n) \geq \left(\frac{\nu}{2} - 1\right) ||u_n||^2 - ||u_n|| - C_3 ||q||_{L^1}, \quad \text{for } n > n_0$$

which implies that (u_n) is bounded. Since $A \colon H_{0,p}^1 \to H_{0,p}^1$ is compact (see Lemma 2.10), (u_n) has a convergent subsequence in $H_{0,p}^1$, and then J satisfies the PS condition on $H_{0,p}^1$ (we proceed as in the proof of Theorem 2.9, Step 4).

From (L_3) and (L_5), we can see that J is an even functional and A is an odd operator.

Step 2. We argue as in the proof of [11, Theorem 1.3]. By contradiction, assume that J has at most a finite number of critical points $\{v_1, -v_1, \ldots, v_m, -v_m\}$. Let a, b two real numbers such that $a < 0$ and $b > 0$ with

$$(2.10) \qquad a < \min\{J(v_1), \ldots, J(v_m), -C_3\|q\|_{L^1}\},$$

$$(2.11) \qquad b > \max\{J(v_1), \ldots, J(v_m)\}.$$

(L_6) implies that

$$(2.12) \qquad \int_0^u I_j(s)\, ds \le |u|^\mu, \quad \text{for all } u \in \mathbb{R}.$$

Given $u \in H^1_{0,p}$, Assumption (L_2) and (2.12) guarantee that

$$
\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} h(t_j) \int_0^{u(t_j)} I_j(s)\, ds - \int_0^{+\infty} q(t) F(t, u(t))\, dt \\
&\le \frac{1}{2}\|u\|^2 + \sum_{j=1}^{\infty} h(t_j) \int_0^{u(t_j)} I_j(s)\, ds - \int_0^{+\infty} q(t)(C_4|u(t)|^\nu - C_5)\, dt \\
&\le \frac{1}{2}\|u\|^2 + \sum_{j=1}^{\infty} h(t_j)|u(t_j)|^\mu - C_4 \int_0^{+\infty} q(t)|u(t)|^\nu\, dt + \int_0^{\infty} C_5 q(t)\, dt \\
&\le \frac{1}{2}\|u\|^2 + \sum_{j=1}^{\infty} h(t_j)\|u\|_\infty^\mu - C_4 \int_0^{+\infty} q(t)|u(t)|^\nu\, dt + C_5\|q\|_{L^1} \\
&\le \frac{1}{2}\|u\|^2 + d^\mu \sum_{j=1}^{\infty} h(t_j)\|u\|^\mu - C_4 \int_0^{+\infty} q(t)|u(t)|^\nu\, dt + C_5\|q\|_{L^1}.
\end{aligned}
$$

As a consequence

$$
J(u) \le \frac{1}{2}\|u\|^2 + d^\mu \sum_{j=1}^{\infty} h(t_j)\|u\|^\mu - C_4 \int_0^{+\infty} q(t)|u(t)|^\nu\, dt + C_5\|q\|_{L^1}.
$$

Consider the unit sphere $S^\infty = \{u \in H^1_{0,p} : \|u\| = 1\}$ in $H^1_{0,p}$. For $u \in S^\infty$ and $\tau > 0$, we have

$$
J(\tau u) \le \frac{1}{2}\tau^2 + d^\mu \sum_{j=1}^{\infty} h(t_j)\tau^\mu - \tau^\nu C_4 \int_0^{+\infty} q(t)|u(t)|^\nu\, dt + C_5\|q\|_{L^1}.
$$

In addition $\lim_{\tau \to +\infty} J(\tau u) = -\infty$. Since $a < 0 = J(\theta)$, then there exist $\tau = \gamma(u)$ and $\delta > 0$ such that

$$(2.13) \qquad \gamma(u) \ge \delta \quad \text{and} \quad J(\gamma(u)u) = a.$$

Now, we show the uniqueness of the function γ satisfying (2.13) (see [7, Lemma 5.1.38]). Indeed, for $u \in S^\infty$ and $\tau > 0$, the assumptions (L_1), (L_6) together

with $\nu > 2$ imply that

$$\frac{d}{d\tau}(J(\tau u)) = (J'(\tau u), u)$$

$$= (\tau u, u) + \sum_{j=1}^{+\infty} h(t_j) I_j(\tau u(t_j)) u(t_j) - \int_0^{+\infty} q(t) f(t, \tau u(t)) u(t) \, dt$$

$$\leq \tau + \frac{1}{\tau} \sum_{j=1}^{+\infty} h(t_j) I_j(\tau u(t_j)) \tau u(t_j) - \frac{\nu}{\tau} \int_0^{+\infty} q(t) F(t, \tau u(t)) \, dt$$

$$+ \frac{\nu C_3}{\tau} \int_0^{+\infty} q(t) \, dt$$

$$\leq \frac{\nu}{\tau} \left(\frac{1}{2} \tau^2 + \frac{\mu}{\nu} \sum_{j=0}^{\infty} h(t_j) \int_0^{\tau u(t_j)} I_j(s) \, ds - \int_0^{+\infty} q(t) F(t, \tau u(t)) \, dt \right)$$

$$+ \frac{\nu C_3 \|q\|_{L^1}}{\tau}$$

$$\leq \frac{\nu}{\tau} \left(\frac{1}{2} \tau^2 + \sum_{j=1}^{\infty} h(t_j) \int_0^{\tau u(t_j)} I_j(s) \, ds - \int_0^{+\infty} q(t) F(t, \tau u(t)) \, dt \right)$$

$$+ \frac{\nu C_3 \|q\|_{L^1}}{\tau}$$

$$= \frac{\nu}{\tau}(J(\tau u)) + \frac{\nu C_3 \|q\|_{L^1}}{\tau} = \frac{\nu}{\tau}(J(\tau u) + C_3 \|q\|_{L^1}).$$

From (2.13) and (2.10) we obtain that

$$\frac{d}{d\tau} J(\tau u)|_{\tau=\gamma(u)} \leq \frac{\nu}{\tau}(J(\gamma(u)u) + C_3\|q\|_{L^1}) = \frac{\nu}{\tau}(a + C_3\|q\|_{L^1}) < 0.$$

Thus, for $u \in H^1_{0,p}$, the continuous function $h(\tau) = J(\tau u)$ is strictly decreasing in the neighbourhood of $\tau = \gamma(u)$. From (2.13), we have that $\gamma(u)$ is unique.

Now we prove that $\gamma \colon S^\infty \to \mathbb{R}$ is continuous. Let $(u_n) \subset S^\infty$, $u_0 \in S^\infty$, and $u_n \to u_0$ in $H^1_{0,p}$, as $n \to \infty$. From (L_2) and (2.12), we have

$$a = J(\gamma(u_n)u_n) = \frac{1}{2}\|\gamma(u_n)u_n\|^2 + \sum_{j=1}^{\infty} h(t_j) \int_0^{\gamma(u_n)u_n} I_j(s) \, ds$$

$$- \int_0^{+\infty} q(t) F(t, \gamma(u_n)u_n(t)) \, dt$$

$$\leq \frac{1}{2}(\gamma(u_n))^2 + d^\mu \sum_{j=1}^{\infty} h(t_j)(\gamma(u_n))^\mu$$

$$- C_4(\gamma(u_n))^\nu \int_0^{+\infty} q(t)|u_n(t)|^\nu \, dt + C_5\|q\|_{L^1}.$$

Hence

$$(2.14) \quad a \leq \frac{1}{2}(\gamma(u_n))^2 + d^\mu \sum_{j=1}^{\infty} h(t_j)(\gamma(u_n))^\mu$$

$$- C_4(\gamma(u_n))^\nu \int_0^{+\infty} q(t)|u_n(t)|^\nu \, dt + C_5||q||_{L^1}.$$

Moreover, $||\sqrt[\nu]{q}u_n||_{L^\nu}^\nu \to ||\sqrt[\nu]{q}u_0||_{L^\nu}^\nu$, as $n \to +\infty$. As a consequence, there exist $C_6 > 0$ and $n_0 \in \mathbb{N}$ such that $\int_0^{+\infty} q(t)|u_n(t)|^\nu \, dt \geq C_6$, for $n \geq n_0$. Therefore, from (2.14) we have

$$a \leq \frac{1}{2}(\gamma(u_n))^2 + d^\mu \sum_{j=1}^{\infty} h(t_j)(\gamma(u_n))^\mu - C_4 C_6(\gamma(u_n))^\nu + C_5||q||_{L^1}, \quad \text{for } n \geq n_0.$$

Then $(\gamma(u_n))$ is bounded, hence it has a least one limiting point. Let w be an arbitrary limiting point of $(\gamma(u_n))$, i.e. there exists $(\gamma(u_{n_k}))$ a subsequence of $(\gamma(u_n))$ such that $\gamma(u_{n_k}) \to w$, as $k \to \infty$. Letting $k \to \infty$ in the equation $J(\gamma(u_{n_k})u_{n_k}) = a$ yields $J(wu_0) = a$. From the uniqueness of $\gamma(u_0)$, we deduce that $w = \gamma(u_0)$; this implies that $\gamma(u_n) \to \gamma(u_0)$ as $n \to +\infty$, and then the map $\gamma \colon S^\infty \to \mathbb{R}$ is continuous.

Let $0 < \varepsilon < \delta$ be such that $J^a \cap B_\varepsilon = \emptyset$, i.e. $J^a \subset H^1_{0,p} \setminus B_\varepsilon$, where $B_\varepsilon = \{u \in H^1_{0,p} : ||u|| < \varepsilon\}$. Define the mapping $\eta \colon [0,1] \times (H^1_{0,p} \setminus B_\varepsilon) \to H^1_{0,p} \setminus B_\varepsilon$ by

$$\eta(s, u) = \begin{cases} (1-s)u + s\gamma\left(\dfrac{u}{||u||}\right)\left(\dfrac{u}{||u||}\right), & u \in (H^1_{0,p} \setminus B_\varepsilon) \setminus J^a, \\ u, & u \in J^a. \end{cases}$$

(See also [7, p. 346], [11, p. 3654], and [21, p. 161]). From the continuity of $\gamma(\,\cdot\,)$, η is a continuous mapping which is a strong deformation retraction from $H^1_{0,p} \setminus B_\varepsilon$ to J^a. From (2.11), one can further construct a strong deformation retraction of $H^1_{0,p}$ in J^b. Then

$$\beta_k = \beta_k(a, b) = \dim H_k(J^b, J^a) = \dim H_k(H^1_{0,p}, H^1_{0,p} \setminus B_\varepsilon).$$

Since $H^1_{0,p} \setminus B_\varepsilon$ is homotopically equivalent to S^∞ and S^∞ is contractible, we deduce that $H^1_{0,p} \setminus B_\varepsilon$ is contractible. Therefore

$$H_k(H^1_{0,p}, H^1_{0,p} \setminus B_\varepsilon) \simeq 0 \quad \text{and} \quad \beta_k = 0, \quad \text{for } k = 0, 1, \ldots$$

As a consequence

$$(2.15) \qquad\qquad \sum_{q=0}^{\infty} (-1)^k \beta_k(a, b) = 0.$$

Recall that a space X is called contractible if there exists a point $x_0 \in X$ such that $Id_X \sim g$, where $g \colon X \to X$ is defined by $g(x) = x_0$ for every $x \in X$.

On the other hand, let $r > 0$ be small enough such that the open balls $B(\theta, r)$, $B(v_i, r)$, and $B(-v_i, r)$ $(i = 1, \ldots, m)$ are mutually disjoint. Then

$$M_k(a, b) = \sum_{i=1}^{m} \dim C_k(J, v_i) + \sum_{i=1}^{m} \dim C_k(J, -v_i) + \dim C_k(J, \theta), \quad k = 0, 1, \ldots$$

By Lemmas 1.5 and 1.7, we have

$$
\begin{aligned}
(2.16) \quad & \sum_{k=0}^{\infty} (-1)^k M_k(a, b) \\
&= \sum_{i=1}^{m} \left[\sum_{k=0}^{\infty} (-1)^k \dim C_k(J, v_i) + \sum_{k=0}^{\infty} (-1)^k \dim C_k(J, -v_i) \right] \\
&\quad + \sum_{k=0}^{\infty} (-1)^k \dim C_k(J, \theta) \\
&= \sum_{i=1}^{m} [\deg(I - A, B(v_i, r), \theta) + \deg(I - A, B(-v_i, r), \theta)] \\
&\quad + \deg(I - A, B(\theta, r), \theta) \\
&= \deg\left(I - A, \bigcup_{i=1}^{m} (B(v_i, r) \cup B(-v_i, r)) \cup B(\theta, r), \theta \right) \\
&= \text{an odd number.}
\end{aligned}
$$

(See the end of the proof of [11, Theorem 1.3] and [21, Theorem 1.3]). Finally, (2.15) and (2.16) imply that

$$\sum_{k \in \mathbb{N}} (-1)^k \beta_k(a, b) \neq \sum_{k \in \mathbb{N}} (-1)^k M_k(a, b),$$

leading to a contradiction with Lemma 1.4. As a consequence, we conclude that J has an infinite number of critical points in $H_{0,p}^1$; therefore problem (1.1) has infinitely many weak solutions in $H_{0,p}^1$. $\qquad\square$

REMARK 2.14. Assumptions (L_4) and (L_6) are used to prove that the functional J is well defined and the operator A is compact. The proof, which is similar to that of Lemma 2.10, is omitted.

EXAMPLE 2.15. The following impulsive boundary value problem has infinitely many weak solutions in $H_{0,p}^1$:

$$
(2.17) \quad \begin{cases}
-(e^t u'(t))' = e^{-t} u^5(t), & t \neq t_j, \ j \in \{1, 2, \ldots\}, \ t > 0, \\
u(0) = u(+\infty) = 0, \\
\Delta(e^{t_j} u'(t_j)) = \dfrac{u(t_j)^3}{1 + (t_j)^2}, & j \in \{1, 2, \ldots\}.
\end{cases}
$$

Indeed, assumptions (L_1)–(L_6) are satisfied:

(L_1) $F(t,u) = u^6/6 = uf(t,u)/6$ and thus $6F(t,u) \leq uf(t,u) + 6C_3$, for any constant $C_3 > 0$,

(L_2) $F(t,u) = u^6/6 \geq C_4|u|^6 - C_5$ for $C_4 = 1/6$ and $C_5 = 1$,

(L_3) $f(t,u)$ is odd,

(L_4) $|f(t,u)| \leq g(u)$ where we may take, e.g., $g(u) = 1 + |u^5|$,

(L_5) $I_j(u) = u^3$ is odd,

(L_6) for $\mu = 5 < \nu = 6$, $5 \int_0^u I_j(s)\,ds = 5u^4/4 \geq u^4$.

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