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GROUND STATE SOLUTIONS FOR A CLASS OF NONLINEAR MAXWELL–DIRAC SYSTEM

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Abstract. This paper is concerned with the following nonlinear Maxwell–Dirac system

$$\begin{cases} -i\displaystyle\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + \omega u - \phi u = F_u(x,u), \\ -\Delta\phi = 4\pi|u|^2, \end{cases}$$

for $x \in \mathbb{R}^3$. The Dirac operator is unbounded from below and above, so the associated energy functional is strongly indefinite. We use the linking and concentration compactness arguments to establish the existence of ground state solutions for this system with asymptotically quadratic nonlinearity.

1. Introduction and main results

We study the following nonlinear Maxwell–Dirac system

$$(1.1)\qquad \begin{cases} -i\displaystyle\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + \omega u - \phi u = F_u(x,u), \\ -\Delta\phi = 4\pi|u|^2, \end{cases}$$

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where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $u \in \mathbb{C}^4$, $\partial_k = \partial/\partial x_k$, $a > 0$, α_1, α_2, α_3 and β are the 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

ϕ is the electron field. In this paper, we are interested in the existence of ground state solutions of system (1.1) with asymptotically quadratic nonlinearity, that is, solutions corresponding to the least energy for the energy functional of system (1.1).

The Maxwell–Dirac system, which describes the interaction of a particle with its self-generated electromagnetic field, plays an important role in quantum electrodynamics. Also it has been used as effective theories in atomic, nuclear and gravitational physics (see [39]). In the past decade, system (1.1) has been studied for a long time and many results are available concerning the Cauchy problem, see for instance, [8], [9], [27], [29], [32], [30], [38] and the references therein. As we known, the existence of stationary solutions of the Maxwell-Dirac system has been an open problem for a long time, see [31, p. 235]. As far as the existence of stationary solutions of system (1.1) is concerned by using variational methods, there is a poineering work by Esteban et al. [23] in which a multiplicity result is studied. After that, Abenda [1] obtained the existence result of solitary wave solutions. And a strong localization result was obtained in [36]. On the other hand, in [28], Garrett Lisi gave numerical evidence of the existence of bounded states by using an axially symmetric ansatz. For more detailed descriptions for equations and systems related to Dirac equations, we refer to the recent review [24] and the references therein.

We emphasize that the works mentioned above mainly concerned with the autonomous system with null self-coupling ($F \equiv 0$). In [12], Chen and Zheng studied system (1.1) with nonlinear self-coupling model ($F \neq 0$), and the existence of least energy stationary solutions of system (1.1) was obtained for the special superquadratic power nonlinearity $F_u(x, u) = a(x)|u|^{p-2}u$ with $2 < p < 3$. Zhang et al. [46] considered the general superquadratic nonlinearity. Besides, for other related topics including the superquadratic singular perturbation problem and concentration phenomenon of semi-classical states, see, for instance [20]–[22] and the references therein.

Inspired by the above works, the purpose of this paper is to consider system (1.1) with non-autonomous asymptotically quadratic nonlinearity. To the best of our knowledge, there has been no work concerning on this case up to now.

We mainly study the existence of ground state solutions via variational methods. Before stating our main result, we first make the following assumptions on the nonlinearity:

(F_1) $F(x,u) \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, \mathbb{R}^+)$ and $F(x,u)$ is 1-periodic in x_k, $k = 1,2,3$;
(F_2) $F_u(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in x;
(F_3) there exists a bounded function $F_\infty \in C(\mathbb{R}^3, \mathbb{R}^+)$ such that $|F_u(x,u) - F_\infty(x)u|/|u| \to 0$ as $|u| \to \infty$ uniformly in x, and $\inf_{x\in\mathbb{R}^3} F_\infty(x) > a + \omega$;
(F_4) $\widetilde{F}(x,u) \geq 0$ for all u, $\widetilde{F}(x,u) \to \infty$ as $|u| \to \infty$, where $\widetilde{F}(x,u) = (1/2)F_u(x,u)u - F(x,u)$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that* $\omega \in (-a,a)$ *and* (F_1)–(F_4) *are satisfied. Then system* (1.1) *has at least one ground state solutions.*

As a motivation we recall that there is a large number of works on existence of stationary solutions of nonlinear Schrödinger–Maxwell system arising in the non-relativistic quantum mechanics, see, for example, [2], [5], [11], [37], [40], [41] and so on. It is quite natural to ask if certain similar results can be obtain for nonlinear Maxwell–Dirac system arising in the relativistic quantum mechanics, we will give an answer for Maxwell–Dirac system in the present paper. From a mathematical viewpoint, the two systems possess different variational structures. Note that, for the Schrödinger–Maxwell system, techniques based on the mountain pass theorem are well applied to the investigation. However, for the Maxwell–Dirac system, the mountain pass structure no longer be satisfied because the associated energy functional is strongly indefinite, the classical critical point theory cannot be applied directly. On the other hand, one of the main difficulties of such problem is the lack of compactness of Sobolev embedding. Hence our problem poses more challenges in the calculus of variation in nature. In order to overcome these difficulties, we will turn to the linking and concentration compactness arguments (see [6], [33] and [34]).

Very recently, there are some works focused on existence of stationary solutions and concentration of semi-classical solutions for nonlinear Dirac equation but not for Maxwell–Dirac system. See, for example [7], [10], [13], [15]–[19], [25], [26], [35], [43]–[45], [47], [48] and the references therein. Compared to the Dirac equations, the Maxwell–Dirac system becomes much more complicated since the effects of nonlocal term. In order to overcome this obstacle, we need more delicate estimates for nonlocal term (see Lemma 3.5).

The remainder of this paper is organized as follows. In Section 2, we formulate the variational setting, and present a critical point theorem required. In Section 3, we will use the linking and concentration compactness principle to prove our main result.

2. Variational setting and abstract theorem

Below by $|\cdot|_q$ we denote the usual L^q- norm, $(\,\cdot\,,\,\cdot\,)_2$ denote the usual L^2 inner product, c, C_i stand for different positive constants. For convenience, let Dirac operator

$$A := -i\sum_{k=1}^{3}\alpha_k\partial_k + a\beta + \omega,$$

and let $\sigma(A)$, $\sigma_c(A)$ be the spectrum of A, the continuous spectrum of A, respectively. It is well known that A is a selfadjoint operator on $L^2 := L^2(\mathbb{R}^3,\mathbb{C}^4)$ with $\mathcal{D}(A)\subset H^1 := H^1(\mathbb{R}^3,\mathbb{C}^4)$. A Fourier analysis shows that $\sigma(A)=\sigma_c(A)=\mathbb{R}\setminus(-a+\omega,a+\omega)$. For $\omega\in(-a,a)$, the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^-\oplus L^+, \qquad u = u^- + u^+$$

such that A is negative definite on L^- and positive definite on L^+. Let $E := \mathcal{D}(|A|^{1/2}) = H^{1/2}$ be equipped with the inner product

$$\langle u,v\rangle = (|A|^{1/2}u,|A|^{1/2}v)_2$$

and the induced norm $\|u\| = \langle u,u\rangle^{1/2}$, where $|A|$ and $|A|^{1/2}$ denote respectively the absolute value of A and the square root of $|A|$. Note that this norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^p for all $q\in[2,3]$ and compactly into $L^q_{\rm loc}$ for all $p\in[1,3)$. It is clear that E possesses the following decomposition

$$E = E^-\oplus E^+ \quad\text{and}\quad E^\pm = E\cap L^\pm.$$

These two subspaces are orthogonal with respect to both $(\,\cdot\,,\,\cdot\,)_2$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ inner products.

Let $\mathcal{D}^{1,2} := \mathcal{D}^{1,2}(\mathbb{R}^3,\mathbb{R})$ be the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R})$ with respect to the norm

$$\|u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^3}|\nabla u|^2dx.$$

It is well known that system (1.1) can be reduced to a single equation with non-local term. Actually, for each $u\in E$, the linear functional T_u in $\mathcal{D}^{1,2}$ defined by

$$T_u(v) = \int_{\mathbb{R}^3}|u|^2v\,dx, \quad v\in\mathcal{D}^{1,2},$$

is continuous. In fact, since $u\in L^q$ for all $q\in[2,3]$, one has $|u|^2\in L^{6/5}$ for all $u\in E$, and Hölder inequality and Sobolev inequality imply that

$$\begin{aligned}(2.1)\quad |T_u(v)| &= \bigg|\int_{\mathbb{R}^3}|u|^2v\,dx\bigg| \\ &\le \bigg(\int_{\mathbb{R}^3}||u|^2|^{6/5}\,dx\bigg)^{5/6}\bigg(\int_{\mathbb{R}^3}|v|^6\,dx\bigg)^{1/6} \le S^{-1/2}||u|^2|_{6/5}\|v\|_{\mathcal{D}}.\end{aligned}$$

where S is the Sobolev embedding constant. It follows from the Lax–Milgram theorem that there exists a unique $\phi_u \in \mathcal{D}^{1,2}$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v\,dx = 4\pi \int_{\mathbb{R}^3} |u|^2 v\,dx, \quad \text{for all } v \in \mathcal{D}^{1,2}, \tag{2.2}$$

that is ϕ_u satisfies the Poisson equation $-\Delta\phi_u = 4\pi|u|^2$ and there holds

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\,dy = \frac{1}{|x|} * |u|^2.$$

By (2.1), (2.2), it is easy to see that

$$\|\phi_u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^3} \phi_u|u|^2\,dx \le S^{-1/2}||u|^2|_{6/5}\|\phi_u\|_{\mathcal{D}} \tag{2.3}$$

and

$$\int_{\mathbb{R}^3} \phi_u|u|^2\,dx \le S^{-1/2}||u|^2|_{6/5}\|\phi_u\|_{\mathcal{D}} \le S^{-1}|u|_{12/5}^4. \tag{2.4}$$

Substituting ϕ_u in (1.1), we are led to the equation

$$-i\sum_{k=1}^{3} \alpha_k\partial_k u + a\beta u + \omega u - \phi_u u = F_u(x,u). \tag{2.5}$$

Next, on E we define the following functional

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma(u) - \Psi(u) \tag{2.6}$$

for $u = u^+ + u^- \in E$, where

$$\Gamma(u) = \frac{1}{4}\int_{\mathbb{R}^3} \phi_u|u|^2\,dx = \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(y)|^2|u(x)|^2}{|x-y|}\,dy\,dx$$

and

$$\Psi(u) = \int_{\mathbb{R}^3} F(x,u)\,dx.$$

Moreover, our hypotheses imply that $\Phi \in C^1(E,\mathbb{R})$, and a standard argument shows that critical points of Φ are solutions of system (1.1) (see [14], [42]).

In order to find critical points of Φ, we shall use the following abstract theorem which is taken from [6] and [14].

Let E be a Banach space with direct sum $E = X \oplus Y$ and corresponding projections P_X, P_Y onto X, Y. Let $\mathcal{S} \subset (X)^*$ be a dense subset, for each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s\colon E \to \mathbb{R}, \quad p_s(u) := |s(x)| + \|y\| \quad \text{for } u = x + y \in E.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the topology induced by semi-norm family $\{p_s\}$, w^* denote the weak*-topology on E^*. Now, some notations are needed. For a functional $\Phi \in C^1(E,\mathbb{R})$ we write $\Phi_a = \{u \in E \mid \Phi(u) \ge a\}$, $\Phi^b = \{u \in E \mid \Phi(u) \le b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Recall that Φ is said to be weakly sequentially lower semicontinuous if for any $u_n \rightharpoonup u$ in E one has $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$, and Φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\Phi'(u_n)w=\Phi'(u)w$ for each $w\in E$. Recall that a sequence $\{u_n\}\subset E$ is said to be a $(C)_c$-sequence if $\Phi(u_n)\to c$ and $(1+\|u_n\|)\Phi'(u_n)\to 0$.

Suppose:

(Φ_0) for any $c\in\mathbb{R}$, superlevel Φ_c is $\mathcal{T_S}$-closed, and $\Phi'\colon(\Phi_c,\mathcal{T_S})\to(E^*,w^*)$ is continuous;

(Φ_1) for any $c>0$, there exists $\xi>0$ such that $\|u\|<\xi\|P_Yu\|$ for all $u\in\Phi_c$;

(Φ_2) there exists $r>0$ such that $\varrho:=\inf\Phi(S_r\cap Y)>0$, where $S_r:=\{u\in E:\|u\|=r\}$.

Now we state the following critical point theorem which will be used later (see [6], [14]).

THEOREM 2.1. *Let* $(\Phi_0)-(\Phi_2)$ *be satisfied and suppose there are* $R>r>0$ *and* $e\in Y$ *with* $\|e\|=1$ *such that* $\sup\Phi(\partial Q)\le\varrho$ *where* $Q:=\{u=x+te\mid x\in X,\ t\ge0,\ \|u\|<R\}$. *Then* Φ *has a* $(C)_c$*-sequence with* $\varrho\le c\le\sup\Phi(Q)$.

3. Proof of the main result

First, let $r>0$, set $B_r:=\{u\in E\mid\|u\|\le r\}$, $S_r:=\{u\in E\mid\|u\|=r\}$. In virtue of the assumptions (F_1)–(F_3), for any $\varepsilon>0$, there exist positive constants r_ε, C_ε such that

$$\begin{cases}|F_u(x,u)|\le\varepsilon|u| & \text{for all } 0\le|u|\le r_\varepsilon,\\ |F_u(x,u)|\le\varepsilon|u|+C_\varepsilon|u|^{p-1} & \text{for all } (x,u),\\ |F(x,u)|\le\varepsilon|u|^2+C_\varepsilon|u|^p & \text{for all } (x,u),\end{cases}\tag{3.1}$$

where $p\in(2,3)$.

Before proving our result, we need some preliminary results.

LEMMA 3.1. Γ *and* Ψ *are non-negative, weakly sequentially lower semi-continuous,* Γ' *and* Ψ' *are weakly sequentially continuous.*

PROOF. The above Lemma is standard because E embeds continuously into L^q for $q\in[2,3]$ and compactly into L^q_{loc} for $q\in[1,3)$ (see [14]). □

LEMMA 3.2. *Let* $(F_1)-(F_3)$ *be satisfied, there exists* $r>0$ *such that* $\varrho:=\inf\Phi(S_r\cap E^+)>0$.

PROOF. Observe that $|u|_p^p\le c_p\|u\|^p$ for all $u\in E$ by Sobolev embedding. For any $u\in E^+$, by (2.4) and (3.1) we have

$$\begin{aligned}\Phi(u)&=\frac12\|u\|^2-\Gamma(u)-\Psi(u)\ge\frac12\|u\|^2-C_1\|u\|^4-c_2\varepsilon\|u\|^2-C_\varepsilon c_p\|u\|^p\\&=\left(\frac12-c_2\varepsilon\right)\|u\|^2-C_1\|u\|^4-C_\varepsilon c_p\|u\|^p.\end{aligned}$$

Since $p \in (2,3)$, choosing suitable $r > 0$ we see that the desired conclusion holds. □

Let $\Lambda := \inf_{x\in\mathbb{R}^3} F_\infty(x)$. By virtue of (F_3), we take a number μ satisfying $a+\omega < \mu < \Lambda$. Thus there exists $e \in E^+$ with $\|e\| = 1$ such that

$$(a+\omega)|e|_2^2 < 1 < \mu|e|_2^2 < \Lambda|e|_2^2. \tag{3.2}$$

Lemma 3.3. *Let* (F_1)–(F_3) *be satisfied, there is* $R_0 > r > 0$*, such that* $\Phi|_{\partial Q} \le 0$*, where* $Q := \{u = u^- + se \mid u^- \in E^-,\ s \ge 0,\ \|u\| \le R_0\}$.

Proof. Suppose to the contrary that there exist $u_n = s_n e + u_n^-$ with $\|u_n\| \to \infty$ such that

$$\frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}(\delta_n^2 - \|v_n^-\|^2) - \frac{1}{4}\int_{\mathbb{R}^3} \frac{\phi_{u_n}|u_n|^2}{\|u_n\|^2} - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^2} \ge 0, \tag{3.3}$$

where $\delta_n = s_n/\|u_n\|$, $v_n = u_n/\|u_n\|$ and $v_n^- = u_n^-/\|u_n\|$. Therefore, we know by (F_1) that $\delta_n \ge \|v_n^-\|$. Since $\delta_n^2 + \|v_n^-\|^2 = 1$, up to a subsequence, $\delta_n \to \delta$ and $v_n^- \rightharpoonup v^-$ in E. Set $v = \delta e + v^-$, it follows from (3.2) that

$$\begin{aligned}\delta^2 - \|v^-\|^2 - \int_{\mathbb{R}^3} F_\infty(x)v^2 &\le \delta^2 - \|v^-\|^2 - \Lambda|v|_2^2 \\ &\le \delta^2(\mu - \Lambda)|e|_2^2 - \|v^-\|^2 - \Lambda|v^-|_2^2 < 0.\end{aligned}$$

Then there exists a bounded set $\Omega \subset \mathbb{R}^3$ such that

$$\delta^2 - \|v^-\|^2 - \int_\Omega F_\infty(x)v^2 < 0. \tag{3.4}$$

Letting $R(x,u) := F(x,u) - (1/2)F_\infty(x)u^2$, then $|R(x,u)| \le C_2|u|^2$ for some $C_2 > 0$ and $R(x,u)/|u|^2 \to 0$ as $|u| \to \infty$ uniformly in x. Hence, by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega \frac{R(x,u_n)}{\|u_n\|^2} = \lim_{n\to\infty} \int_\Omega \frac{R(x,u_n)}{|u_n|^2}|v_n|^2 = 0. \tag{3.5}$$

Thus (3.3)–(3.5) imply that

$$\begin{aligned}0 &\le \lim_{n\to\infty} \left(\frac{1}{2}(\delta_n^2 - \|v_n^-\|^2) - \frac{1}{4}\int_{\mathbb{R}^3} \frac{\phi_{u_n}|u_n|^2}{\|u_n\|^2} - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^2}\right) \\ &\le \lim_{n\to\infty} \left(\frac{1}{2}(\delta_n^2 - \|v_n^-\|^2) - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}\right) \\ &\le \frac{1}{2}\left(\|\delta\|^2 - \|v^-\|^2 - \int_\Omega F_\infty(x)v^2\right) < 0.\end{aligned}$$

Now the desired conclusion is obtained from this contradiction. □

Combining Lemmas 3.1–3.3 and Theorem 2.1 we obtain

Lemma 3.4. *Suppose that* (F_1)–(F_3) *are satisfied. Then for the functional* Φ*, there exists a* $(C)_c$*-sequence* $\{u_n\}$ *with* $c > 0$.

PROOF. With $X = E^-$ and $Y = E^+$. Clearly, Φ satisfies (Φ_1) because $\Gamma, \Psi \geq 0$. In virtue of Lemma 3.1, we see that (Φ_0) is satisfied. Lemma 3.2 implies that (Φ_2) holds. Lemma 3.3 shows that Φ possesses the linking structure of Theorem 2.1. Therefore, there exists a sequence $\{u_n\}$ satisfying

$$\Phi(u_n) \to c > 0 \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \to 0. \qquad \square$$

In the following, we discuss the properties of the $(C)_c$-sequences. Since the effect of nonlocal term $\Gamma(u)$, it is difficult to verify the boundedness of the $(C)_c$-sequence for the functional Φ. Motivated by Ackermann [3], we give a delicate estimate for the norm of $\Gamma'(u)$, it is very important in our arguments.

LEMMA 3.5. *For any $u \in E \setminus \{0\}$, there exists $C > 0$ such that*

$$\Gamma'(u)u > 0 \quad \textit{and} \quad \|\Gamma'(u)\|_{E^*} \leq C\left(\sqrt{\Gamma'(u)u} + \Gamma'(u)u\right),$$

where E^* denotes the dual space of E.

PROOF. Clearly, $\Gamma'(u)u = 4\Gamma(u) > 0$. Now we show the second conclusion. Since Γ is the unique nonlocal term in Φ, from the argument in Ackermann [3] (see also [4]), we have

$$\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2\right)|v|^2\,dx \leq C_3\sqrt{\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2\right)|u|^2\,dx \int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |v|^2\right)|v|^2\,dx}$$

for all $u, v \in E$ and some $C_3 > 0$. Hence using this, (2.4) and Hölder inequality, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2\right)|uv|\,dx \\
&\quad\leq \left(\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2\right)|u|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2)|v|^2\,dx\right)^{1/2} \\
&\quad\leq C_4\left(\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2)|u|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2)|u|^2\,dx\right)^{1/4} \\
&\qquad\times\left(\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |v|^2\right)|v|^2\,dx\right)^{1/4} \\
&\quad\leq C_5\left(\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2\right)|u|^2\,dx\right)^{3/4}\|v\|,
\end{aligned}$$

which implies that

$$|\Gamma'(u)v| \leq C_5(\Gamma'(u)u)^{3/4}\|v\| \leq C\left(\sqrt{\Gamma'(u)u} + \Gamma'(u)u\right)\|v\|.$$

This shows the second conclusion. $\square$

LEMMA 3.6. *Suppose that (F_1)–(F_4) are satisfied. Then any $(C)_c$-sequence of Φ is bounded.*

Proof. Let $\{u_n\} \subset E$ be such that

$$\Phi(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \to 0. \tag{3.6}$$

Then, there is constant $C_6 > 0$ such that we have

$$C_6 \geq \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n = \Gamma(u_n) + \int_{\mathbb{R}^3} \widetilde{F}(x, u_n). \tag{3.7}$$

Suppose to the contrary that $\{u_n\}$ is unbounded. Setting $v_n := u_n/\|u_n\|$, then $\|v_n\| = 1$ and $|v_n|_s \leq c_s\|v_n\| = c_s$ for all $s \in [2, 3]$. After passing to a subsequence, we can assume that $v_n \rightharpoonup v$ in E. Observe that

$$\Phi'(u_n)(u_n^+ - u_n^-) = \|u_n\|^2\bigg(1 - \frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} - \int_{\mathbb{R}^3} \frac{F_u(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|}\bigg).$$

Hence

$$\frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} + \int_{\mathbb{R}^3} \frac{F_u(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|} \to 1. \tag{3.8}$$

Set $h(r) := \inf\{\widetilde{F}(x, u) \mid x \in \mathbb{R}^3 \text{ and } u \in \mathbb{C}^4 \text{ with } |u| \geq r\}$ for $r \geq 0$. By (F_4), $h(r) \to \infty$ as $r \to \infty$. For $0 \leq a < b$, let $\Omega_{n(a,b)} := \{x \in \mathbb{R}^3 \mid a \leq |u_n(x)| < b\}$ and

$$C_a^b := \inf\bigg\{\frac{\widetilde{F}(x, u)}{|u|^2} \;\bigg|\; x \in \mathbb{R}^3 \text{ and } u \in \mathbb{C}^4 \text{ with } a \leq |u(x)| < b\bigg\}.$$

By (3.7), it is easy to prove that

$$|\Omega_{n(b,\infty)}| \leq \frac{C_6}{h(b)} \to 0 \tag{3.9}$$

as $b \to \infty$ uniformly in n, and for any fixed $0 < a < b$,

$$\int_{\Omega_{n(a,b)}} |v_n|^2 = \frac{1}{\|u_n\|^2}\int_{\Omega_{n(a,b)}} |u_n|^2 \leq \frac{C_6}{C_a^b\|u_n\|^2} \to 0 \tag{3.10}$$

as $n \to \infty$.

Now, let $0 < \varepsilon < 1/4$. By (3.1), for any $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that $|F_u(x, u_n)| \leq \varepsilon|u_n|$, for all $|u_n| \leq a_\varepsilon$. Consequently,

$$\int_{\Omega_{n(0,a_\varepsilon)}} \frac{F_u(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|} \leq \int_{\Omega_{n(0,a_\varepsilon)}} \varepsilon|v_n^+ - v_n^-||v_n| \leq \varepsilon|v_n|_2^2 \leq \varepsilon \tag{3.11}$$

for all n. From (F_3), we can deduce that there is $c > 0$ such that $|F_u(x, u)| \leq c|u|$ for all (x, u). Hence, by (3.9) and Hölder inequality, we can take large b_ε such that

$$\begin{aligned}\int_{\Omega_{n(b_\varepsilon,\infty)}} \frac{F_u(x, u_n)(v_n^+ - v_n^-)|v_n|}{|u_n|} &\leq c\int_{\Omega_{n(b_\varepsilon,\infty)}} |v_n^+ - v_n^-||v_n| \\ &\leq |\Omega_{n(b_\varepsilon,\infty)}|^{1/6}\bigg(\int_{\Omega_{n(b_\varepsilon,\infty)}} |v_n^+ - v_n^-|^2\bigg)^{1/2}\bigg(\int_{\Omega_{n(b_\varepsilon,\infty)}} |v_n|^3\bigg)^{1/3} \leq \varepsilon\end{aligned} \tag{3.12}$$

for all n. By (3.10), there is $n_0 > 0$ such that, for all $n \geq n_0$,

$$\begin{aligned}(3.13)\quad &\int_{\Omega_{n(a_\varepsilon,b_\varepsilon)}} \frac{F_u(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|} \\ &\leq c\int_{\Omega_{n(a_\varepsilon,b_\varepsilon)}} |v_n^+ - v_n^-||v_n| \leq c|v_n|_2 \bigg(\int_{\Omega_{n(a_\varepsilon,b_\varepsilon)}} |v_n|^2\bigg)^{1/2} \leq \varepsilon.\end{aligned}$$

Next we deal with the nonlocal term. From (3.7) we easily know

$$\frac{\Gamma(u_n)}{\|u_n\|} \to 0, \quad \text{as } n \to \infty.$$

Moreover, by Lemma 3.5, we have

$$\begin{aligned}(3.14)\quad \bigg|\frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\bigg| &\leq \frac{\|\Gamma'(u_n)\|_{E^*}\|u_n^+ - u_n^-\|}{\|u_n\|^2} \\ &\leq C_7\bigg|\frac{\Big(\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n\Big)\|u_n^+ - u_n^-\|}{\|u_n\|^2}\bigg| \\ &\leq C_8\bigg|\frac{\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n}{\|u_n\|}\bigg| \\ &= C_9\bigg(\frac{1}{\sqrt{\|u_n\|}}\sqrt{\frac{4\Gamma(u_n)}{\|u_n\|}} + \frac{4\Gamma(u_n)}{\|u_n\|}\bigg) \leq \varepsilon\end{aligned}$$

for all $n \geq n_0$. Now the combination of (3.11)–(3.14) shows that

$$\limsup_{n\to\infty}\bigg(\frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} + \int_{\mathbb{R}^3} \frac{F_u(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|}\bigg) \leq 4\varepsilon < 1,$$

which contradicts (3.8). Therefore, $\{u_n\}$ is bounded in E. □

Let $\mathcal{K} := \{u \in E \mid \Phi'(u) = 0, u \neq 0\}$ be the set of nontrivial critical points of Φ.

LEMMA 3.7. *Suppose that* (F_1)–(F_4) *are satisfied. Then system* (1.1) *has a nontrivial solution, i.e.* $\mathcal{K} \neq \emptyset$.

PROOF. Lemma 3.4 implies that the existence of a $(C)_c$-sequence $\{u_n\}$, where $c > 0$. By Lemma 3.6, $\{u_n\}$ is bounded in E. Let

$$(3.15)\qquad \delta := \overline{\lim_{n\to\infty}} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |u_n|^2 dx.$$

If $\delta = 0$, by Lions' concentration compactness principle in [34] or [42, Lemma 1.21], then $u_n \to 0$ in L^p for any $p \in (2,3)$. Therefore, it follows from (2.4) and (3.1) that

$$\int_{\mathbb{R}^3} F(x,u_n)dx \to 0, \quad \int_{\mathbb{R}^3} F_u(x,u_n)u_n dx \to 0 \quad \text{and} \quad \Gamma(u_n) \to 0$$

as $n \to \infty$. Consequently,

$$c = \lim_{n\to\infty}\left(\Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n\right)$$
$$= \lim_{n\to\infty}\left(\Gamma(u_n) + \int_{\mathbb{R}^3}\left(\frac{1}{2}F_u(x,u_n)u_n - F(x,u_n)\right)dx\right) = 0.$$

This is a contradiction. Hence $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^3$ such that

$$\int_{B(k_n,1+\sqrt{3})} |u_n|^2\,dx > \frac{\delta}{2}.$$

Let us define $v_n(x) = u_n(x+k_n)$ so that

$$\int_{B(0,1+\sqrt{3})} |v_n|^2\,dx > \frac{\delta}{2}. \tag{3.16}$$

Since Φ and Φ' are $\mathbb{Z}^3$-translation invariant, we obtain $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c \quad \text{and} \quad (1+\|v_n\|)\Phi'(v_n) \to 0. \tag{3.17}$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n \to v$ in L^s_{loc}, for all $2 \le s < 3$ and $v_n \to v$ almost everywhere in $\mathbb{R}^3$. Hence it follow from (3.16) and (3.17) that $\Phi'(v) = 0$ and $v \neq 0$. This shows that $v \in \mathcal{K}$ is a nontrivial of system (1.1). □

Proof of Theorem 1.1. Lemma 3.7 shows that $\mathcal{K}$ is not an empty set. Let $m := \inf\{\Phi(u) \mid u \in \mathcal{K}\setminus\{0\}\}$ be the least energy of Φ. First of all, we claim that

$$\theta := \inf\{\|u\| \mid u \in \mathcal{K}\} > 0. \tag{3.18}$$

Indeed, for any $u \in \mathcal{K}$, it holds

$$0 = \Phi'(u)(u^+ - u^-) = \|u\|^2 - \Gamma'(u)(u^+ - u^-) - \Psi'(u)(u^+ - u^-)$$

jointly with (2.4), (3.1) which implies that

$$\|u\|^2 \le \varepsilon\|u\|^2 + C_{10}\|u\|^4 + C_\varepsilon\|u\|^p.$$

Choosing ε small enough, we see easily that $\|u\| > 0$ for each $u \in \mathcal{K}$. Therefore, $\theta > 0$.

Suppose that $\{u_n\} \subset \mathcal{K}$ such that $\Phi(u_n) \to m$ as $n \to \infty$. Then $\{u_n\}$ is a $(C)_m$-sequence. By Lemma 3.6, $\{u_n\}$ is bounded. For this sequence $\{u_n\}$, we denote δ as in (3.15). If $\delta = 0$, then $u_n \to 0$ L^p for all $p \in (2,3)$. Now, for any $\varepsilon > 0$, using (2.4) and (3.1) we have

$$\int_{\mathbb{R}^3} F_u(x,u_n)(u_n^+ - u_n^-)\,dx \to 0 \quad \text{and} \quad \Gamma'(u_n)(u_n^+ - u_n^-) \to 0$$

as $n \to \infty$. Consequently,

$$\|u_n\|^2 = \Phi'(u_n)(u_n^+ - u_n^-) + \Gamma'(u_n)(u_n^+ - u_n^-) + \int_{\mathbb{R}^3} F_u(x,u_n)(u_n^+ - u_n^-)\,dx \to 0$$

as $n \to \infty$. This contradicts with (3.18). Therefore, $\delta > 0$. After a suitable $\mathbb{Z}^3$-translation, a subsequence of $\{u_n\}$ converges weakly to some $u_0 \in \mathcal{K}$. By Fatou's lemma and (F_4), we have

$$\begin{aligned}\Phi(u_0) &= \Phi(u_0) - \frac{1}{2}\Phi'(u_0)u_0 = \Gamma(u_0) + \int_{\mathbb{R}^3} \widetilde{F}(x, u_0)\, dx \\ &\leq \varliminf_{n\to\infty} \bigg(\Gamma(u_n) + \int_{\mathbb{R}^3} \widetilde{F}(x, u_n)\, dx\bigg) = \varliminf_{n\to\infty} \bigg(\Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n\bigg) = m.\end{aligned}$$

Hence $u_0 \in \mathcal{K}$ with $\Phi(u_0) = m$, and Theorem 1.1 is proved. □

References

[1] S. Abenda, *Solitary waves for Maxwell–Dirac and Coulomb–Dirac models*, Ann. Inst. H. Poincaré **68** (1998), 229–244.

[2] A. Ambrosetti, *On Schrödinger–Poisson systems*, Milan J. Math. **76** (2008), 257–274.

[3] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), 423–443.

[4] ______, *A Cauchy–Schwarz type inequality for bilinear integrals on positive measures*, Proc. Amer. Math. Soc. **133** (2005), 2647–2656.

[5] A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger–Maxwell equations*, J. Math. Anal. Appl. **345** (2008), 90–108.

[6] T. Bartsch and Y.H. Ding, *Deformation theorems on non-metrizable vector spaces and applications to critical point theory*, Math. Nachr. **279** (2006), 1267–1288.

[7] ______, *Solutions of nonlinear Dirac equations*, J. Differential Equations **226** (2006), 210–249.

[8] J.M. Chadam, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell–Dirac system in one space dimension*, J. Funct. Anal. **13** (1973), 173–184.

[9] J.M. Chadam and R.T. Glassey, *On the Maxwell–Dirac equations with zero magnetic field and their solutions in two space dimension*, J. Math. Anal. Appl. **53** (1976), 495–507.

[10] T. Cazenave and L. Vazquez, *Existence of local solutions of a classical nonlinear Dirac field*, Comm. Math. Phys. **105** (1986), 35–47.

[11] G. Cerami and G. Vaira, *Positive solutions for some non-autonomous Schrödinger–Poisson systems*, J. Differential Equations **248** (2010), 521–543.

[12] G.Y. Chen and Y.Q. Zheng, *Stationary solutions of non-autonomous Maxwell–Dirac systems*, J. Differential Equations **255** (2013), 840–864.

[13] Y.H. Ding, *Semi-classical ground states concentrating on the nonlinear potential for a Dirac equation*, J. Differential Equations **249** (2010), 1015–1034.

[14] ______, *Variational Methods for Strongly Indefinite Problems*, World Scientific Press, 2008.

[15] Y.H. Ding and X.Y. Liu, *Semi-classical limits of ground states of a nonlinear Dirac equation*, J. Differential Equations **252** (2012), 4962–4987.

[16] ______, *On semiclassical ground states of a nonlinear Dirac equation*, Rev. Math. Phys. **10** (2012), 1250029.

[17] Y.H. Ding and B. Ruf, *Solutions of a nonlinear Dirac equation with external fields*, Arch. Rational Mech. Anal. **190** (2008), 57–82.

[18] ______, *Existence and concentration of semi-classical solutions for Dirac equations with critical nonlinearites*, SIAM J. Math. Anal. **44** (2012), 3755–3785.

[19] Y.H. Ding and J.C. Wei, *Stationary states of nonlinear Dirac equations with general potentials*, Rev. Math. Phys. **20** (2008), 1007–1032.

[20] Y.H. Ding, J.C. Wei and T. Xu, *Existence and concentration of semi-classical solutions for a nonlinear Maxwell–Dirac system*, J. Math. Phys. **54** (2013), 061505.

[21] Y.H. Ding and T. Xu, *On the concentration of semi-classical states for an onlinear Dirac–Klein–Gordon system*, J. Differential Equations **256** (2014), 1264–1294.

[22] ______, *On semi-classical limits of ground states of a nonlinear Maxwell–Dirac system*, Calc. Var. Partial Differentail Equations **51** (2014), 17–44.

[23] M.J. Esteban, V. Georgiev and E. Séré, *Stationary solutions of the Maxwell–Dirac and the Klein–Gordon–Dirac equations*, Calc. Var. Partial Differential Equations **4** (1996), 265–281.

[24] M.J. Esteban, M. Lewin and E. Séré, *Variational methods in relativistic quantum mechanics*, Bull. Amer. Math. Soc. **45** (2008), 535–593.

[25] M.J. Esteban and E. Séré, *Stationary states of nonlinear Dirac equations: A variational approach*, Comm. Math. Phys. **171** (1995), 323–350.

[26] ______, *An overview on linear and nonlinear Dirac equations*, Discrete Cotin. Dyn. Syst. **8** (2002), 281–397.

[27] M. Flato, J. Simon and E. Taffin, *On the global solutions of the Maxwell–Dirac equations*, Commun. Math. Phys. **113** (1987), 21–49.

[28] A. Garrett Lisi, *A solitary wave solution of the Maxwell–Dirac equations*, J. Phys. A Math. Gen. **28** (1995), 5385–5392.

[29] V. Georgiev, *Small amplitude solutions of Maxwell–Dirac equations*, Indiana Univ. Math. J. **40** (1991), 845–883.

[30] R.T. Glassey and J.M. Chadam, *Properties of the solutions of the Cauchy problem for the classical coupled Maxwell–Dirac equations in one space dimension*, Proc. Amer. Math. Soc. **43** (1974), 373–378.

[31] W.T. Grandy Jr, *Relativistic Quantum Mechanics of Leptonsand Fields*, Fundam. Theor. Phys., vol. 41, Kluwer Academic Publishers Group, Dordrecht, 1991.

[32] L. Gross, *The Cauchy problem for the coupled Maxwell–Dirac equations*, Commun. Pure Appl. Math. **19** (1966), 1–5.

[33] W. Kryszewki and A. Szulkin, *Generalized linking theorem with an application to semilinear Schrödinger equation*, Adv. Differential Equations **3** (1998), 441–472.

[34] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case*, part 2, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 223–283.

[35] F. Merle, it Existence of stationary states for nonlinear Dirac equations, J. Differential Equations **74** (1988), 50–68.

[36] C.J. Radford, *The stationary Maxwell–Dirace quations*, J. Phys. A Math. Gen. **36** (2003), 5663–5681.

[37] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), 665–674.

[38] C. Sparber and P. Markowich, *Semiclassical asymptotics for the Maxwell–Dirac system*, J. Math. Phys. **44** (2003), 4555–4572.

[39] B. Thaller, *The Dirac Equation*, Texts and Monographs in Physics, Springer, Berlin, 1992.

[40] X.H. Tang, *New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation*, Adv. Nonlinear Studies **14** (2014), 349–361.

[41] ______, *New condition on nonlinearity for a periodic Schrödinger equation having zero as spectrum*, J. Math. Anal. Appl. **413** (2014), 392–410.

[42] M. Willem, *Minimax Theorems*, Birkhäuser, Berlin, 1996.

[43] M.B. Yang and Y.H. Ding, *Stationary states for nonlinear Dirac equations with superlinear nonlinearities*, Topol. Methods Nonlinear Anal. **39** (2012), 175–188.

[44] F.K. Zhao and Y.H. Ding, *Infinitely many solutions for a class of nonlinear Dirac equations without symmetry*, Nonlinear Anal. **70** (2009), 921–935.

[45] J. Zhang, W.P. Qin and F.K. Zhao, *Multiple solutions for a class of nonperiodic Dirac equations with vector potentials*, Nonlinear Anal. **75** (2012), 5589–5600.

[46] J. Zhang, X.H. Tang and W. Zhang, *Ground states for nonlinear Maxwell–Dirac system with magnetic field*, J. Math. Anal. Appl. **421** (2015), 1573–1585.

[47] ———, *Ground state solutions for nonperiodic Dirac equation with superquadratic nonlinearity*, J. Math. Phys. **54** (2013), 101502.

[48] ———, *On ground state solutions for superlinear Dirac equation*, Acta Math. Sci. **34** (2014), 840–850.

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