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PERIODIC BIFURCATION PROBLEMS FOR FULLY NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS VIA AN INTEGRAL OPERATOR APPROACH: THE MULTIDIMENSIONAL DEGENERATION CASE

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Abstract. We consider a T-periodically perturbed autonomous functional differential equation of neutral type. We assume the existence of a T-periodic limit cycle x_0 for the unperturbed autonomous system. We also assume that the linearized unperturbed equation around the limit cycle has the characteristic multiplier 1 of geometric multiplicity 1 and algebraic multiplicity greater than 1. The paper deals with the existence of a branch of T-periodic solutions emanating from the limit cycle. The problem of finding such a branch is converted into the problem of finding a branch of zeros of a suitably defined bifurcation equation $P(x,\varepsilon)+\varepsilon Q(x,\varepsilon)=0$. The main task of the paper is to define a novel equivalent integral operator having the property that the T-periodic adjoint Floquet solutions of the unperturbed linearized operator correspond to those of the equation $P'(x_0(\theta),0)=0$, $\theta\in[0,T]$. Once this is done it is possible to express the condition for the existence of a branch of zeros for the bifurcation equation in terms of a multidimensional Malkin bifurcation function.

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1. Introduction

In [26] Malkin developed a perturbation theory to study the existence, uniqueness and stability of bifurcating periodic solutions from a limit cycle x_0 of an autonomous system when it is perturbed by a T-periodic nonlinear term of small amplitude. The system is of the form

$$\dot{x} = \phi(x) + \varepsilon\psi(t, x, \varepsilon), \tag{1.1}$$

where $\phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $\psi \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$, ψ is T-periodic with respect to time and $\varepsilon \geq 0$ is a small parameter. The main tool for this study was the so-called Malkin bifurcation function

$$f_0(\theta) = \int_0^T \langle z_0(\tau), \psi(\tau - \theta, x_0(\tau), 0)\rangle \, d\tau.$$

Here $\langle \cdot\, , \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$ and z_0 is a T-periodic solution of

$$\dot{z} = (\phi'(x_0(t)))^* z,$$

the adjoint system of the linearized unperturbed system

$$\dot{y} = \phi'(x_0(t))y,$$

which is assumed to have the characteristic multiplier 1 simple.

In [26] it is shown that if the bifurcation function f_0 has a simple zero $\theta_0 \in [0, T]$, then there exists a branch of solutions x_ε, $\varepsilon \geq 0$ small, emanating from $x_0(\theta_0)$. In [27] Malkin extended the perturbation theory to periodically perturbed non autonomous T-periodic systems. The results obtained by Malkin in these papers have been also proved independently by Loud [22]. Since these pioneering papers [26], [27], [22] a relevant bibliography has been devoted to the refinement and development in various directions of the results contained in these papers, sometimes leading to different expressions and employ of the bifurcation function. From the huge bibliography on the subject we mention, in the sequel, some of the papers which are more closely related to the interest and methods employed in this paper.

Under the regularity assumptions on the functions ϕ and ψ of (1.1) indicated above, in [16] the problem of finding a branch of T-periodic solutions originating from x_0 is reduced to the problem of finding a branch of zeros of a bifurcation equation of the form

$$P(x) + \varepsilon Q(x, \varepsilon) = 0, \tag{1.2}$$

where $P\colon \mathbb{R}^n \to \mathbb{R}^n$ and $Q\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ are given by $P(x) = \Pi_0(x) - x$ and $Q(x, \varepsilon) = (\Pi_\varepsilon(x) - \Pi_0(x))/\varepsilon$ with $P'(x_0)$ singular, here Π_ε is the singular Poincaré map associated to (1.1). By means of the classical implicit function theorem, it is shown that, under the usual assumption of the existence of a simple zero of the Malkin bifurcation function associated to (1.1), the bifurcation equation (1.2) has a branch of zeros. The same approach has permitted in [20] to show the existence of branches of T-periodic solutions along prescribed directions at some point of the limit cycle.

A significant example of how to associate a suitable abstract bifurcation function to solve an infinite dimensional bifurcation problem is given in [21]. Precisely, for a class of periodically perturbed autonomous equations, the authors introduce a novel version of the usual equivalent integral operator, i.e. the operator whose fixed points are the T-periodic solutions of the considered problem and viceversa.

The resulting bifurcation equation still has the form (1.2). To this equation it is possible to associate a suitable Malkin bifurcation function with the property that if it has a simple zero then (1.2) has a branch of zeros. This is ensured by [17, Theorem 2], which is the infinite dimensional version of [16, Theorem 1].

The aforementioned novel equivalent integral operator is constructed in such a way that the condition that the characteristic multiplier 1 of the unperturbed linearized equation is simple ensures that 0 is a simple eigenvalue of $P'(x_0(\theta))$, $\theta \in [0,T]$. In the infinite dimensional case, this property is not verified if we build the bifurcation equation (1.2) from the usual equivalent integral operator, see [21] for the details.

If the functions ϕ and ψ in (1.1) satisfy less regularity conditions not allowing the employ of the classical implicit function theorem or one of its variants, then, the Malkin bifurcation function can be usefully employed to prove that the topological degree of suitably defined operators, whose fixed points are periodic solutions of the considered equation and viceversa, is different from zero, see [5], [8], [11], [14], [15], [23]. The behavior of the bifurcating periodic solutions, when the perturbation vanishes, has been studied in [24] and [25].

In all the papers cited above, when one deals with the existence of bifurcation of T-periodic solutions, the crucial assumption is that the characteristic multiplier 1 of the linearized unperturbed equation is simple. In other words, the eigenvalue 1 of the translation operator along the trajectories of the linearized unperturbed equation over the period is simple.

Aim of this paper is to provide an application of the method, based on the definition of a novel equivalent integral operator, to the relevant case when the linearized unperturbed equation does not possess T-periodic solutions linearly independent with $e_0 := \dot{x}_0$, but it possesses T-periodic adjoint Floquet solutions. Namely, the case when the eigenvalue 1 of the translation operator has geometric multiplicity 1 and algebraic multiplicity $m > 1$.

To illustrate by a concrete example the method mentioned above we consider a class of functional differential equations of neutral type of the form

$$\dot{x}(t) = \phi(x(t-\varepsilon), \dot{x}(t-\varepsilon)) + \varepsilon\psi(t, x(t-\varepsilon), \dot{x}(t-\varepsilon), \varepsilon), \tag{1.3}$$

where $\phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, ψ is T-periodic in time and $\varepsilon \in [0,1]$ is the perturbation parameter.

We assume that (1.3) when $\varepsilon = 0$ has a T-periodic limit cycle x_0. The assumptions on ϕ and ψ will be precised later. The interest for the study of the conditions ensuring the existence of bifurcation of periodic solutions for differential equations with delay goes back to the 60–80's of the twentieth century. Very recently, new contributions to the existence, bifurcation and stability of periodic solutions for functional differential equations have been presented in several papers, see e.g. [9], [10], [13], [29], [31]–[33] as well as the extensive references therein.

The study of the existence, bifurcation and stability of periodic solutions for delayed differential equations is of particular relevance for the control of vibrations, resonance and other harmful phenomena in various mechanical systems, as well as for the dynamical analysis of cellular neural networks. Indeed, in the control of mechanical systems serious problems arise from the unavoidable time delays in both controller and actuators. Moreover, the presence of switching delay in neuron amplifiers is a particularly harmful source of potential instability for the dynamics of cellular neural network, among an huge bibliography on these topics we cite here [12] and [6].

In all the previous cited references the delay, when it is finite, is a given fixed positive number or a periodic function of the time. Furthermore, the employed methods are of topological nature, mainly fixed point theorems and topological degree together with different bifurcation theorems. In the present paper the delay $\varepsilon > 0$ is considered as an effect of the T-periodic perturbation of the autonomous system, in other words it disappears as the perturbation vanishes and our method differs from those of the cited papers, since it is based on the Malkin bifurcation function. It is opinion of the authors that the abstract result, namely Theorem 2, can be successfully applied to (1.3) also in the case when the delay is a given fixed positive number. At present the involved calculation of the related Malkin bifurcation function has not been performed. This may be matter of future work.

In [7], for the equation (1.3), the existence of the bifurcation of T-periodic solutions $x_\varepsilon, \varepsilon \geq 0$ small, emanating from x_0 was proved under the assumptions that the eigenvalue 1 of the translation operator is simple, namely $m = 1$. This paper follows the approach described in [16] by associating to (1.3) a bifurcation equation along the lines indicated in [1]. Moreover, under the assumption that $m > 1$ and the geometric multiplicity is equal to 1, in [19] for ϕ and ψ in (1.3) of the form

$$\begin{aligned}
\phi(x(t-\varepsilon), \dot{x}(t-\varepsilon)) &= f(x(t), x(t-\varepsilon h)) + a\dot{x}(t-\varepsilon h), \\
\psi(t, x(t-\varepsilon), \dot{x}(t-\varepsilon), \varepsilon) &= g(t, x(t), x(t-\varepsilon h), \varepsilon) + b(t)\dot{x}(t-\varepsilon h),
\end{aligned}$$

a first result on the existence of bifurcation of T-periodic solutions from the limit cycle x_0 has been proved.

The bifurcation results in [7] and [19] make use of the two extensions to infinite dimensional spaces: [7, Theorem 2.1] and [19, Theorem 3.2] of the result formulated for the respective bifurcation equation in [16, Theorem 1] in finite dimension. In particular, since the bifurcation equation in [19] is defined, as in [16] by means of the translation operator, the use of [19, Theorem 3.2] requires that the translation operator along the trajectories of (1.3) is defined in the space $W_2^1[-h,0]$ of the absolutely continuous functions with derivative in $L_2[-h,0]$. Indeed, this space allows to calculate the Floquet vectors of the adjoint equation as required by Theorem 3.2 of [19], while this computation is quite problematic in $C^1[-h,0]$. Moreover, the restrictive assumption on the form of the functions ϕ and ψ is due to the necessity to prove the differentiability of the translation operator in $W_2^1[-h,0]$ with respect to both the small parameter $\varepsilon \geq 0$ and the space variable.

Coming back to the present paper, here we closely follow the method outlined in [21] to introduce a bifurcation equation $P(x,\varepsilon) + \varepsilon Q(x,\varepsilon) = 0$ satisfying the assumptions of [18, Theorem 1] and [19, Theorem 3.2] which are formulated in terms of a multidimensional Malkin bifurcation function. In particular, in order to define this bifurcation function it is necessary that there exists a one-to-one correspondence between the T-periodic adjoint Floquet solutions of the linearized unperturbed equation and those of the equation $P'(x_0(\theta),0) = 0$, $\theta \in [0,T]$. Therefore, to this aim we define an appropriate novel equivalent integral operator with such a property, this is the main point of the paper. Indeed, as showed in [28], the usual equivalent integral operator does not necessarily satisfy this property. As already observed the same problem was faced in [21]. On the other hand, for the translation operator along the trajectories of the linearized unperturbed equation the adjoint vectors of the eigenvalue 1 correspond to the T-periodic adjoint Floquet solutions of this equation. This fact turns out to be quite useful in [19] where the bifurcation equation is built upon the translation operator.

We would like also to mention the papers [2]–[4] where the bifurcation of periodic solutions for differential equations with delay is obtained under the assumption that the geometric and algebraic multiplicity of the eigenvalue 1 of the translation operator are equal and greater than 1. Therefore, the existence of periodic adjoint Floquet solutions is not allowed. Moreover, under the same assumption, Malkin in [27] proved the uniqueness of T-periodic solutions x_ε, $\varepsilon \geq 0$ small, and their asymptotic stability for smooth T-periodic non autonomous perturbed differential system. The branch originates from the normally non degenerate manifold S of the initial states of the periodic solutions of the unperturbed system. In the case when the perturbation is only Lipschitz, bifurcation of isolated branches of periodic solutions from S is shown in [5]. Finally, for ordinary differential equations Rhouma and Chicone in [30] treated the case when S is not normally non degenerate, thus allowing the existence of adjoint Floquet solutions.

The paper is organized as follows. In Section 2 we give a precise formulation of the considered bifurcation problem. In Section 3 we provide an useful representation formula for the T-periodic adjoint Floquet solutions of the linearized unperturbed equation of (1.3). In Section 4 we introduce the equivalent integral operator which allows to convert the problem of the existence of T-periodic solutions to (1.3) in the problem of finding the fixed points of the integral operator. In Section 5, starting from the equivalent integral operator, we define a novel equivalent integral operator which has the required properties concerning the adjoint vectors of the corresponding linearized unperturbed equation. In Section 6 we determine the eigenvector and the adjoint vectors of the adjoint operators of the linearized unperturbed equation corresponding to both the equivalent integral operator and the novel equivalent integral operator. Finally, in Section 7 all the results of the previous Sections permit to verify the conditions under which Theorem 3.2 of [19] guarantees the existence of the sough-after bifurcation in terms of a multidimensional Malkin bifurcation function.

2. Formulation of the problem

In this paper we consider a nonlinear autonomous functional differential equation perturbed by a nonautonomous T-periodic nonlinear perturbation of small amplitude. We assume that both the right hand side of the autonomous equation and the perturbation depend on the derivative of the state. Moreover, we suppose that the perturbation introduces a small delay in time both in the state and its derivative that disappears as the perturbation vanishes. The resulting equation turns out to be a functional differential equation of neutral type of the form

$$\dot{x}(t) = \phi(x(t-\varepsilon), \dot{x}(t-\varepsilon)) + \varepsilon\psi(t, x(t-\varepsilon), \dot{x}(t-\varepsilon), \varepsilon), \tag{2.1}$$

where $\phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, are continuous functions, ψ is T-periodic in time and $\varepsilon \in [0,1]$ is the perturbation parameter. We also assume that

$$\|\phi(x, y_1) - \phi(x, y_2)\| \leq K\|y_1 - y_2\|,$$

for some $0 < K < 1$, whenever $x \in E$ and

$$\|\psi(t, x, y_1, \varepsilon) - \psi(t, x, y_2, \varepsilon)\| \leq L\|y_1 - y_2\|,$$

for some $L > 0$ uniformly with respect to the other variables. Let $\varepsilon_0 > 0$ such that $K + \varepsilon_0 L = q < 1$. Since in this paper we are interested in small values of $\varepsilon \geq 0$, without loss of generality, we may assume from now on that $\varepsilon \in [0, \varepsilon_0]$. We assume that the unperturbed system, namely (2.1) with $\varepsilon = 0$, has a unique T-periodic solution x_0, that is

$$\dot{x}_0(t) = \phi(x_0(t), \dot{x}_0(t)), \quad x_0(t) = x_0(t+T), \quad t \in \mathbb{R}. \tag{2.2}$$

Moreover, we assume that $\phi \in C^2(U)$, where U is a neighbourhood of the set $\{(t, x_0(t), \dot{x}_0(t), \varepsilon) : t \in [0,T], \varepsilon \in [0, \varepsilon_0]\}$. Since the equation (2.2) is autonomous the function $x_0^\theta(t) := x_0(t+\theta)$ is also a solution to (2.2), for any $\theta \in [0,T]$, i.e. the set of shifts $x_0^\theta(\,\cdot\,)$, $\theta \in [0,T]$, represents the family Γ of T-periodic solutions to (2.2). Therefore the linearized unperturbed equation

$$\dot{x}(t) = \phi'_{(1)}(x_0^\theta(t), \dot{x}_0^\theta(t))x(t) + \phi'_{(2)}(x_0^\theta(t), \dot{x}_0^\theta(t))\dot{x}(t) \tag{2.3}$$

has $\Gamma' := \{\dot{x}_0^\theta(\,\cdot\,) := \dot{x}_0(\,\cdot\, + \theta) : \theta \in [0,T]\}$ as the family of nontrivial T-periodic solutions.

Let $a_\theta(t) := \phi'_{(1)}(x_0^\theta(t), \dot{x}_0^\theta(t))$ and $b_\theta(t) := \phi'_{(2)}(x_0^\theta(t), \dot{x}_0^\theta(t))$, whenever $\theta \in [0,T]$. Hence (2.3) takes the form

$$\dot{x}(t) = a_\theta(t)x(t) + b_\theta(t)\dot{x}(t). \tag{2.4}$$

Now, we assume that the equation (2.4) has the set of T-periodic adjoint Floquet solutions given by

$$v_j^\theta(t) = \sum_{i=0}^{j} \frac{t^{j-i}}{(j-i)!\,T^{j-i}}\, e_i^\theta(t), \quad j = 1, \ldots, m, \ m \leq n-1,$$

where $e_0^\theta(t) := \dot{x}_0^\theta(t) \not\equiv 0$ for any $\theta \in [0,T]$ and e_j^θ are T-periodic functions.

We pose the following:

Problem 2.1. To find conditions to guarantee the existence of a branch $x_\varepsilon, \varepsilon \geq 0$ small, of T-periodic solutions to (2.1) emanating from the T-periodic function $x_0^{\theta_0} \in \Gamma$ for some $\theta_0 \in [0,T]$.

3. A representation formula for the T-periodic adjoint Floquet solutions

We need the following result.

Lemma 3.1. *Let*

$$v_j(t) = \sum_{i=0}^{j} \frac{t^{j-i}}{(j-i)!\,T^{j-i}}\, e_i(t), \quad j = 1, \ldots, m, \ m \leq n-1,$$

where e_j are T-periodic functions and e_0 is a T-periodic solution of the linear equation $\dot x = A(t)x$. If v_j is a solution of $\dot x = A(t)x$, then e_j solves the equation

$$\dot x(t) = A(t)x(t) - \frac{1}{T}e_{j-1}(t), \quad j = 1, \dots, m. \tag{3.1}$$

Proof. We proceed by induction, consider $j = 1$ thus $v_1(t) = te_0(t)/T + e_1(t)$. Since v_1 is a solution of $\dot x = A(t)x$ we obtain

$$\frac{1}{T}\, e_0(t) + \frac{t}{T}\,\dot e_0(t) + \dot e_1(t) = \frac{t}{T}\, A(t)e_0(t) + A(t)e_1(t),$$

hence

$$\dot e_1(t) = A(t)e_1(t) - \frac{1}{T}\, e_0(t).$$

Assume now that the statement holds for $j = 1, \dots, k$, $k < m$. We prove that it is true for $j = k+1$. Since

$$v_{k+1}(t) = \sum_{i=0}^{k+1} \frac{t^i}{i!\,T^i}\, e_{k+1-i}(t)$$

is a solution of $\dot x = A(t)x$ we have

$$\begin{aligned}\dot v_{k+1}(t) &= \sum_{i=1}^{k+1} \frac{t^{i-1}}{(i-1)!\,T^i}\, e_{k+1-i}(t) + \sum_{i=1}^{k+1} \frac{t^i}{i!\,T^i}\,\dot e_{k+1-i}(t) + \dot e_{k+1}(t)\\ &= \sum_{i=1}^{k} \frac{t^i}{i!\,T^{i+1}}\, e_{k-i}(t) + \frac{1}{T}\, e_k(t) + \sum_{i=1}^{k+1} \frac{t^i}{i!\,T^i}\,\dot e_{k+1-i}(t) + \dot e_{k+1}(t)\\ &= A(t)e_{k+1}(t) + \sum_{i=1}^{k+1} \frac{t^i}{i!\,T^i}\, A(t)e_{k+1-i}(t).\end{aligned}$$

Since by induction

$$\frac{t^i}{i!\,T^i}\frac{1}{T}\, e_{k-i}(t) + \frac{t^i}{i!\,T^i}\,\dot e_{k+1-i}(t) = \frac{t^i}{i!\,T^i}\, A(t)e_{k+1-i}(t)$$

for $i = 1, \dots, k$ and

$$\frac{t^{k+1}}{(k+1)!\,T^{k+1}}\,\dot e_0(t) = \frac{t^{k+1}}{(k+1)!\,T^{k+1}} A(t)e_0(t),$$

we obtain

$$\dot e_{k+1}(t) = A(t)e_{k+1}(t) - \frac{1}{T}\, e_k(t),$$

that is (3.1) for $j = k+1$. □

Remark 3.2. Consider the equation (2.4). The conditions on ψ imply that $\|b_\theta(t)\| < 1$ for any $t, \theta \in [0, T]$, thus the matrix $(I - b_\theta(t))$ is invertible for any $t, \theta \in [0, T]$. Let $A_\theta(t) := (I - b_\theta(t))^{-1} a_\theta(t)$, then (2.4) can be rewritten in the form $\dot x = A_\theta(t)x$. By Lemma 3.1 we have

$$\dot e_j^\theta(t) = A_\theta(t)e_j^\theta(t) - \frac{1}{T}\, e_{j-1}^\theta(t),$$

that is

$$\dot{e}_j^\theta(t) = a_\theta(t)e_j^\theta(t) + b_\theta(t)\dot{e}_j^\theta(t) + b_\theta(t)\frac{1}{T}e_{j-1}^\theta(t) - \frac{1}{T}\,e_{j-1}^\theta(t), \quad j = 1, \dots, m.$$

As a consequence of Lemma 3.1 we have the following result.

COROLLARY 3.3. *If m is the highest order of the T-periodic adjoint Floquet solutions then the equation $\dot{x} = A(t)x - e_m/T$ does not have T-periodic solutions.*

PROOF. Arguing by contradiction assume that e_{m+1} is a T-periodic solution to $\dot{x} = A(t)x - e_m/T$. Consider the function

$$v_{m+1}(t) := \sum_{i=0}^{m+1} \frac{t^i}{i!T^i}\, e_{m+1-i}(t).$$

If we show that it is a T-periodic solution to $\dot{x} = A(t)x$, then we obtain a contradiction. Indeed,

$$\dot{v}_{m+1}(t) = \sum_{i=1}^{m+1} \frac{t^{i-1}}{(i-1)!T^i}\, e_{m+1-i}(t) + \sum_{i=1}^{m+1} \frac{t^i}{i!T^i}\, \dot{e}_{m+1-i}(t) + \dot{e}_{m+1}(t).$$

Since $0 \le m+1-i \le m$ for $i = 1, \dots, m+1$, by assumption v_{m+1-i} is a T-periodic solution of $\dot{x} = A(t)x$, thus Lemma 3.1 implies that

$$\dot{e}_{m+1-i}(t) = A(t)e_{m+1-i}(t) - \frac{1}{T}\, e_{m-i}(t).$$

Therefore

$$\begin{aligned}\dot{v}_{m+1}(t) &= A(t)e_{m+1}(t) - \frac{1}{T}\, e_m(t) + \sum_{i=0}^{m} \frac{t^i}{i!\,T^{i+1}}\, e_{k-i}(t) \\ &\quad + \sum_{i=1}^{m} \frac{t^i}{i!\,T^i}[A(t)e_{m+1-i}(t) - \frac{1}{T}\, e_{m-1}(t)] + \frac{t^{m+1}}{(m+1)!T^{m+1}}\, A(t)e_0(t) \\ &= A(t)e_{m+1}(t) + \sum_{i=1}^{m+1} \frac{t^i}{i!\,T^i}\, A(t)e_{m+1-i}(t) = A(t)v_{m+1}(t).\end{aligned}$$ □

In the sequel we will denote e_j^θ simply by e_j. We can prove the following.

LEMMA 3.4. *For any $t \in [0,T]$, $\{e_j(t)\}_{j=0}^m$ and $\{(I - b(t))e_j(t)\}_{j=0}^m$ are two sets of linearly independent vectors of $\mathbb{R}^n$.*

PROOF. Let $A(t) = (I - b(t))^{-1}a(t)$, we have that

$$\dot{e}_j(t) = A(t)e_j(t) - \frac{1}{T}\, e_{j-1}(t) \quad \text{for any} \quad j = 1, \dots, m, \quad \text{and} \quad \dot{e}_0(t) = A(t)e_0(t).$$

Consider the linear combination

$$\sum_{j=0}^{m} \alpha_j e_j(t) = 0, \quad t \in [0,T].$$

Since e_j is differentiable at any point t we have

$$\sum_{j=0}^{m} \alpha_j \dot{e}_j(t) = 0, \quad t \in [0,T].$$

Hence

$$\alpha_0 A(t)e_0(t) + \sum_{j=1}^{m} \alpha_j A(t)e_j(t) - \frac{1}{T}\sum_{j=1}^{m} \alpha_j e_{j-1}(t) = 0, \quad t \in [0,T].$$

Thus

$$\sum_{j=1}^{m} \alpha_j e_{j-1}(t) = 0, \quad t \in [0,T],$$

or equivalently

$$\sum_{j=0}^{m-1} \alpha_{j+1} e_j(t) = 0, \quad t \in [0,T]. \tag{3.2}$$

Deriving (3.2) we have

$$\sum_{j=0}^{m-1} \alpha_{j+1} \dot{e}_j(t) = 0, \quad t \in [0,T].$$

and the same argument as before shows that

$$\sum_{j=0}^{m-2} \alpha_{j+2} e_j(t) = 0, \quad t \in [0,T].$$

Repeating this procedure m times we get $\alpha_m e_0(t) = 0$, $t \in [0,T]$.

On the other hand $e_0 \not\equiv 0$ and e_0 is a solution of $\dot{e}_0(t) = A(t)e_0(t)$, hence $e_0(t) \neq 0$, for any $t \in [0,T]$. This implies that $\alpha_m = 0$ whenever $t \in [0,T]$. Analogously, we can show that $\alpha_j = 0$, for $j = 1,\ldots,m-1$. That is $\{e_j(t)\}_{j=0}^m$ is a set of linearly independent vectors in $\mathbb{R}^n$ whenever $t \in [0,T]$. Moreover, since the matrix $(I - b(t))$ is invertible for any $t \in [0,T]$ we have the linear independence in $\mathbb{R}^n$ also of the vectors $\{(I-b(t))e_j(t)\}_{j=0}^m$, for any $t \in [0,T]$. □

We need the following general result.

Lemma 3.5. *Assume that $\{y_j\}_{j=0}^m$ are continuous functions from $[a,b]$ to $\mathbb{R}^n$, where $m \leq n-1$. Assume the existence of $\widehat{t} \in [a,b)$ such that the vectors $\{y_j(\widehat{t})\}_{j=0}^m$ are linearly independent in $\mathbb{R}^n$. Then there exists $\widehat{\tau} > 0$ such that $\widehat{t} + \widehat{\tau} < b$ and for any $\tau \in [0,\widehat{\tau}]$ the vectors $\{\int_{\widehat{t}}^{\widehat{t}+\tau} y_j(s)\,ds\}_{j=0}^m$ are linearly independent in $\mathbb{R}^n$.*

Proof. By assumption the rank of the matrix $\{y_{k,j}(\widehat{t})\}$, $k = 0, \dots, n-1$, $j = 0, \dots, m$, is m. Thus, we can assume without loss of generality, that

$$\det \begin{bmatrix} y_{0,0}(\widehat{t}) & \dots & y_{0,m}(\widehat{t}) \\ \vdots & \ddots & \vdots \\ y_{m,0}(\widehat{t}) & \dots & y_{m,m}(\widehat{t}) \end{bmatrix} \neq 0.$$

Arguing by contradiction, we assume the existence of a sequence $\tau_p \to 0+$ as $p \to +\infty$ such that

$$\det \begin{bmatrix} \displaystyle\int_{\widehat{t}}^{\widehat{t}+\tau_p} y_{0,0}(s)\,ds & \dots & \displaystyle\int_{\widehat{t}}^{\widehat{t}+\tau_p} y_{0,m}(s)\,ds \\ \vdots & \ddots & \vdots \\ \displaystyle\int_{\widehat{t}}^{\widehat{t}+\tau_p} y_{m,0}(s)\,ds & \dots & \displaystyle\int_{\widehat{t}}^{\widehat{t}+\tau_p} y_{m,m}(s)\,ds \end{bmatrix} = 0.$$

By the mean value theorem for integrals we obtain

$$\tau_p^{m+1} \det \begin{bmatrix} y_{0,0}(\widehat{t} + h_{0,0}\tau_p) & \dots & y_{0,m}(\widehat{t} + h_{0,m}\tau_p) \\ \vdots & \ddots & \vdots \\ y_{m,0}(\widehat{t} + h_{m,0}\tau_p) & \dots & y_{m,m}(\widehat{t} + h_{m,m}\tau_p) \end{bmatrix} = 0.$$

for some $h_{k,j} \geq 0$, $k = 0, \dots, m$, $j = 0, \dots, m$. Passing to the limit as $\tau_p \to 0+$ we obtain a contradiction. $\square$

Remark 3.6. By Lemma 3.4 and Lemma 3.5 we obtain the existence of $t_0 \in (0, T)$ such that the vectors $\{\int_0^{t_0} (I - b(s)) e_j(s)\,ds\}_{j=0}^m$ are linearly independent in $\mathbb{R}^n$.

4. The equivalent integral operator

Let $F_\varepsilon \colon C^1(T) \to C^1(T)$, $\varepsilon \in [0, \varepsilon_0]$, the integral operator defined, for any $y \in C^1(T)$, as follows

$$\begin{aligned} (4.1)\quad (F_\varepsilon y)(t) := y(0) &+ \int_0^t [\phi(y(s-\varepsilon), \dot{y}(s-\varepsilon)) + \varepsilon\psi(s, y(s-\varepsilon), \dot{y}(s-\varepsilon), \varepsilon)]\,ds \\ &- \frac{t_0}{T}\left(\frac{t}{t_0} - 1\right) \int_0^T [\phi(y(s-\varepsilon), \dot{y}(s-\varepsilon)) + \varepsilon\psi(s, y(s-\varepsilon), \dot{y}(s-\varepsilon), \varepsilon)]\,ds, \end{aligned}$$

for $t \in [0, T]$, where $0 < t_0 < T$ is given in Remark 3.6. Here $C^1(T)$ denotes the Banach space of continuously differentiable T-periodic functions $y \colon \mathbb{R} \to \mathbb{R}^n$. Assume that $x(t)$ is a T-periodic solution to (2.1), consider the extension of x from $[0, T]$ to $\mathbb{R}$ by T-periodicity and denote this extension by y, clearly $y \in C^1(T)$, then from (4.1) we have

$$(F_\varepsilon y)(t) = y(0) + \int_0^t \dot{y}(s)\,ds - \frac{t_0}{T}\left(\frac{t}{t_0} - 1\right) \int_0^T \dot{y}(s)\,ds = y(t), \quad t \in [0, T],$$

that is y is a fixed point of F_ε. Vice versa, suppose that y is a fixed point of F_ε, hence for any $t \in [0,T]$ we have

$$
\begin{aligned}
(4.2)\quad y(t) = y(0) &+ \int_0^t [\phi(y(s-\varepsilon), \dot y(s-\varepsilon)) + \varepsilon\psi(s, y(s-\varepsilon), \dot y(s-\varepsilon), \varepsilon)]\, ds \\
&- \frac{t_0}{T}\bigg(\frac{t}{t_0} - 1\bigg) \int_0^T [\phi(y(s-\varepsilon), \dot y(s-\varepsilon)) + \varepsilon\psi(s, y(s-\varepsilon), \dot y(s-\varepsilon), \varepsilon)]\, ds,
\end{aligned}
$$

thus,

$$
\begin{aligned}
(4.3)\quad \dot y(t) = \phi(y(t-\varepsilon), \dot y(t-\varepsilon)) &+ \varepsilon\psi(t, y(t-\varepsilon), \dot y(t-\varepsilon), \varepsilon) \\
&- \frac{1}{T} \int_0^T [\phi(y(s-\varepsilon), \dot y(s-\varepsilon)) + \varepsilon\psi(s, y(s-\varepsilon), \dot y(s-\varepsilon), \varepsilon)]\, ds,
\end{aligned}
$$

Put $t = 0$ in (4.2). We obtain

$$
(4.4)\qquad \frac{t_0}{T} \int_0^T [\phi(y(s-\varepsilon), \dot y(s-\varepsilon)) + \varepsilon\psi(s, y(s-\varepsilon), \dot y(s-\varepsilon), \varepsilon)]\, ds = 0.
$$

Replacing (4.4) into (4.3) we obtain that $y \in C^1(T)$ is a T-periodic solution to (2.1). Therefore, to solve $y = F_\varepsilon y$ is equivalent to determine T-periodic solutions of (2.1), for this F_ε is called the equivalent integral operator.

Remark 4.1. By [1, Lemma 4.4.4] F_ε is condensing with constant q, $0<q<1$, with respect to the Hausdorff measure of non compactness. By [1, Theorem 1.5.9] the Fréchet derivative of F_ε is also q-condensing.

Consider now the operator $F_0\colon C^1(T) \to C^1(T)$ defined by

$$
(F_0 y)(t) = y(0) + \int_0^t \phi(y(s), \dot y(s))\, ds - \frac{t_0}{T}\bigg(\frac{t}{t_0} - 1\bigg) \int_0^T \phi(y(s), \dot y(s))\, ds,
$$

for $t \in [0,T]$. For every $x_0^\theta \in \Gamma$, we denote by $y_\theta \in C^1(T)$ the extension of x_0^θ from $[0,T]$ to $\mathbb{R}$ by T-periodicity, we have that $F_0 y_\theta = y_\theta$ whenever $\theta \in [0,T]$. Since F_0 is Fréchet differentiable and y_θ is differentiable with respect to θ we get

$$
\begin{aligned}
\bigg(\frac{d}{d\theta} y_\theta\bigg)(t) = \bigg(\frac{d}{d\theta} y_\theta\bigg)(0) &+ \int_0^t \bigg[a_\theta(s)\bigg(\frac{d}{d\theta} y_\theta\bigg)(s) + b_\theta(s)\frac{d}{dt}\bigg(\frac{d}{d\theta} y_\theta\bigg)(s)\bigg]\, ds \\
&- \frac{t_0}{T}\bigg(\frac{t}{t_0} - 1\bigg) \int_0^T \bigg[a_\theta(s)\bigg(\frac{d}{d\theta} y_\theta\bigg)(s) + b_\theta(s)\bigg(\frac{d}{d\theta} y_\theta\bigg)(s)\bigg]\, ds,
\end{aligned}
$$

hence $\frac{d}{d\theta} y_\theta$ is a fixed point of $F_0'(y_\theta)$ for any $\theta \in [0,T]$, and so $F_0'(y_\theta)$ is the equivalent integral operator for (2.3). As a consequence Γ' coincides with the set of all the fixed points of the operator $F_0'(y_\theta)$, thus $0 \in \sigma(I - F_0'(y_\theta))$. Moreover, since $F_0'(y_\theta)$ is q-condensing, with $0 < q < 1$, 0 is an eigenvalue of finite multiplicity, see [1, Theorem 2.6.11]. The assumption on its multiplicity is crucial for the calculation of the Malkin bifurcation function to associate to the problem, indeed this calculation requires the knowledge of all the adjoint vectors of the equivalent integral operator and of its adjoint operator. As it is shown in [18] the T-periodic adjoint Floquet solutions of the linearized unperturbed equation correspond to the adjoint vectors of the translation operator over the period along the trajectories of (2.4). Unfortunately, such a relationship may fail in the case of the equivalent integral operator defined in (4.1), see [28]. To solve this problem we follow the method introduced in [21] consisting in defining a novel equivalent integral operator for which the above property holds true. For this the first step is to calculate the adjoint of the operator $F_0'(y_\theta)\colon C^1(T) \to C^1(T)$. To this regard observe that all our subsequent results are formulated only in terms of the eigenvector and of the corresponding Floquet adjoint vectors of the operator $F_0'(y_\theta)^*$. These vectors are elements of $C^1(T)^*$. On the other hand, they coincide with those of the operator $F_0'(y_\theta)^*$ when it is defined in the space $W_2^1(T)^* = W_2^1(T)$. Therefore, throughout the paper, in order to calculate these vectors we use the space $W_2^1(T)$ of absolutely continuous T-periodic functions with derivative in $L_2(0,T)$. We denote by $\langle x, y\rangle_w := \langle x(0), y(0)\rangle + \int_0^T \langle \dot{x}(s), \dot{y}(s)\rangle\, ds$ the inner product in $W_2^1(T)$ and for the sake of simplicity we will omit θ in the notations for $a_\theta(s)$ and $b_\theta(s)$.

For any $x, y \in W_2^1(T)$ we have

$$
\begin{aligned}
\langle F_0'(y_\theta)x, y\rangle_w &= \langle (F_0'(y_\theta)x(0), y(0)\rangle + \int_0^T \left\langle \left[\frac{d}{ds}(F_0'(y_\theta))x\right](s), \dot{y}(s)\right\rangle ds \\
&= \left\langle x(0) + \frac{t_0}{T}\int_0^T (a(s)x(s) + b(s)\dot{x}(s))\, ds, y(0)\right\rangle \\
&\quad + \int_0^T \left\langle a(t)x(t) + b(t)\dot{x}(t) - \frac{1}{T}\int_0^T (a(s)x(s) + b(s)\dot{x}(s))\, ds, \dot{y}(t)\right\rangle dt \\
&= \langle x(0), y(0)\rangle + \int_0^T \left\langle a(s)x(s), \frac{t_0}{T}y(0)\rangle\, ds + \int_0^T \langle b(s)\dot{x}(s), \frac{t_0}{T}y(0)\right\rangle ds \\
&\quad + \int_0^T \langle a(s)x(s), \dot{y}(s)\rangle\, ds + \int_0^T \langle b(s)\dot{x}(s), \dot{y}(s)\rangle\, ds \\
&\quad - \int_0^T \left\langle \int_0^T a(s)x(s) + b(s)\dot{x}(s))\, ds, \frac{1}{T}\dot{y}(t)\right\rangle dt \\
&= \langle x(0), y(0)\rangle + \int_0^T \left\langle x(s), a^*(s)\frac{t_0}{T}y(0)\right\rangle ds + \int_0^T \left\langle \dot{x}(s), b^*(s)\frac{t_0}{T}y(0)\right\rangle ds \\
&\quad + \int_0^T \langle x(s), a^*(s)\dot{y}(s)\rangle\, ds + \int_0^T \langle \dot{x}(s), b^*(s)\dot{y}(s)\rangle\, ds.
\end{aligned}
$$

We now calculate the terms containing x. Since $x \in W_2^1(T)$, it can be represented as follows

$$
x(s) = x(0) + \int_0^s \dot{x}(t)\, dt - \frac{s}{T}\int_0^T \dot{x}(t)\, dt.
$$

Therefore,

$$
\begin{aligned}
&\int_0^T \left\langle x(s), a^*(s)\frac{t_0}{T}\, y(0)\right\rangle ds \\
&\quad= \int_0^T \left\langle x(0) + \int_0^s \dot{x}(t)\,dt - \frac{s}{T}\int_0^T \dot{x}(t)\,dt, a^*(s)\frac{t_0}{T}\, y(0)\right\rangle ds \\
&\quad= \left\langle x(0), \int_0^T a^*(s)\frac{t_0}{T}\, y(0)\,ds\right\rangle + \int_0^T \left\langle \int_0^s \dot{x}(t)\,dt, a^*(s)\frac{t_0}{T}\, y(0)\right\rangle ds \\
&\qquad - \int_0^T \left\langle \int_0^T \dot{x}(t)\,dt, \frac{s}{T}a^*(s)\frac{t_0}{T}y(0)\right\rangle ds \\
&\quad= \left\langle x(0), \int_0^T a^*(s)\frac{t_0}{T}y(0)\,ds\right\rangle + \int_0^T \left\langle \dot{x}(t), \int_t^T a^*(s)\frac{t_0}{T}\, y(0)\,ds\right\rangle dt \\
&\qquad - \int_0^T \left\langle \dot{x}(t), \int_0^T \frac{s}{T}\, a^*(s)\frac{t_0}{T}\, y(0)\,ds\right\rangle dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \langle x(s), a^*(s)\dot{y}(s)\rangle\, ds \\
&\quad= \int_0^T \left\langle x(0) + \int_0^s \dot{x}(t)\,dt - \frac{s}{T}\int_0^T \dot{x}(t)\,dt, a^*(s)\dot{y}(s)\right\rangle ds \\
&\quad= \left\langle x(0), \int_0^T a^*(s)\dot{y}(s)\,ds\right\rangle + \int_0^T \left\langle \dot{x}(s), \int_s^T a^*(t)\dot{y}(t)\,dt\right\rangle ds \\
&\qquad - \int_0^T \left\langle \dot{x}(s), \int_0^T \frac{t}{T}\, a^*(t)\dot{y}(t)\,dt\right\rangle ds.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&\langle x, (F_0'(y_\theta))^* y\rangle_w \\
&= \langle x(0), y(0)\rangle + \left\langle x(0), \int_0^T a^*(s)\frac{t_0}{T}\, y(0)\,ds\right\rangle + \left\langle x(0), \int_0^T a^*(s)\dot{y}(s)\,ds\right\rangle \\
&\quad + \int_0^T \left\langle \dot{x}(t), \int_t^T a^*(s)\frac{t_0}{T}\, y(0)\,ds\right\rangle dt - \int_0^T \left\langle \dot{x}(t), \int_0^T \frac{s}{T}\, a^*(s)\frac{t_0}{T}\, y(0)\,ds\right\rangle dt \\
&\quad + \int_0^T \left\langle \dot{x}(s), b^*(s)\frac{t_0}{T}\, y(0)\right\rangle ds + \int_0^T \langle \dot{x}(s), b^*(s)\dot{y}(s)\rangle\, ds \\
&\quad + \int_0^T \left\langle \dot{x}(s), \int_s^T a^*(t)\dot{y}(t)\,dt\right\rangle ds - \int_0^T \left\langle \dot{x}(s), \int_0^T \frac{t}{T}\, a^*(t)\dot{y}(t)\,dt\right\rangle ds \\
&\quad + \int_0^T \left\langle \dot{x}(s), -\frac{1}{T}\int_0^T b^*(t)\left(\frac{t_0}{T}y(0) + \dot{y}(t)\right) dt\right\rangle ds,
\end{aligned}
$$

where the last added term is zero. Thus,

$$
[(F_0'(y_\theta))^* y](0) = y(0) + \int_0^T a^*(s)\left(\frac{t_0}{T}\, y(0) + \dot{y}(s)\right) ds,
$$

$$\left[\frac{d}{dt}(F_0'(y_\theta))^*y\right](t) = \int_t^T a^*(s)\left(\frac{t_0}{T}y(0)+\dot{y}(s)\right)ds + b^*(t)\left(\frac{t_0}{T}y(0)+\dot{y}(t)\right)$$
$$-\frac{1}{T}\int_0^T\left[\int_t^T a^*(s)\left(\frac{t_0}{T}\,y(0)+\dot{y}(s)\right)ds + b^*(t)\left(\frac{t_0}{T}\,y(0)+\dot{y}(t)\right)\right]dt.$$

5. A novel equivalent integral operator

In this Section we introduce a novel integral operator $\widehat{F}_\varepsilon, \varepsilon \geq 0$ small, with the property that $\widehat{F}_0'(y_\theta)$ has exactly m adjoint vectors corresponding to the m T-periodic solutions of the linearized unperturbed equation (2.4). Therefore, it enjoys the same property of the translation operator along the trajectories of (2.1). This novel integral operator is obtained by a suitable modification of the equivalent integral operator defined by means of (2.4). As already noticed, in general, the equivalent integral operator defined in (4.1) does not have the previous required property. An analogous construction has been proposed in [21] when the characteristic multiplier of the unperturbed linearized equation is simple. To our best knowledge this is the first time that such a construction is performed in the case of the existence of T-periodic adjoint Floquet solutions.

Let $\mathcal{F}\colon C^1(T) \to C^1(T)$ be the operator defined by

$$(\mathcal{F}x)(t) := \frac{1}{T}\int_0^t (I-b(s))x(s)\,ds - \frac{t_0}{T^2}\left(\frac{t}{t_0}-1\right)\int_0^T (I-b(s))x(s)\,ds,$$

We need the following result.

Lemma 5.1. *There exists $\tau > 0$ such that $t_0+\tau < T$ and*

$$\left\{\int_{t_0}^{t_0+\tau} e_j(s)\,ds\right\}_{j=0}^m \quad \text{and} \quad \left\{\int_{t_0}^{t_0+\tau} (\mathcal{F}e_j)(s)\,ds\right\}_{j=0}^m$$

are two sets of linearly independent vectors of $\mathbb{R}^n$.

Proof. By Lemma 3.4 and Remark 3.6, $\{e_j(t_0)\}_{j=0}^m$ and $\{(\mathcal{F}e_j)(t_0)\}_{j=0}^m$ are two sets of linearly independent vectors of $\mathbb{R}^n$, hence from Lemma 3.5 we derive the assertion. □

Consider now the linearly independent set of vectors $\left\{\int_{t_0}^{t_0+\tau}(\mathcal{F}e_j)(s)\,ds\right\}_{j=0}^m$. If $m < n-1$, then we complete this set by adding vectors $\{h_j\}_{j=m+1}^{n-1}$ in such a way that $\left\{\int_{t_0}^{t_0+\tau}(\mathcal{F}e_j)(s)\,ds\right\}_{j=0}^m \cup \{h_j\}_{j=m+1}^{n-1}$ is a basis of $\mathbb{R}^n$. Let $\{f_j\}_{j=0}^m \cup \{k_j\}_{j=m+1}^{n-1} \subset \mathbb{R}^n$ be such that

$$\left\langle \int_{t_0}^{t_0+\tau}(\mathcal{F}e_i)(s)\,ds, f_j\right\rangle = \delta_{i,j} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \tag{5.1}$$

with $i,j = 0,\ldots,m$.

$$\langle h_j, f_i\rangle = 0, \quad j = m+1,\ldots,n-1,\ i = 0,\ldots,m. \tag{5.2}$$

$$(5.3)\qquad \left\langle \int_{t_0}^{t_0+\tau} (\mathcal{F}e_i)(s)\,ds, k_j \right\rangle = 0, \quad i = 0,\dots,m,\ j = m+1,\dots,n-1.$$

$$(5.4)\qquad \langle h_i, k_j\rangle = \delta_{i,j}, \quad i = m+1,\dots,n-1,\ j = m+1,\dots,n-1.$$

Moreover, complete the set of vectors $\big\{\int_{t_0}^{t_0+\tau} e_j(s)\,ds\big\}_{j=0}^{m}$ by adding vectors $\{r_j\}_{j=m+1}^{n-1}$ such that their union is a basis of $\mathbb{R}^n$.

Finally, for any $t \in [0,T]$, define the function $\xi(t)\colon \mathbb{R}^n \to C^1(T)$ as follows

$$\xi(t)\cdot\, := \sum_{j=0}^{m} \langle\,\cdot\,, f_j\rangle((\mathcal{F}e_j)(t) - e_j(t)) + \sum_{j=m+1}^{n-1} \langle\,\cdot\,, k_j\rangle \frac{1}{\tau}(h_j - r_j).$$

We can now define a novel integral operator $\widehat{F}_\varepsilon\colon C^1(T) \to C^1(T)$ as follows

$$(5.5)\qquad (\widehat{F}_\varepsilon x)(t) = (F_\varepsilon x)(t) - \xi(t)\int_{t_0}^{t_0+\tau} [(F_\varepsilon x)(s) - x(s)]\,ds, \quad t \in [0,T].$$

The following result holds.

Theorem 5.2. *$\widehat{F}_\varepsilon$ is equivalent to F_ε. Moreover, e_j, $j = 1,\dots,m$, are the only adjoint vectors of $\widehat{F}_0'(y_\theta)$, whenever $\theta \in [0,T]$, that is*

$$\widehat{F}_0'(y_\theta)e_j = e_j + e_{j-1}, \quad j = 1,\dots,m,\ \theta \in [0,T].$$

Proof. First we prove that $1 \notin \sigma\big(\int_{t_0}^{t_0+\tau} \xi(s)\,ds\big)$. For this, suppose that $y \in \mathbb{R}^n$ is an eigenvector of $\big(\int_{t_0}^{t_0+\tau} \xi(s)\,ds\big)$, thus $y = \sum_{i=0}^{m} \alpha_i g_i + \sum_{i=m+1}^{n-1} \beta_i h_i$, where $g_i = \int_{t_0}^{t_0+\tau} (\mathcal{F}e_i)(s)\,ds$ and $\int_{t_0}^{t_0+\tau} \xi(s)\,y\,ds = y$, or, equivalently,

$$\begin{aligned}\int_{t_0}^{t_0+\tau} \bigg(\sum_{j=0}^{m}\bigg\langle \sum_{i=0}^{m}\alpha_i g_i + \sum_{i=m+1}^{n-1}\beta_i h_i, f_j\bigg\rangle((\mathcal{F}e_j)(t) - e_j(t)) \\ + \sum_{j=m+1}^{n-1}\bigg\langle \sum_{i=0}^{m}\alpha_i g_i + \sum_{i=m+1}^{n-1}\beta_i h_i, k_j\bigg\rangle \frac{1}{\tau}(h_j - r_j)\bigg)\,dt = \sum_{i=0}^{m}\alpha_i g_i + \sum_{i=m+1}^{n-1}\beta_i h_i,\end{aligned}$$

or

$$(5.6)\qquad \begin{aligned}\sum_{j=0}^{m}\bigg\langle \sum_{i=0}^{m}\alpha_i g_i + \sum_{i=m+1}^{n-1}\beta_i h_i, f_j\bigg\rangle\bigg(g_j - \int_{t_0}^{t_0+\tau} e_j(t)\,dt\bigg) \\ + \sum_{j=m+1}^{n-1}\bigg\langle \sum_{i=0}^{m}\alpha_i g_i + \sum_{i=m+1}^{n}\beta_i h_i, k_j\bigg\rangle(h_j - r_j) = \sum_{i=0}^{m}\alpha_i g_i + \sum_{i=m+1}^{n-1}\beta_i h_i.\end{aligned}$$

From (5.1)–(5.4) we have that (5.6) can be rewritten in the form

$$\sum_{j=0}^{m}\alpha_j g_j + \sum_{j=m+1}^{n-1}\beta_j h_j = \sum_{j=0}^{m}\alpha_j\bigg(g_j - \int_{t_0}^{t_0+\tau} e_j(t)\,dt\bigg) + \sum_{j=m+1}^{n-1}\beta_j(h_j - r_j),$$

and so

$$\sum_{j=0}^{m}\alpha_j\int_{t_0}^{t_0+\tau}e_j(t)\,dt+\sum_{j=m+1}^{n-1}\beta_j r_j=0,$$

which implies $\alpha_j=0$, $j=0,\dots,m$, $\beta_j=0$,$j=m+1\dots,n-1$. That is $y=0$, hence $1\notin\sigma\big(\int_{t_0}^{t_0+\tau}\xi(s)\,ds\big)$.

We now prove the equivalence of the equations:

$$F_\varepsilon y=y, \tag{5.7}$$

$$\widehat{F}_\varepsilon y=y, \tag{5.8}$$

where F_ε is defined in (4.1) and $\widehat{F}_\varepsilon$ in (5.5), $\varepsilon\in[0,\varepsilon_0]$. Assume that $y\in C^1(T)$ is such that (5.7) holds, then $\int_{t_0}^{t_0+\tau}[(F_\varepsilon y)(s)-y(s)]\,ds=0$, and so (5.8) is satisfied. Vice versa, if (5.8) holds we have that

$$(F_\varepsilon y)(t)-y(t)-\xi(t)\int_{t_0}^{t_0+\tau}[(F_\varepsilon y)(s)-y(s)]\,ds=0,$$

integrating from t_0 and $t_0+\tau$ we obtain

$$\int_{t_0}^{t_0+\tau}[(F_\varepsilon y)(s)-y(s)]\,ds-\int_{t_0}^{t_0+\tau}\xi(s)\,ds\int_{t_0}^{t_0+\tau}[(F_\varepsilon y)(s)-y(s)]\,ds=0.$$

But $1\notin\sigma(\int_{t_0}^{t_0+\tau}\xi(s)\,ds)$, hence $\int_{t_0}^{t_0+\tau}[(F_\varepsilon y)(s)-y(s)]\,ds=0$, i.e. (5.7) is satisfied. In particular, from the equivalence between $\widehat{F}_\varepsilon$ and F_ε we have $\widehat{F}_0'(y_\theta)e_0=e_0$. Let us show that $\widehat{F}_0'(y_\theta)e_j=e_j+e_{j-1}$ with $j=1,\dots,m$ and $\theta\in[0,T]$. For this observe that

$$\xi(t)\int_{t_0}^{t_0+\tau}(\mathcal{F}e_j)(s)\,ds=(\mathcal{F}e_j)(t)-e_j(t)\quad\text{for } j=0,\dots,m. \tag{5.9}$$

Abusing notation in what follows we denote $F_0'(y_\theta)$ and $\widehat{F}_0'(y_\theta)$ simply by G and $\widehat{G}$, respectively. Consider, for $j=1,\dots,m$, the equation $\widehat{G}e_j=e_j+e_{j-1}$. That is,

$$(Ge_j)(t)-\xi(t)\int_{t_0}^{t_0+\tau}[(Ge_j)(s)-e_j(s)]\,ds=e_j(t)+e_{j-1}(t), \tag{5.10}$$

for $j=1,\dots,m$ and $t\in[0,T]$. By Lemma 3.1 and Remark 3.2 we have that

$$e_j(t)=a(t)e_j(t)+b(t)\dot e_j(t)-\frac{1}{T}\,(I-b(t))e_{j-1}(t). \tag{5.11}$$

Associating to (5.11) the equivalent integral operator we obtain

$$\begin{aligned}e_j(t)=\,&e_j(0)+\int_0^t[a(s)e_j(s)+b(s)\dot e_j(s)-\frac{1}{T}(I-b(s))e_{j-1}(s)]\,ds\\&-\frac{t_0}{T}\left(\frac{t}{t_0}-1\right)\int_0^T\left[a(s)e_j(s)+b(s)\dot e_j(s)-\frac{1}{T}(I-b(s))e_{j-1}(s)\right]ds,\end{aligned}$$

or, equivalently,

$$e_j(t) = (Ge_j)(t) - \frac{1}{T}\int_0^t (I - b(s))e_{j-1}(s)\,ds + \frac{t_0}{T^2}\left(\frac{t}{t_0} - 1\right)\int_0^T (I - b(s))e_{j-1}(s)\,ds, \tag{5.12}$$

i.e. $e_j(t) = (Ge_j)(t) - (\mathcal{F}e_{j-1})(t)$. Thus

$$\xi(t)\int_{t_0}^{t_0+\tau}[(Ge_j)(s)-e_j(s)]\,ds = \xi(t)\int_{t_0}^{t_0+\tau}(\mathcal{F}e_{j-1})(s)\,ds = (\mathcal{F}e_{j-1})(t)-e_{j-1}(t),$$

and so (5.10) takes the form $(Ge_j)(t) - (\mathcal{F}e_{j-1})(t) + e_{j-1}(t) = e_j(t) + e_{j-1}(t)$, or $(Ge_j)(t)-(\mathcal{F}e_{j-1})(t) = e_j(t)$, that is (5.12). In conclusion, (5.10) is verified since (5.12) has been proved. It remains only to show they there are no other adjoint vectors to $\widehat{G}$. Arguing by contradiction, assume that there exists $u \in C^1(T)$ such that $\widehat{G}u = u + e_m$; that is

$$(Gu)(t) - \xi(t)\int_{t_0}^{t_0+\tau}[(Gu)(s) - u(s)]ds = u(t) + e_m(t). \tag{5.13}$$

Integrating (5.13) from t_0 and $t_0 + \tau$ we obtain

$$\int_{t_0}^{t_0+\tau}[(Gu)(s) - u(s)]\,ds - \int_{t_0}^{t_0+\tau}\xi(s)\,ds\int_{t_0}^{t_0+\tau}[(Gu)(s) - u(s)]\,ds = \int_{t_0}^{t_0+\tau}e_m(s)\,ds,$$

which implies that

$$\int_{t_0}^{t_0+\tau}[(Gu)(s) - u(s)]\,ds = \left(I - \int_{t_0}^{t_0+\tau}\xi(s)\,ds\right)^{-1}\int_{t_0}^{t_0+\tau}e_m(s)\,ds. \tag{5.14}$$

Substituting (5.14) into (5.13) we get

$$(Gu)(t) - \xi(t)\left(I - \int_{t_0}^{t_0+\tau}\xi(s)\,ds\right)^{-1}\int_{t_0}^{t_0+\tau}e_m(s)\,ds = u(t) + e_m(t). \tag{5.15}$$

From (5.9) we have

$$\xi(t)\int_{t_0}^{t_0+\tau}(\mathcal{F}e_m)(s)\,ds = (\mathcal{F}e_m)(t) - e_m(t).$$

Thus

$$\int_{t_0}^{t_0+\tau}\xi(s)\,ds\int_{t_0}^{t_0+\tau}(\mathcal{F}e_m)(s)\,ds = \int_{t_0}^{t_0+\tau}(\mathcal{F}e_m)(s)\,ds - \int_{t_0}^{t_0+\tau}e_m(s)\,ds,$$

and so

$$\left(I - \int_{t_0}^{t_0+\tau}\xi(s)\,ds\right)^{-1}\int_{t_0}^{t_0+\tau}e_m(s)\,ds = \int_{t_0}^{t_0+\tau}(\mathcal{F}e_m)(s)\,ds. \tag{5.16}$$

Substituting (5.16) into (5.15) we get

$$(Gu)(t) - \xi(t) \int_{t_0}^{t_0+\tau} (\mathcal{F}e_m)(s)\, ds = u(t) + e_m(t),$$

hence, from (5.9), $(Gu)(t) - (\mathcal{F}e_m)(t) + e_m(t) = u(t) + e_m(t)$, namely

$$(Gu)(t) - (\mathcal{F}e_m)(t) = u(t). \tag{5.17}$$

Observe that (5.17) is the equivalent integral equation for the unperturbed linearized equation

$$\dot{u}(t) = a(t)u(t) + b(t)\dot{u}(t) - \frac{1}{T}(I - b(t))e_m(t),$$

which contradicts Corollary 3.3. □

6. Eigenvector and adjoint vectors

6.1. The eigenvector of G^*. By assumption the eigenvalue $1 \in \sigma(G)$ has geometric multiplicity one with corresponding eigenvector $e_0 := \dot{x}_0$. In this Section we determine the unique solution of the equation $y = G^*y$, where $G^*\colon W_2^1(T) \to W_2^1(T)$ has been calculated in Section 4. We have that

$$\begin{aligned} y(t) = (G^*y)(t) &= y(0) + \int_0^T a^*(s)\Big(\frac{t_0}{T}y(0) + \dot{y}(s)\Big)\, ds \\ &+ \int_0^t \Big[\int_r^T a^*(s)\Big(\frac{t_0}{T}y(0) + \dot{y}(s)\Big)\, ds + b^*(r)\Big(\frac{t_0}{T}y(0) + \dot{y}(r)\Big)\Big]\, dr \\ &- \frac{t}{T}\int_0^T \Big[\int_t^T a^*(s)\Big(\frac{t_0}{T}y(0) + \dot{y}(s)\Big)\, ds + b^*(t)\Big(\frac{t_0}{T}y(0) + \dot{y}(t)\Big)\Big]\, dt. \end{aligned} \tag{6.1}$$

Taking the derivative we obtain

$$\begin{aligned} \dot{y}(t) &= \int_t^T a^*(s)\Big(\frac{t_0}{T}y(0) + \dot{y}(s)\Big)\, ds + b^*(t)\Big(\frac{t_0}{T}y(0) + \dot{y}(t)\Big) \\ &- \frac{1}{T}\int_0^T \Big[\int_t^T a^*(s)\Big(\frac{t_0}{T}y(0) + \dot{y}(s)\Big)\, ds + b^*(t)\Big(\frac{t_0}{T}y(0) + \dot{y}(t)\Big)\Big]\, dt. \end{aligned} \tag{6.2}$$

Let $v(t) := t_0 y(0)/T + \dot{y}(t)$, then $\dot{y}(t) = v(t) - t_0 y(0)/T$ and $\int_0^T v(t)\, dt = t_0 y(0)$. Hence

$$\dot{y}(t) = v(t) - \frac{1}{T}\int_0^T v(s)\, ds. \tag{6.3}$$

Replacing (6.3) into (6.2) we obtain

$$\begin{aligned} v(t) - \frac{1}{T}\int_0^T v(s)\, ds = &\int_t^T a^*(s)v(s)\, ds + b^*(t)v(t) \\ &- \frac{1}{T}\int_0^T \Big[\int_t^T a^*(s)v(s)\, ds + b^*(t)v(t)\Big]\, dt, \end{aligned}$$

namely

$$(6.4)\quad (I-b^*(t))v(t)-\frac{1}{T}\int_0^T [v(s)-b^*(s)v(s)]\,ds = \int_t^T a^*(s)v(s)\,ds-\frac{1}{T}\int_0^T\int_t^T a^*(s)v(s)\,ds\,dt.$$

Let $w(t):=(I-b^*(t))v(t)$, then (6.4) becomes

$$w(t)-\frac{1}{T}\int_0^T w(s)\,ds=\int_t^T a^*(s)(I-b^*(s))^{-1}w(s)\,ds - \frac{1}{T}\int_0^T\int_t^T a^*(s)(I-b^*(s))^{-1}w(s)\,ds\,dt,$$

that is

$$(6.5)\quad w(t)-\int_t^T a^*(s)(I-b^*(s))^{-1}w(s)\,ds = \frac{1}{T}\int_0^T\bigg(w(s)-\int_s^T a^*(r)(I-b^*(r))^{-1}w(r)\,dr\bigg)\,ds.$$

Put $\eta(t):=w(t)-\int_t^T a^*(s)(I-b^*(s))^{-1}w(s)\,ds$. Thus (6.5) takes the form

$$\eta(t)=\frac{1}{T}\int_0^T \eta(s)\,ds,$$

Hence, $\eta(t)$ is a constant. Thus

$$(6.6)\qquad \dot w(t)+a^*(t)(I-b^*(t))^{-1}w(t)=0$$

is the adjoint equation of $\dot x(t)=(I-b(t))^{-1}a(t)x(t)$, hence (6.6) has a unique T-periodic solution v_0 and we have

$$\begin{aligned} v(t)&=(I-b^*(t))^{-1}v_0(t),\\ \dot y(t)&=(I-b^*(t))^{-1}v_0(t)-\frac{1}{T}\int_0^T (I-b^*(s))^{-1}v_0(s)\,ds, \end{aligned}$$

i.e.

$$y(t)=y(0)+\int_0^t (I-b^*(s))^{-1}v_0(s)\,ds-\frac{t}{T}\int_0^T (I-b^*(s))^{-1}v_0(s)\,ds.$$

and so, recalling that $y(0)=(1/t_0)\int_0^T v(s)\,ds$, we get

$$(6.7)\quad y(t)=\frac{1}{t_0}\int_0^T (I-b^*(s))^{-1}v_0(s)\,ds + \int_0^t (I-b^*(s))^{-1}v_0(s)\,ds-\frac{t}{T}\int_0^T (I-b^*(s))^{-1}v_0(s)\,ds.$$

Observe that $y(T) = y(0)$, i.e. y is T-periodic. Since v_0 is the unique T-periodic solution to (6.6), the solution to $G^*y = y$ is also unique and, by substituting (6.7) in $G^*y = y$, we obtain

$$\int_0^t \left[v_0(s) - \int_s^T a^*(r)(I - b^*(r))^{-1} v_0(r)\, dr \right] ds \\ = \frac{t}{T} \int_0^T \left(v_0(s) - \int_s^T a^*(r)(I - b^*(r))^{-1} v_0(r)\, dr \right) ds.$$

which holds true in virtue of (6.5). Hence, the unique solution of $G^*y = y$ is given by (6.7).

Consider now the equation

$$(I - G^*)y = z_0, \tag{6.8}$$

where z_0 is a given twice differentiable T-periodic function such that (6.8) is solvable. In the following we calculate the solution to (6.8). For this, we rewrite (6.8) as

$$\begin{aligned} z_0(t) = y(t) - y(0) &- \int_0^T a^*(s)\left(\frac{t_0}{T}y(0) + \dot{y}(s)\right) ds \\ &- \int_0^t \left[\int_r^T a^*(s)\left(\frac{t_0}{T}y(0) + \dot{y}(s)\right) ds + b^*(r)\left(\frac{t_0}{T}y(0) + \dot{y}(r)\right)\right] dr \\ &+ \frac{t}{T}\int_0^T \left[\int_t^T a^*(s)\left(\frac{t_0}{T}y(0) + \dot{y}(s)\right) ds + b^*(t)\left(\frac{t_0}{T}y(0) + \dot{y}(t)\right)\right] dt. \end{aligned} \tag{6.9}$$

Since z_0 is T-periodic, then y is also T-periodic. As before, let $v(t) := t_0 y(0)/T + \dot{y}(t)$, then (6.3) holds. Therefore, from (6.9) we have

$$\dot{y}(t) = \int_t^T a^*(s)\left(\frac{t_0}{T}y(0) + \dot{y}(s)\right) ds + b^*(t)\left(\frac{t_0}{T}y(0) + \dot{y}(t)\right) \\ - \frac{1}{T}\int_0^T \left[\int_t^T a^*(s)\left(\frac{t_0}{T}y(0) + \dot{y}(s)\right) ds + b^*(t)\left(\frac{t_0}{T}y(0) + \dot{y}(t)\right)\right] dt + \dot{z}_0(t),$$

or, equivalently,

$$v(t) - \frac{1}{T}\int_0^T v(t)\, dt = \int_t^T a^*(s)v(s)\, ds + b^*(t)v(t) \\ - \frac{1}{T}\int_0^T \left[\int_t^T a^*(s)v(s)\, ds + b^*(t)v(t) \right] dt + \dot{z}_0(t).$$

Thus

$$(I - b^*(t))v(t) - \frac{1}{T}\int_0^T (I - b^*(t))v(t)\, dt \\ = \int_t^T a^*(s)v(s)\, ds - \frac{1}{T}\int_0^T \int_t^T a^*(s)v(s)\, ds + \dot{z}_0(t).$$

Again, let $w(t) = (I - b^*(t))v(t)$, hence

$$
\begin{aligned}
(6.10)\quad w(t) - \int_t^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds \\
= \frac{1}{T}\int_0^T \bigg(w(t) - \int_t^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds\bigg)\,dt + \dot z_0(t).
\end{aligned}
$$

Taking the derivative of (6.10) we get

$$
(6.11)\qquad \dot w(t) + a^*(t)(I - b^*(t))^{-1} w(t) - \ddot z_0(t) = 0.
$$

Let $v_1(t)$ be the solution od (6.11) satisfying the initial condition

$$
(6.12)\qquad w(0) = w(T) - z_0(0).
$$

Since (6.10) is equivalent to (6.8) and (6.11) is obtained by differentiation from (6.10), to show that the solvability of (6.8) implies the solvability of the Cauchy problem (6.11)–(6.12) it is enough to verify that (6.12) follows from (6.10). Indeed, we have

$$
\begin{aligned}
w(0) - \int_0^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds \\
&= \frac{1}{T}\int_0^T \bigg(w(t) - \int_t^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds\bigg)\,dt + \dot z_0(0), \\
w(T) &= \frac{1}{T}\int_0^T \bigg(w(t) - \int_t^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds\bigg)\,dt + \dot z_0(T),
\end{aligned}
$$

from the T-periodicity of z_0 we have $\dot z(0) = \dot z(T)$. Therefore,

$$
(6.13)\qquad w(0) - \int_0^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds = w(T).
$$

From (6.9) it follows that

$$
(6.14)\qquad -\int_0^T a^*(s)(I - b^*(s))^{-1} w(s)\,ds = z_0(0).
$$

Thus (6.13) and (6.14) give (6.12), and we have

$$
\begin{aligned}
v(t) &= (I - b^*(t))^{-1} v_1(t), \\
\dot y(t) &= (I - b^*(t))^{-1} v_1(t) - \frac{1}{T}\int_0^T (I - b^*(s))^{-1} v_1(s)\,ds, \\
y(0) &= \frac{1}{t_0}\int_0^T (I - b^*(s))^{-1} v_1(s)\,ds
\end{aligned}
$$

and finally,

$$
\begin{aligned}
(6.15)\quad y(t) = \int_0^t (I - b^*(s))^{-1} v_1(s)\,ds \\
- \frac{t}{T}\int_0^T (I - b^*(s))^{-1} v_1(s)\,ds + \frac{1}{t_0}\int_0^T (I - b^*(s))^{-1} v_1(s)\,ds.
\end{aligned}
$$

Note that the function given by (6.15) is T-periodic. As we have done for $G^*y = y$, by substituting (6.15) into (6.8) and using (6.10) we can verify that y is a solution to (6.8).

In conclusion, we have proved the following results.

Proposition 6.1. *The unique eigenvector of the operator G^* corresponding to the eigenvalue 1 is given by*

$$y(t) = \int_0^t (I - b^*(s))^{-1} v_0(s)\, ds \\ - \frac{t}{T}\int_0^T (I - b^*(s))^{-1} v_0(s)\, ds + \frac{1}{t_0}\int_0^T (I - b^*(s))^{-1} v_0(s)\, ds,$$

where v_0 is the T-periodic solution of the adjoint equation

$$\dot{w}(t) + a^*(t)(I - b^*(t))^{-1} w(t) = 0.$$

Proposition 6.2. *Assume that the equation $(I - G^*)y = z_0$ is solvable for a given T-periodic twice differentiable function z_0, then the solution is given by*

$$y(t) = \int_0^t (I - b^*(s))^{-1} v_1(s)\, ds \\ - \frac{t}{T}\int_0^T (I - b^*(s))^{-1} v_1(s)\, ds + \frac{1}{t_0}\int_0^T (I - b^*(s))^{-1} v_1(s)\, ds,$$

where v_1 is the solution of the Cauchy problem

$$\begin{cases} \dot{w}(t) + a^*(t)(I - b^*(t))^{-1} w(t) - \ddot{z}_0(t) = 0, \\ w(0) = w(T) - z_0(0). \end{cases}$$

6.2. The eigenvector and the adjoint vectors of $\widehat{G}^*$. In the previous section we have determined the eigenvector of G^*, the adjoint operator of G. In this section we calculate the eigenvector and the corresponding adjoint vectors of $\widehat{G}^*$, namely the adjoint operator of $\widehat{G} := \widehat{F_0}'(y_\theta)$, $\theta \in [0, T]$. For this, from the definition of the novel integral operator $\widehat{F_\varepsilon}$, we have to calculate the adjoint operator Ξ^* of Ξ defined by

$$(\Xi x)(t) := \xi(t) \int_{t_0}^{t_0+\tau} x(s)\, ds.$$

As pointed out in Section 3 in order to calculate the eigenvector and the adjoint vectors we can consider the operators as defined in $W_2^1(T)$ rather than in $C^1(T)$. For $x, y \in W_2^1(T)$ we have

$$\langle \Xi x, y\rangle_w = \langle (\Xi x)(0), y(0)\rangle + \int_0^T \left\langle \frac{d}{dt}\left[\xi(t)\int_{t_0}^{t_0+\tau} x(s)\, ds\right], \dot{y}(t) \right\rangle dt \\ = \left\langle \xi(0)\int_{t_0}^{t_0+\tau} x(s)\, ds, y(0)\right\rangle + \int_0^T \left\langle \sum_{j=0}^m \left\langle \int_{t_0}^{t_0+\tau} x(s)\, ds, f_j\right\rangle \dot{b}_j(t), \dot{y}(t)\right\rangle dt$$

$$
\begin{aligned}
&= \bigg\langle \sum_{j=0}^{m} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \bigg\rangle b_j(0) + \sum_{j=m+1}^{n-1} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, k_j \bigg\rangle c_j, y(0) \bigg\rangle \\
&\quad + \int_0^T \bigg\langle \sum_{j=0}^{m} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \bigg\rangle \dot b_j(t), \dot y(t) \bigg\rangle \,dt,
\end{aligned}
$$

where $b_j(t) := (\mathcal{F}e_j)(t) - e_j(t)$, $j = 0,\ldots,m$, and $c_j := (h_j - r_j)/\tau$, $j = m+1,\ldots,n-1$.

We calculate the three last terms separately. For the first term we have

$$
\begin{aligned}
\bigg\langle \sum_{j=0}^{m} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \rangle b_j(0), y(0) \rangle &= \sum_{j=0}^{m} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \bigg\rangle \langle b_j(0), y(0)\rangle \\
&= \sum_{j=0}^{m} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \langle b_j(0), y(0)\rangle \bigg\rangle \\
&= \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \\
&= \int_{t_0}^{t_0+\tau} \bigg\langle x(s), \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \,ds \\
&= \int_0^T \bigg\langle x(s), \chi_{[t_0,t_0+\tau]}(s) \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \,ds \\
&= \int_0^T \bigg\langle x(0) + \int_0^s \dot x(t)\,dt - \frac{s}{T}\int_0^T \dot x(t)\,dt, \chi_{[t_0,t_0+\tau]}(s) \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \,ds \\
&= \bigg\langle x(0), \tau \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \\
&\quad + \int_0^T \bigg\langle \dot x(t), \int_t^T \chi_{[t_0,t_0+\tau]}(s)\,ds \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \,dt \\
&\quad - \int_0^T \bigg\langle \dot x(t), \int_0^T \frac{s}{T} \chi_{[t_0,t_0+\tau]}(s)\,ds \sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \bigg\rangle \,dt.
\end{aligned}
$$

The second term gives

$$
\begin{aligned}
&\bigg\langle \sum_{j=m+1}^{n-1} \bigg\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, k_j \rangle c_j, y(0) \bigg\rangle \\
&\qquad = \int_0^T \bigg\langle x(s), \chi_{[t_0,t_0+\tau]}(s) \sum_{j=m+1}^{n-1} k_j \langle c_j, y(0)\rangle \bigg\rangle \,ds \\
&\qquad = \bigg\langle x(0), \tau \sum_{j=m+1}^{n-1} k_j \langle c_j, y(0)\rangle \bigg\rangle
\end{aligned}
$$

$$+\int_0^T \left\langle \dot{x}(t), \int_t^T \chi_{[t_0,t_0+\tau]}(s)ds \sum_{j=m+1}^{n-1} k_j\langle c_j, y(0)\rangle \right\rangle dt$$
$$-\int_0^T \left\langle \dot{x}(t), \int_0^T \frac{s}{T}\chi_{[t_0,t_0+\tau]}(s)\,ds \sum_{j=m+1}^{n-1} k_j\langle c_j, y(0)\rangle \right\rangle dt.$$

Finally, we calculate the third term

$$\int_0^T \left\langle \sum_{j=0}^{m} \left\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \right\rangle \dot{b}_j(t), \dot{y}(t) \right\rangle dt$$
$$= \int_0^T \sum_{j=0}^{m} \left\langle \left\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, f_j \right\rangle \dot{b}_j(t), \dot{y}(t) \right\rangle dt$$
$$= \int_0^T \left\langle \int_{t_0}^{t_0+\tau} x(s)\,ds, \sum_{j=0}^{m} f_j\langle \dot{b}_j(t), \dot{y}(t)\rangle \right\rangle dt$$
$$= \int_0^T \int_0^T \left\langle x(s), \chi_{[t_0,t_0+\tau]}(s)\,ds \sum_{j=0}^{m} f_j\langle \dot{b}_j(t), \dot{y}(t)\rangle\,ds \right\rangle dt$$
$$= \int_0^T \left\langle x(s), \chi_{[t_0,t_0+\tau]}(s) \sum_{j=0}^{m} f_j \int_0^T \langle \dot{b}_j(t), \dot{y}(t)\rangle\,dt \right\rangle ds$$
$$= \left\langle x(0), \tau \sum_{j=0}^{m} f_j \int_0^T \langle \dot{b}_j(t), \dot{y}(t)\rangle\,dt \right\rangle$$
$$+\int_0^T \left\langle \dot{x}(t), \int_t^T \chi_{[t_0,t_0+\tau]}(s)\,ds \sum_{j=0}^{m} f_j \int_0^T \langle \dot{b}_j(r), \dot{y}(r)\rangle\,dr \right\rangle dt$$
$$-\int_0^T \left\langle \dot{x}(t), \int_0^T \frac{s}{T}\chi_{[t_0,t_0+\tau]}(s)\,ds \sum_{j=0}^{m} f_j \int_0^T \langle \dot{b}_j(r), \dot{y}(r)\rangle\,dr \right\rangle dt.$$

Observe that

$$\int_t^T \chi_{[t_0,t_0+\tau]}(s)\,ds = \tau\chi_{[0,t_0]}(t) + (t_0+\tau-t)\chi_{(t_0,t_0+\tau]}(t),$$

$$\text{(6.16)}\quad \int_0^t \tau\chi_{[0,t_0]}(s)\,ds + \int_0^t (t_0+\tau-s)\chi_{(t_0,t_0+\tau]}(s)\,ds$$
$$= \left[\tau(t-t_0) - \frac{\tau^2}{2}\right]\chi_{[0,t_0]}(t) + \left[\tau(t-t_0) + tt_0 - \frac{1}{2}(t^2+t_0^2+\tau^2)\right]\chi_{(t_0,t_0+\tau]}(t)$$
$$+\tau t_0 + \frac{\tau^2}{2} := p(t),$$

and

$$\int_0^T \frac{s}{T}\chi_{[t_0,t_0+\tau]}(s)\,ds = \frac{\tau}{2T}(2t_0+\tau).$$

Summing up

$$(\Xi^* y)(0) = \tau\bigg(\sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle + \sum_{j=m+1}^{n-1} k_j \langle c_j, y(0)\rangle + \sum_{j=0}^{m} f_j \int_0^T \langle \dot b_j(s), \dot y(s)\rangle\, ds\bigg);$$

and

$$\begin{aligned}\bigg[\frac{d}{dt}(\Xi^* y)\bigg](t) = \int_t^T & \chi_{[t_0,t_0+\tau]}(s)\, ds \bigg[\sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \\ &+ \sum_{j=m+1}^{n-1} k_j \langle c_j, y(0)\rangle + \sum_{j=0}^{m} f_j \int_0^T \langle \dot b_j(r), \dot y(r)\rangle\, dr\bigg] \\ &- \int_0^T \frac{s}{T}\chi_{[t_0,t_0+\tau]}(s)\, ds \bigg[\sum_{j=0}^{m} f_j \langle b_j(0), y(0)\rangle \\ &+ \sum_{j=m+1}^{n-1} k_j \langle c_j, y(0)\rangle + \sum_{j=0}^{m} f_j \int_0^T \langle \dot b_j(r), \dot y(r)\rangle\, dr\bigg],\end{aligned}$$

and finally, we have

$$(\Xi^* y)(t) = \tau R_0 + p(t) R_0 - \frac{\tau t}{2T}(2t_0 + \tau) R_0,$$

where

$$R_0 = \sum_{j=0}^{m} f_j \langle b_j, y\rangle_w + \sum_{j=m+1}^{n} k_j \langle c_j, y\rangle_w. \tag{6.17}$$

We can summarize all the previous calculations as follows.

Proposition 6.3. *The adjoint operator $\Xi^*\colon W_2^1(T) \to W_2^1(T)$ is given by*

$$(\Xi^* y)(t) = \bigg[\tau + p(t) - \frac{\tau t}{2T}(2t_0 + \tau)\bigg] R_0, \quad t \in [0, T],$$

where p and R_0 are defined in (6.16) *and* (6.17)*, respectively.*

We now calculate the eigenvector of $\widehat G^*$ and its adjoint vectors. For this, recall that Theorem 5.2 ensures that the operator $\widehat G := \widehat F_0'(y_\theta)$, $\theta \in [0, T]$, has only one linearly independent eigenvector e_0 corresponding to the eigenvalue 1 and exactly $m \le n - 1$ adjoint vectors $e_1, \ldots, e_m$, namely

$$\widehat G e_0 = e_0 \quad \text{and} \quad \widehat G e_j = e_j + e_{j-1}, \quad j = 1, \ldots, m.$$

Since $\widehat G$ is q-condensing, with $0 < q < 1$ the eigenvalue 1 of $\widehat G$ is of finite multiplicity, thus the eigenspace corresponding to the eigenvalue 1 of the adjoint operator $\widehat G^*$ has the same structure, namely there exist $z_0, \ldots, z_m$ such that

$$\widehat G^* z_0 = z_0, \tag{6.18}$$

$$\widehat G^* z_j = z_j + z_{j-1}, \quad j = 1, \ldots, m. \tag{6.19}$$

Since, by definition, $\widehat{G} = G - \Xi(G - I)$, (6.18) can be rewritten as

$$(G^* - (G^* - I)\Xi^*)z_0 = z_0$$

or, equivalently,

$$(G^* - I)(I - \Xi^*)z_0 = 0. \tag{6.20}$$

By Proposition 6.1 the equation $(G^* - I)z = 0$ is uniquely solvable and we have the explicit expression of the solution. Denote this solution by $\widetilde{z}_0$. $(I - \Xi)$ is invertible, in fact if y is such that $(I - \Xi)y = 0$, that is

$$y(t) - \xi(t)\int_{t_0}^{t_0+\tau} y(s)\,ds = 0. \tag{6.21}$$

Integrating (6.21) from t_0 and $t_0 + \tau$ we get

$$\int_{t_0}^{t_0+\tau} y(s)\,ds - \int_{t_0}^{t_0+\tau} \xi(s)\,ds \int_{t_0}^{t_0+\tau} y(s)\,ds = 0,$$

Since $1 \notin \sigma\big(\int_{t_0}^{t_0+\tau} \xi(s)\,ds\big)$ we have that

$$\int_{t_0}^{t_0+\tau} y(s)\,ds = 0. \tag{6.22}$$

Replace (6.22) into (6.21) to obtain $y(t) = 0$, for any $t \in [0, T]$. Thus $(I - \Xi^*)$ is also invertible, as a consequence $z_0 = (I - \Xi^*)^{-1}\widetilde{z}_0$ is a solution of (6.20).

We have the following result.

PROPOSITION 6.4. *The solution of the equation*

$$(I - \Xi^*)x = x_0 \tag{6.23}$$

for any $x_0 \in W_2^1(T)$, is T-periodic and it is given by

$$x(t) = \left[\tau + p(t) - \frac{\tau t}{2T}(2t_0 + \tau)\right]\left(\sum_{j=0}^{m} f_j\alpha_j + \sum_{j=m+1}^{n-1} k_j\beta_j\right) + x_0(t), \quad t \in [0, T],$$

where $p(t)$ is defined by (6.16) *and α_j, $j = 0, \dots, m$; β_j, $j = m + 1, \dots, n - 1$ are known real numbers.*

PROOF. Since $(I - \Xi^*)$ is invertible, the equation (6.23) is uniquely solvable. By Proposition 6.3, (6.23) can be rewritten as follows:

$$x(t) = \left[\tau + p(t) - \frac{\tau t}{2T}(2t_0 + \tau)\right] \cdot \left(\sum_{j=0}^{m} f_j\langle b_j, x\rangle_w + \sum_{j=m+1}^{n-1} k_j\langle c_j, x\rangle_w\right) + x_0(t), \tag{6.24}$$

Multiplying (6.24) by b_j, $j = 0, \dots, m$, and c_j, $j = m + 1, \dots, n - 1$ in $W_2^1(T)$ we obtain a system of linear algebraic equations with respect to the unknown

$\langle b_j, x\rangle_w$, $j = 0,\ldots,m$ and $\langle c_j, x\rangle_w$, $j = m+1,\ldots,n-1$, which is solvable since (6.23) is solvable. Denote the solutions of this system by

$$\begin{aligned} \alpha_j &= \langle b_j, x\rangle_w, \quad j = 0,\ldots,m, \\ \beta_j &= \langle c_j, x\rangle_w, \quad j = m+1,\ldots,n-1. \end{aligned} \tag{6.25}$$

Substituting (6.25) into (6.24) we obtain

$$x(t) = \bigg[\tau + p(t) - \frac{\tau t}{2T}(2t_0+\tau)\bigg]\bigg(\sum_{j=0}^{m} f_j\alpha_j + \sum_{j=m+1}^{n-1} k_j\beta_j\bigg) + x_0(t).$$

Let $\widetilde{p}(t) := \tau + p(t) - (\tau t/(2T))(2t_0+\tau)$, since $x_0(t)$ is T-periodic and $\widetilde{p}(0) = \widetilde{p}(T) = \tau$ we obtain $x(0) = x(T)$. Rewrite (6.19) in the equivalent form

$$(G^* - I)(I - \Xi^*)z_j = z_{j-1}, \quad j = 1,\ldots,m. \tag{6.26}$$

For $j = 1$ we have that z_0 satisfies the condition of Proposition 6.2, then $(G^*-I)z = -z_0$ has a solution $\widetilde{z}_1$, thus $z_1 = (I-\Xi^*)^{-1}\widetilde{z}_1$ is a solution of (6.26) with $j = 1$. Applying Proposition 6.4 one can easily determine z_1. Observe that z_1 satisfies again the condition of Proposition 6.2, thus we can proceed by using the same arguments as before to determine z_j for $j = 2,\ldots,m$. □

We now give a very simple example to illustrate the procedure presented in the proof of Proposition 6.4. We consider the case when $n = 2, m = 1$, thus we have that $\xi(t)y = \langle y, f_0\rangle b_0(t) + \langle y, f_1\rangle b_1(t)$ and (6.24) becomes

$$x(t) - \widetilde{p}(t)(f_0\langle b_0, x\rangle_w + f_1\langle b_1, x\rangle_w) = x_0(t). \tag{6.27}$$

Let $a_{i,j} := \langle \widetilde{p} f_i, b_j\rangle_w$ with $i = 0,1$, $j = 0,1$. Therefore

$$\begin{aligned} a_{i,j} = {}&\bigg\langle \tau f_i, \frac{t_0}{T^2}\int_0^T (I - b(s))e_j(s)\,ds - e_j(0)\bigg\rangle \\ &+ \int_0^T \bigg\langle \dot{\widetilde{p}}(t) f_i, \frac{1}{T}(I - b(t))x(t) - \frac{1}{T^2}\int_0^T (I-b(s))e_j(s)\,ds - e_j(t)\bigg\rangle\,dt. \end{aligned}$$

Multiplying (6.27) by b_0 and b_1 in the space $W_2^1(T)$ we obtain the system

$$\begin{cases} (1 - a_{0,0})\langle b_0, x\rangle_w - a_{1,0}\langle b_1, x\rangle_w = \langle b_0, x_0\rangle_w, \\ -a_{0,1}\langle b_0, x\rangle_w + (1 - a_{1,1})\langle b_1, x\rangle_w = \langle b_1, x_0\rangle_w. \end{cases}$$

This system has a unique solution $\alpha_0 = \langle b_0, x_0\rangle_w, \alpha_1 = \langle b_1, x_0\rangle_w$. In conclusion,

$$x(t) = \widetilde{p}(t)(\alpha_0 f_0 + \alpha_1 f_1) + x_0(t).$$

7. Existence of bifurcation

Consider the novel equivalent integral operator defined in (5.5), that is

$$(\widehat{F}_\varepsilon x)(t) = (F_\varepsilon x)(t) - \xi(t)\int_{t_0}^{t_0+\tau} [(F_\varepsilon x)(s) - x(s)]ds, \quad t \in [0,T],$$

where F_ε is given in (4.1), i.e.

$$\begin{aligned}(F_\varepsilon x)(t) = x(0) &+ \int_0^t [\phi(x(s-\varepsilon),\dot{x}(s-\varepsilon)) + \varepsilon\psi(s,x(s-\varepsilon),\dot{x}(s-\varepsilon),\varepsilon)]\,ds \\ &- \frac{t_0}{T}\left(\frac{t}{t_0}-1\right)\int_0^T [\phi(x(s-\varepsilon),\dot{x}(s-\varepsilon)) + \varepsilon\psi(s,x(s-\varepsilon),\dot{x}(s-\varepsilon),\varepsilon)]\,ds.\end{aligned}$$

Define $P\colon W_2^1(T)\times[0,1]\to W_2^1(T)$ and $Q\colon W_2^1(T)\times[0,1]\to W_2^1(T)$ as follows

$$P(x,\varepsilon)(t) := \widetilde{P}(x,\varepsilon)(t) - x(t) - \xi(t)\int_{t_0}^{t_0+\tau} [\widetilde{P}(x,\varepsilon)(s) - x(s)]\,ds,$$

and

$$Q(x,\varepsilon)(t) := \widetilde{Q}(x,\varepsilon)(t) - \xi(t)\int_{t_0}^{t_0+\tau} \widetilde{Q}(x,\varepsilon)(s)\,ds,$$

where

$$\begin{aligned}\widetilde{P}(x,\varepsilon)(t) :=\,& x(0) + \int_0^t \phi(x(s-\varepsilon),\dot{x}(s-\varepsilon))\,ds \\ &- \frac{t_0}{T}\left(\frac{t}{t_0}-1\right)\int_0^T \phi(x(s-\varepsilon),\dot{x}(s-\varepsilon))\,ds,\end{aligned}$$

and

$$\begin{aligned}\widetilde{Q}(x,\varepsilon)(t) :=\,& \int_0^t \psi(s,x(s-\varepsilon),\dot{x}(s-\varepsilon),\varepsilon)\,ds \\ &- \frac{t_0}{T}\left(\frac{t}{t_0}-1\right)\int_0^T \psi(s,x(s-\varepsilon),\dot{x}(s-\varepsilon),\varepsilon)\,ds.\end{aligned}$$

Therefore, $\widehat{F}_\varepsilon x = x$ can be rewritten in the form of the following bifurcation equation

$$P(x,\varepsilon) + \varepsilon Q(x,\varepsilon) = 0. \tag{7.1}$$

In this section our aim is to verify that the conditions of ([19, Theorem 3.2]) are satisfied for the bifurcation equation (7.1). As a consequence, we obtain the existence of a branch of T-periodic solutions to (2.1) bifurcating from the limit cycle x_0. For this, we adopt the abstract setting of ([19, Theorem 3.2]), namely we consider $P,Q\colon E\times[0,1]\to E$, where E is a Banach space. We assume that the equation $P(x,0)=0$ has a one-dimensional manifold of solution $\widehat{\Gamma} := \{x(\theta) : \theta\in[0,T]\}, T\in\mathbb{R}$. In the following $R'_{(i)}(x,\varepsilon)$ will denote the derivative of $R(x,\varepsilon)$ with respect to the i-variable, $i\in\{1,2\}$.

We assume the following conditions:

(c_1) P and Q are continuous operators in both the variables in a neighbourhood U of $\widehat{\Gamma}\times[0,1]$.

(c_2) For any $\varepsilon\in[0,1]$ there exist $P'_{(1)}(x,\varepsilon)$ and $P''_{(1,1)}(x,\varepsilon)$ in a neighbourhood V of $\widehat{\Gamma}$, continuous in both the variables in U.

(c_3) For any $\varepsilon\in[0,1]$ there exists $P'_{(2)}(x(\theta),\varepsilon)$ and $P''_{(2,1)}(x(\theta),\varepsilon)$ continuous in ε, whenever $\theta\in[0,T]$.

(c_4) There exists $P''_{(1,2)}(x(\theta),0)$, whenever $\theta\in[0,T]$.

(c_5) For any $\varepsilon\in[0,1]$ there exists $Q'_{(1)}(x,\varepsilon)$ in V continuous in both the variables in U.

(c_6) There exists $Q'_{(2)}(x(\theta),0)$ whenever $\theta\in[0,T]$.

(c_7) $x(\theta)$ is twice differentiable for any $\theta\in[0,T]$ and $x'(\theta)\not\equiv 0$, for any $\theta\in[0,T]$.

(c_8) The operator $I-P'_{(1)}(x(\theta),0)$ is q-condensing with respect to the Hausdorff measure of non compactness with constant $0<q<1$, $\theta\in[0,T]$.

(c_9) The zero eigenvalue of the operator $P'_{(1)}(x(\theta),0)$, $\theta\in[0,T]$, has geometric multiplicity 1 and algebraic multiplicity $m>1$, $m\le n-1$. We denote by $e_0(\theta)$ the corresponding unique linearly independent eigenvector $x'_0(\theta)$ and by $e_j(\theta)$, $j=1,\dots,m$, the adjoint vectors, i.e. for any $\theta\in[0,T]$,

$$\begin{aligned}P'_{(1)}(x(\theta),0)e_0(\theta)&=0,\\ P'_{(1)}(x(\theta),0)e_j(\theta)&=e_{j-1}(\theta)\quad j=1,\dots,m.\end{aligned}$$

Without loss of generality, compare ([19, Lemma 3.1]), we may assume that, for any $\theta\in[0,T]$,

$$\begin{aligned}\langle e_j(\theta),z_{m-j}(\theta)\rangle&\neq 0,\quad j=1,\dots,m,\\ \langle e_j(\theta),z_i(\theta)\rangle&=0,\quad i,j=1,\dots,m,\ i\neq m-j,\end{aligned}$$

where $z_0(\theta)$ is the eigenvector of the adjoint operator $(P'_{(1)}(x(\theta),0))^*$ and $z_j(\theta)$, $j=1,\dots,m$, are the adjoint vectors of $(P'_{(1)}(x(\theta),0))^*$.

Finally, denote by $\pi(\theta)\colon E\to\operatorname{span}\{e_0(\theta),\dots,e_m(\theta)\}$ the Riesz projector associated to $P'_{(1)}(x(\theta),0)$ and by $\alpha_j(\theta)$, $j=0,\dots,m$, the coefficients of the decomposition of $e'_0(\theta)$, namely

$$e'_0(\theta)=\alpha_0(\theta)e_0(\theta)+\sum_{j=1}^m\alpha_j(\theta)e_j(\theta)+\widetilde{y}(\theta),$$

where $\widetilde{y}(\theta)\in(I-\pi(\theta))E$. Define

$$\begin{aligned}y_0(\theta)&:=-\,(P'_{(1)}(x(\theta),0)|_{(I-\pi(\theta))E})^{-1}(P'_{(2)}(x(\theta),0)+Q(x(\theta),0)),\\ A(\theta)&:=\pi(\theta)P''_{(1,2)}(x(\theta),0)e_0(\theta)\\ &\qquad+\pi(\theta)P''_{(1,1)}(x(\theta),0)e_0(\theta)y_0(\theta)+\pi(\theta)Q'_{(1)}(x(\theta),0)e_0(\theta),\end{aligned}$$

$$\begin{aligned}R(\theta) :=\ &\pi(\theta)P''_{(1,2)}(x(\theta),0)y_0(\theta)\\&+\frac{1}{2}\pi(\theta)P''_{(2,2)}(x(\theta),0)+\frac{1}{2}\pi(\theta)P''_{(1,1)}(x(\theta),0)y_0(\theta)y_0(\theta)\\&+\pi(\theta)Q'_{(1)}(x(\theta),0)y_0(\theta)+\pi(\theta)Q'_{(2)}(x(\theta),0).\end{aligned}$$

We are now in the position to define the generalized multidimensional Malkin bifurcation function as follows

$$M_\theta(\lambda_0,\ldots,\lambda_m)=-\frac{1}{2}\lambda_0^2\sum_{j=1}^m\alpha_j(\theta)e_{j-1}(\theta)+A(\theta)\lambda_0+\sum_{j=1}^m\lambda_je_{j-1}(\theta)+R(\theta).$$

The following result holds:

THEOREM 7.1 ([19, Theorem 3.2]). *Assume conditions* (c_1)–(c_9). *Let* $\theta_0\in[0,T]$ *be such that*

$$\pi(\theta_0)(Q(x(\theta_0),0)+P'_{(2)}(x(\theta_0),0))=0.$$

If the system $M_{\theta_0}(\lambda_0,\ldots,\lambda_m)=0$ *is solvable with respect to* λ_j, $j=0,\ldots,m$, *and for the solution* $(\mu_0,\ldots,\mu_m)$ *the condition*

$$\det\bigg(\frac{\partial}{\partial\lambda_0}M_{\theta_0}(\mu_0,\ldots,\mu_m),\ldots,\frac{\partial}{\partial\lambda_m}M_{\theta_0}(\mu_0,\ldots,\mu_m)\bigg)\neq 0$$

holds. Then equation (7.1) *has a solution of the form*

$$x_\varepsilon=x(\theta_0)+\varepsilon\mu_0e_0(\theta_0)+\varepsilon^2\sum_{j=1}^m\mu_je_j(\theta_0)+\varepsilon y_0(\theta_0)+O(\varepsilon^3). \tag{7.2}$$

PROOF. The results of all the previous Sections allow to easily verify that the assumptions (c_1)–(c_9) are fulfilled by the operators P and Q. For the sake of conciseness we refer to [19] to see how the conditions on the multidimensional Malkin bifurcation function are used in the proof. □

REMARK 7.2. We recall that in the case when $m=1$ the corresponding bifurcation result for (2.1) was obtained in [7].

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