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COINCIDENCE OF MAPS ON TORUS FIBRE BUNDLES OVER THE CIRCLE

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Abstract. The main purpose of this work is to study coincidences of fibre-preserving self-maps over the circle S^1 for spaces which are fibre bundles over S^1 and the fibre is the torus T. We classify all pairs of self-maps over S^1 which can be deformed fibrewise to a pair of coincidence free maps.

1. Introduction

Given a fibration $M \xrightarrow{p} S^1$ and fibre-preserving maps $f,g\colon M \to M$ over S^1, the question is if the pair (f,g) can be deformed by fibrewise homotopy over S^1 to a coincidence free pair (f',g').

This problem was motivated by the case in that $f = \mathrm{Id}$, and in this case, the question is if the map g can be deformed by fibrewise homotopy over S^1 to a fixed point free map g', which has been considered by many authors, among them see [4], [6], [8] and [9].

Let us consider fibre-preserving maps $f,g\colon M \to M$, where M is a fibre bundle over the circle S^1 and the fibre is a closed surface S. These fibre bundles are obtained from the space $S \times [0,1]$ by identifying the points $(x,0)$ with the points $(\phi(x),1)$, where ϕ is a homeomorphism of the surface S. The cases when $f = \mathrm{Id}$ and the fibre S is either the torus T or the Klein bottle K, were completely solved in [8] and [9], respectively.

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In this work, we consider the fibre $S = T$. We denote the total space by $M(\phi)$. We investigate when given fibre-preserving maps $f, g\colon M(\phi) \to M(\phi)$ over S^1, i.e. $p \circ f = p$ and $p \circ g = p$, the pair (f, g) can be deformed by fibrewise homotopy over S^1 to a coincidence free pair (f', g').

The set of homotopy classes of the pairs (f, g) such that $(f|_T, g|_T)$ can be deformed to a coincidence free pair is given by Theorem 3.6.

This paper is organized into four sections. In Section 2 we prove that our problem is equivalent to the existence of a section. This is given by Theorem 2.2. We show that to find this section it is equivalent to find a lifting in an algebraic diagram. This is the Proposition 2.10. We also present some results on the torus T and fibre bundles over S^1 and fibre T. These results include the Nielsen number of a pair of maps of the torus and the fundamental group of the spaces $M(\phi)$, $M(\phi) \times_{S^1} M(\phi)$ and $M(\phi) \times_{S^1} M(\phi) \setminus \Delta$ where Δ is the diagonal in $M(\phi) \times_{S^1} M(\phi)$, which is the pullback of $p\colon M(\phi) \to M(\phi)$ by $p\colon M(\phi) \to M(\phi)$.

In Section 3 we classify all T-bundles over S^1. This is the Proposition 3.4. We also obtain a presentation for the fundamental groups of $M(\phi)$, $M(\phi) \times_{S^1} M(\phi)$ and $M(\phi) \times_{S^1} M(\phi) \setminus \Delta$.

In Section 4, we present a necessary and sufficient condition for the existence of the lifting in the diagram

$$\begin{tikzcd}
 & \pi_1(\mathcal{F}) \simeq \pi_2(T, T \setminus 1) \arrow[d] \\
 & \pi_1(E_{S^1}(M(\phi))) \simeq \pi_1(M(\phi) \times_{S^1} M(\phi) \setminus \Delta) \arrow[d, "q_\#"] \\
\pi_1(M(\phi)) \arrow[r, "(f,g)_\#"'] \arrow[ur, dashed, "\psi"] & \pi_1(M(\phi) \times_{S^1} M(\phi))
\end{tikzcd}$$

with base points suitable. These conditions are related to existence of solutions of a system of equations involving the presentation of the groups above.

In Section 5, we classify all the pairs of maps (f, g), which can be deformed, by a fibrewise homotopy over S^1, to a pair of coincidence free maps (f', g'), which is Theorem 5.1.

2. Preliminary and general results

2.1. Coincidence theory. Let $f, g\colon X \to Y$ be maps between finite CW-complexes. Denote by $\mathrm{Coin}(f, g) = \{x \in X \mid f(x) = g(x)\}$.

Suppose that x_1, x_2 are in $\mathrm{Coin}(f, g)$. Then we say that x_1, x_2 are Nielsen equivalent according to f and g if there exists a path $\sigma\colon [0, 1] \to X$ such that $\sigma(0) = x_1$, $\sigma(1) = x_2$ and $f \circ \sigma$ is homotopic to $g \circ \sigma$ relative to end points.

We have that the above relation is an equivalence relation. So the set $\mathrm{Coin}(f,g)$ is partioned in equivalence classes of this relation, called coincidence classes.

A coincidence class $\mathcal{F}$ is called essential if given x in $\mathcal{F}$ and homotopies $\{f_t\}$, $\{g_t\}$ of $f=f_0$ and $g=g_0$ there exist x' in $\mathrm{Coin}(f_1,g_1)$ and a path $\gamma\colon [0,1]\to X$ with $\gamma(0)=x$, $\gamma(1)=x'$ such that $f_t\circ\gamma$ is homotopic to $g_t\circ\gamma$ relative to end points.

The coincidence Nielsen number $N(f,g)$ of f and g is defined as the number of essential coincidence classes. We have that $N(f,g)$ is a homotopic invariant, finite, and is a lower bound for the set $\mathrm{Coin}(f',g')$ of each pair maps f',g' homotopic to f and g, respectively. For more details see [7] and [18].

2.2. Presentation of groups. Let A and G be groups with presentations given by $A=\langle Y|S\rangle$ and $G=\langle X,R\rangle$. Let

$$1\longrightarrow A\xrightarrow{l}\widetilde{G}\xrightarrow{\mu}G\longrightarrow 1$$

be a fixed extension of G by A.

Let us denote $\widetilde{Y}=\{l(y)\mid y\in Y\}$. Let $\widetilde{S}=\{\widetilde{s}\mid s\in S\}$ be the set of words obtained of S by changing y to $\widetilde{y}=l(y)$. For each $x\in X$ we choose $\widetilde{x}\in\widetilde{G}$ such that $\mu(\widetilde{x})=x$. Take $\widetilde{X}=\{\widetilde{x}\mid x\in X\}$.

We also consider for each $r\in R$ the word $\widetilde{r}$ in $\widetilde{X}$ obtained from r substituting x by $\widetilde{x}$. Now, each $\widetilde{r}$ is annulated by μ, because by the hypothesis $\mu(\widetilde{x})=x$. Therefore, for each $r\in R$ we have $\widetilde{r}\in\ker(\mu)=\mathrm{Im}\, l$. Since $\mathrm{Im}\, l$ is generated by the set $\widetilde{Y}$, each $\widetilde{r}$ can be written as a word, namely μ_r, in $\widetilde{y}$. Let us denote $\widetilde{R}=\{\widetilde{r}\mu_r^{-1}|r\in R\}$. Since $\mathrm{Im}\, l$ is a normal subgroup of $\widetilde{G}$, each conjugate $\widetilde{x}\widetilde{y}\widetilde{x}^{-1}$, where $\widetilde{x}\in\widetilde{X}$ and $\widetilde{y}\in\widetilde{Y}$, belongs to $\mathrm{Im}\, l$. Therefore $\widetilde{x}\widetilde{y}\widetilde{x}^{-1}$ is a word w_{xy} in $\widetilde{y}$. Let us denote $\widetilde{T}=\{\widetilde{x}\widetilde{y}\widetilde{x}^{-1}w_{xy}^{-1}\mid x\in X \text{ and } y\in Y\}$. With the above notation we have

Theorem 2.1. *The group* $\widetilde{G}=\langle\widetilde{X},\widetilde{Y}|\widetilde{R},\widetilde{S},\widetilde{T}\rangle$.

The proof of this theorem can be found in [11, Chapter 13].

2.3. The general problem. Let (F_i,M_i,B,p_i) be fibre bundles and let $f,g\colon M_1\to M_2$ be fibre-preserving maps over B, i.e. $p_2\circ f=p_1$ and $p_2\circ g=p_1$. When the pair (f,g) is deformable over B to a pair of coincidence free maps (f',g') by a fibrewise homotopy over B?

We will give a formulation for the problem through a geometric diagram. Now we define some spaces which are used in this work.

Let $M_2\times_B M_2=\{(x,y)\in M\times M\mid p_2(x)=p_2(y)\}$ be the pullback of $p_2\colon M_2\to B$ by $p_2\colon M_2\to B$. By [4] the inclusion $M_2\times_B M_2\setminus\Delta\hookrightarrow M_2\times_B M_2$, where Δ is the diagonal in $X=M_2\times_B M_2$, is replaced by the fibration $\mathcal{F}\to E_B(M_2)\xrightarrow{q} M_2\times_B M_2$, whose fibre is denoted by $\mathcal{F}$, $E_B(M_2)=\{(x,w)\in$

$A \times X^I \mid i(x) = w(0)\}$ has the same homotopy type from $A = M_2 \times_B M_2 \setminus \Delta$ and q is given by $q(x, w) = w(1)$.

The next theorem gives an equivalent condition to our problem.

THEOREM 2.2. *The pair of maps (f, g) over B can be deformed to a coincidence free pair (f', g') by fibrewise homotopy over B if and only if there exists a section σ in the diagram*

$$\begin{tikzcd} & \mathcal{F} \arrow[d] & \mathcal{F} \arrow[d] \\ & E_B(f,g) \arrow[r, "\overline{q}_{(f,g)}"] \arrow[d, "q_{(f,g)}"] & E_B(M_2) \arrow[d, "q"] \\ M_1 \arrow[r, "1"'] \arrow[ur, dashed, "\sigma"] & M_1 \arrow[r, "(f,g)"'] & M_2 \times_B M_2 \end{tikzcd} \tag{2.1}$$

where $q_{(f,g)}\colon E_B(f, g) \to M_1$ is the induced fibration from q by (f, g).

PROOF. If exists σ on the above diagram, then we have a map $\theta\colon M_1 \to E_B(M_2)$ given by $\theta = \overline{q}_{(f,g)} \circ \sigma$. Hence the map $H\colon M_1 \times I \to M_2 \times_B M_2$ given by $H(x, s) = \theta_2(x)(s)$ gives a fibrewise homotopy over B between a coincidence free pair and (f, g). Here $\theta(x) = (\theta_1(x), \theta_2(x))$.

Suppose that there exists a homotopy $H\colon M_1 \times [0, 1] \to M_2 \times_B M_2$ such that $H_1 = (f, g)$ and $H_0 = (f', g')$ where $f'(x) \neq g'(x)$, for all $x \in M_1$. We have that $G_0(x) = (H_0(x), c_{H_0(x)})$ belongs to $(M_2 \times_B M_2 \setminus \Delta) \times (M_2 \times_B M_2)^I$ where $c_{H_0(x)}$ is the constant path in $M_2 \times_B M_2 \setminus \Delta \subset M_2 \times_B M_2$ given by $c_{H_0(x)}(t) = H_0(x)$ and $q \circ G_0(x) = c_{H_0(x)}(1) = H_0(x)$. Since q is a fibration, then there exists a homotopy $G\colon M_1 \times [0, 1] \to E_B(M_2)$ which is the lifting of H. We define $\sigma(x) = (x, G_1(x))$. Now $\sigma(x)$ belongs to $E_B(f, g)$ because $q \circ G_1(x) = H_1(x) = (f(x), g(x))$. $\square$

The following proposition relates our problem with a geometric diagram.

PROPOSITION 2.3. *With the above notation we have that the pair of maps (f, g) can be deformed to a coincidence free pair (f', g') by fibrewise homotopy over B if and only if there exists a map $h\colon M_1 \to M_2 \times_B M_2 \setminus \Delta$ which makes the diagram*

$$\begin{tikzcd} & M_2 \times_B M_2 \setminus \Delta \arrow[d, "i"] \\ M_1 \arrow[r, "(f,g)"'] \arrow[ur, dashed, "h"] & M_2 \times_B M_2 \end{tikzcd}$$

homotopy commutative.

Proof. Suppose that there exists a homotopy $H\colon M_1 \times [0,1] \to M_2 \times_B M_2$ such that $p_2 \circ H_t(x) = p_2(x)$, for all $x \in M_1$, for all $t \in [0,1]$; $H_0(x) = (f(x), g(x))$ and $H_1(x) = (f'(x), g'(x))$, where $f'(x) \neq g'(x)$, for all $x \in M_1$.

We define $h\colon M_1 \to M_2 \times_B M_2 \setminus \Delta$ by $h(x) = (f'(x), g'(x))$. Since $p_2 \circ f'(x) = p_1(x)$, $p_2 \circ g'(x) = p_1(x)$ and $f'(x) \neq g'(x)$, for all $x \in M_1$, then $h(x) \in M_2 \times_B M_2 \setminus \Delta$ and $i \circ h$ is homotopic to (f,g), i.e. the diagram is homotopy commutative.

Let $H\colon M_1 \times [0,1] \to M_2 \times_B M_2$ be the homotopy making the diagram homotopy commutative: $H_0 = (f,g)$, $H_1 = ih$. Now H is a fibrewise homotopy between (f,g) and a coincidence free pair. $\square$

Remark 2.4. (a) The fibre $\mathcal{F}$ has homotopy groups

$$\pi_{j-1}(\mathcal{F}) \cong \pi_j(M_2 \times_B M_2, M_2 \times_B M_2 \setminus \Delta) = \pi_j(X, A)$$

(see [5]).

(b) If the fibre F_2 and the spaces M_2 and B are closed manifolds, $\pi_{j-1}(\mathcal{F}) \cong \pi_j(X,A) \cong \pi_j(F_2, F_2 \setminus x)$ where x is a point in F_2 (see [4, Proposition 2.1, p. 53]).

(c) Under conditions (b), if $\dim F_2 = k > 2$, the classical obstruction theory can be used to find obstructions. The primary obstruction occurs to extend the section from $(k-1)$-skeleton to the k-skeleton of M_2 and this obstruction $\mathcal{O}_B(f,g) \in H^k(M_2; \{\pi_{k-1}(\mathcal{F})\})$. There may be other obstructions and the essential reason to apply the obstruction theory is that $\pi_1(M_2)$ acts in $\pi_{k-1}(\mathcal{F})$ independently of the base point of $\mathcal{F}$, because $\pi_1(\mathcal{F}) = 0$ (see [4, Proposition 2.2, p. 54]).

(d) Under conditions (b), when the dimension of the fibre is 2 and $F_2 \neq S^2$, $\mathbb{R}P^2$, there exists a well defined coefficient local system given by the action of $\pi_1(M_2)$ in $H_1(\mathcal{F})$, and so we can define the abelianized obstruction $\mathcal{A}_B(f,g) \in H^2(M_2; \{H_1(\mathcal{F})\})$ (see [15]).

(e) When the fibre is the sphere S^2, then $\pi_2(S^2, S^2 \setminus x)$ is isomorphic to $\mathbb{Z}$ and the obstruction theory is applied because $\pi_1(M_2)$ acts in $\pi_1(\mathcal{F}) \simeq \mathbb{Z}$ independently of the base point of $\mathcal{F}$.

2.4. Torus fibre bundle over S^1. In this subsection $M_1 = M_2 = M$ is a torus-bundle over S^1 and we will obtain types of torus-bundle over S^1 using homeomorphism of T.

Let $\phi\colon T \to T$ be a homeomorphism which has one fixed point denoted by x_0. Without loss of generality we can assume this hypothesis because if $\phi\colon T \to T$ is a homeomorphism with $\phi(x_1) = y_1$, then it follows from [16, Lemma 5.4, chapter 5] that there exists a homeomorphism $h\colon T \to T$ isotopic to the identity Id such that $h(y_1) = x_1$.

Let $H\colon T \times [0,1] \to T$ be isotopy between h and Id with $H_0 = h$ and $H_1 = \mathrm{Id}$. Defining $G\colon T \times [0,1] \to T$ by $G_t(x) = H_t(\phi(x))$ we have that G is an isotopy between $h \circ \phi$ and ϕ. We observe that $h \circ \phi(x_1) = h(y_1) = x_1$. Therefore every homeomorphism $\phi\colon T \to T$ is isotopic to a homeomorphism preserving base point.

We denote by $M(\phi)$ the quotient space obtained from $T \times [0,1]$ where we identify $(x,0)$ with $(\phi(x),1)$. The elements of $M(\phi)$ we denote by $\langle x,t\rangle$.

We have that $T \to M(\phi) \xrightarrow{p} S^1 = I/_{0\sim 1}$ is a trivial locally fibre bundle where p is the projection given by $p(\langle x,t\rangle) = \langle t\rangle$.

Proposition 2.5. *Let $\phi_1, \phi_2\colon T \to T$ be two homeomorphisms. Then $M(\phi_1)$ is homeomorphic to $M(\phi_2)$ by a fibre-preserving homeomorphism over S^1 if and only if ϕ_1 is isotopic to a conjugate of ϕ_2.*

Proof. Suppose that ϕ_1 and $h \circ \phi_2 \circ h^{-1}$ are isotopic. So we have a map $G\colon T \times I \to T$ such that $G(\,\cdot\,,0) = \phi_1$ and $G(\,\cdot\,,1) = h \circ \phi_2 \circ h^{-1}$. Let $G'(\,\cdot\,,t) = h^{-1} \circ G(\,\cdot\,,t),\ t \in \{0,1\}$. We have

$$G'(\,\cdot\,,1) \circ \phi_1 = h^{-1} \circ h \circ \phi_2 \circ h^{-1} \circ \phi_1 = \phi_2 \circ h^{-1} \circ \phi_1 = \phi_2 \circ G'(\,\cdot\,,0).$$

Hence we have a homeomorphism over S^1 between $M(\phi_1)$ and $M(\phi_2)$ given by $\langle x,t\rangle \to \langle G'(x,t),t\rangle$. For the converse suppose that there exists a fibre-preserving homeomorphism over S^1 which we will denote by $h\colon M(\phi_1) \to M(\phi_2)$. Then $h\langle x,t\rangle = \langle h_1(x,t),t\rangle$ and $h_1(\,\cdot\,,1)\circ\phi_1 = \phi_2\circ h_1(\,\cdot\,,0)$. We define $G\colon T\times I \to T$ by $G(x,t) = h_1(x,1)^{-1}\circ\phi_2\circ h_1(x,t)$. Then $G(x,0) = h_1(x,1)^{-1}\circ\phi_2 h_1(x,0) = \phi_1(x)$ and $G(x,1) = h_1(x,1)^{-1}\circ\phi_2 h_1(x,1)$ and so ϕ_1 is isotopic to a conjugate of ϕ_2.$\square$

Corollary 2.6. *The classes of T-bundles over S^1 are classified by the conjugacy classes of isotopic classes of homeomorphism which preserve base point.*

Proof. As observed above we have that every homeomorphism $\phi\colon T \to T$ is isotopic to a homeomorphism which is base point preserving. So from Proposition 2.5 the result follows. $\square$

We use some homeomorphisms of the torus to describe all T-bundles over S^1. We also use the group structure of the torus T in order to simplify the analysis of our algebraic problem.

Let T be defined as the quotient space $\mathbb{R}\times\mathbb{R}/\mathbb{Z}\times\mathbb{Z} \simeq T$ and we denote by $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ the elements of $\mathbb{R}\times\mathbb{R}$ and T, respectively.

Let ϕ be a homeomorphism of T induced by a linear operator in $\mathbb{R}^2$ that preserves $\mathbb{Z}\times\mathbb{Z}$. We identify ϕ with the matrix of a linear operator with integer coefficients and determinant either 1 or -1.

Then $M(\phi)$ is the quotient space of $T\times[0,1]$, where we are identifying $\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right],0\right)$ with $\left(\left[\phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right],1\right)$.

The class of the element $\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right],t\right)$ in the quotient is denoted by $\left\langle\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right],t\right\rangle$.

As observed above, the space $M(\phi)$ is a fibre bundle over the circle S^1, where the fibre is the torus T and the projection map $p\colon M(\phi) \to S^1$, is given by

$$p\left(\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, t \right\rangle\right) = \langle t \rangle \in [0,1]/0 \sim 1 \simeq S^1.$$

2.5. The algebraic problem. In this subsection we will show that the existence of a section over the 2-skeleton gives a deformation to a coincidence free map. This will follow from the fact that the fibre (= torus) is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$ space.

We have that $M(\phi)$ is a CW-complex because it is a quotient space of the CW-complex, $T \times [0,1]$ by the CW-subcomplex, $\mathcal{Q}$, given by $\mathcal{Q} = \{(x,0) \sim (\phi(x),1) \mid x \in T\}$.

If $f\colon M(\phi) \to M(\phi)$ is a map over S^1, we define $f_0\colon T \to T$ by $f_0(x) = y$ if $f(\langle x,0 \rangle) = \langle y,0 \rangle$. This map is well defined since $\langle y_1,0 \rangle = \langle y_2,0 \rangle$ if and only if $y_1 = y_2$.

Proposition 2.7. *The map f_0 satisfies the condition that $\phi \circ f_0 \circ \phi^{-1}$ is homotopic to f_0. Conversely, if $f_0\colon T \to T$ is a map which satisfies the condition that $\phi \circ f_0 \circ \phi^{-1}$ is homotopic to f_0, then there exists a map $f\colon M(\phi) \to M(\phi)$ over S^1 such that f restricted to the fibre is f_0.*

Proof. Define $f_1(x) = y$ if $f(\langle x,1 \rangle) = \langle y,1 \rangle$. Since $\langle x,0 \rangle = \langle \phi(x),1 \rangle$, it follows that $\langle f_0(x),0 \rangle = \langle f_1 \circ \phi(x),1 \rangle$, which implies that $f_1 = \phi \circ f_0 \circ \phi^{-1}$. Now we observe that if $t \notin \{0,1\}$ then $f(\langle x,t \rangle) = \langle g(x,t),t \rangle$. Extending g to a map $\overline{g}\colon T \times I \to T$, by continuity, we have $\overline{g}(x,0) = f_0(x)$ and $\overline{g}(x,1) = f_1(x)$, and the first part follows. For the converse, we define $f\colon M(\phi) \to M(\phi)$ by $f(\langle x,t \rangle) = \langle H(x,t),t \rangle$, where H is the homotopy between f_0 and $\phi \circ f_0 \circ \phi^{-1}$. □

Proposition 2.8. *If $M = M(\phi)$, then there exists a cross section σ (see diagram (2.1)) over M if and only if it exists over the 2-skeleton of M.*

Proof. Basically this follows from the fact that in these conditions the theoretical fibre has homotopy groups equal to zero, except at level 1 (see Remarks 2.4(b)). Therefore, for the construction of the cross section according to Theorem 2.2, once constructed in the 2-skeleton, all the other obstructions are equal to zero. □

For the next proposition we are going to need Theorem 4.3.1 from [1, p. 265], which says:

Theorem 2.9 (Criterion for 2-extendability). *Let (X, L) be a relative* CW-*complex, and $p\colon \widetilde{X} \to X$ a fibration with fibre $F \subset \widetilde{X}$. Further, let X, L and F be path-connected. A section $u\colon L \to \widetilde{L}$, where $\widetilde{L} = p^{-1}(L)$, can be extended to* *a section u' over the 2-skeleton $X_2 = L \cup X^2$ exactly when $i_\#\colon \pi_1(F) \to \pi_1(\widetilde{X})$ is injective and when there is a homomorphism θ making the diagram*

$$\begin{tikzcd}
\pi_1(\widetilde{L}) \arrow[r] & \pi_1(\widetilde{X}) \arrow[dr, "p_\#"] & \\
\pi_1(L) \arrow[u, "u_\#"] \arrow[r] & \pi_1(X) \arrow[u, "\theta"] \arrow[r, "\mathrm{Id}"'] & \pi_1(X)
\end{tikzcd}$$

commutative. We can take such u' that $u'_\# = \theta$.

Given $q\colon E \to Y$ a fibration with fibre F path-connected and a map $f\colon X \to Y$, then construct the geometric pullback $E^* = \{(x,y) \in X \times E \mid f(x) = q(y)\}$

$$\begin{tikzcd}
E^* \arrow[r, "q_2"] \arrow[d, "q_1"'] & E \arrow[d, "q"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We also can construct the algebraic pullback

$$\pi_1(X) \sqcup \pi_1(E) = \{(\alpha,\beta) \in \pi_1(X) \times \pi_1(E) \mid f_\#(\alpha) = q_\#(\beta)\}.$$

We observe that if i_1, given in the homotopy exact sequence of the fibration q

$$\cdots \longrightarrow \pi_1(F) \xrightarrow{i_1} \pi_1(E) \xrightarrow{q_\#} \pi_1(Y) \longrightarrow \cdots$$

is injective, then $\pi_1(E^*)$ is isomorphic to $\pi_1(X) \sqcup \pi_1(E)$ because we have the following diagram

$$\begin{tikzcd}
\pi_1(F) \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_1(E^*) \arrow[r, "q_{1\#}"] \arrow[d, "(q_{1\#},q_{2\#})"', "\approx"] & \pi_1(X) \arrow[d, "\mathrm{Id}"] \\
\pi_1(F) \arrow[r, "i_2"'] & \pi_1(X) \sqcup \pi_1(E) \arrow[r, "p_1"'] & \pi_1(X)
\end{tikzcd}$$

commutative, where $i_2(\beta) = (1, i_1(\beta))$ and $p_1(\alpha,\beta) = \alpha$.

The following proposition reduces our problem to the existence of a lifting in the corresponding diagram of fundamental groups.

Proposition 2.10. *There is a cross section σ (see diagram (2.1)) over the 2-skeleton of $M(\phi)$ if and only if the following diagram of fundamental groups, admits a lifting ψ:*

$$
\begin{tikzcd}
 & \pi_1(\mathcal{F}) \simeq \pi_2(T, T \setminus 1) \arrow[d] \\
 & \pi_1(E_{S^1}(M(\phi))) \simeq \pi_1(M(\phi) \times_{S^1} M(\phi) \setminus \Delta) \arrow[d, "q_\#"] \\
\pi_1(M(\phi)) \arrow[r, "(f,g)_\#"'] \arrow[ur, dashed, "\psi"] & \pi_1(M(\phi) \times_{S^1} M(\phi))
\end{tikzcd}
\tag{2.2}
$$

Proof. First we observe that there exists a homomorphism ψ which makes the diagram of fundamental groups commutative if and only if there exists a homomorphism θ which makes the diagram

$$
\begin{tikzcd}
 & \pi_1(E_{S^1}(f,g)) \arrow[rr, "\overline{q}_{(f,g)\#}"] \arrow[d, "q_{(f,g)\#}"] & & \pi_1(E_{S^1}(M(\phi))) \arrow[d, "q_\#"] \\
\pi_1(M(\phi)) \arrow[r, "\mathrm{Id}"'] \arrow[ur, dashed, "\theta"] & \pi_1(M(\phi)) \arrow[rr, "(f,g)_\#"'] \arrow[urr, dashed, "\psi"] & & \pi_1(M(\phi)) \times_{S^1} \pi_1(M(\phi))
\end{tikzcd}
$$

commutative, because if there exists θ, then it is sufficient to define $\psi = \overline{q}_{(f,g)\#} \circ \theta$. As observed before Proposition 2.10 we have that $\pi_1(E_{S^1}(f,g))$ is isomorphic to $\pi_1(M(\phi)) \sqcup \pi_1(E_{S^1}(M(\phi)))$ and therefore if there exists ψ it is sufficient to define $\theta(x) = (x, \psi(x))$.

Now we suppose that there exists a cross section σ in the diagram (2.1), so there exists $\theta = \sigma_\#$ in the diagram above and therefore there exists ψ.

Now, if the diagram (2.2) of fundamental groups, admits a lifting ψ, by the remark above there exists a homomorphism θ which makes the diagram commutative. So, from the theorem (criterion for 2-extendibility) there exists a cross section σ in the diagram (2.1). $\square$

We remark that our problem is equivalent to an algebraic problem given by Proposition 2.10. In this way, we should compute the homomorphisms and groups in the diagram (2.2).

We consider $M(\phi) \times_{S^1} M(\phi)$ the pullback of $p\colon M(\phi) \to S^1$ by $p\colon M(\phi) \to S^1$ and $p_i\colon M(\phi) \times_{S^1} M(\phi) \to M(\phi)$, $i = 1, 2$, the projections on the first and second coordinates, respectively.

It is easy to see that each element of $M(\phi) \times_{S^1} M(\phi)$ is represented by $(\langle x, t\rangle, \langle y, t\rangle)$ where $x, y \in T$.

For the calculation of the groups given in the diagram (2.2) we will reproduce Propositions 1.7–1.9 from [8, pp. 5–6]:

Proposition 2.11. *The fundamental group $\pi_1(M(\phi), \langle x_0, 0\rangle)$ is isomorphic to the semi-direct product $\pi_1(T) \rtimes \mathbb{Z}$. Further the action $\mathbb{Z} \xrightarrow{\Gamma} \mathrm{Aut}(\Pi_1(T))$*

which comes from the section $s_0\colon S^1 \to M(\phi)$ *defined by* $s_0(t) = \langle x_0, t\rangle$*, is given by* $c\cdot\alpha = c\alpha c^{-1} = \phi_\#(\alpha)$*, where* $c = p_\#\langle s_0\rangle$ *is the generator of* $\pi_1(S^1)$*.*

PROOF. From the homotopy long exact sequence of the fibration, we have a short exact sequence $1 \to \pi_1(T) \to \pi_1(M(\phi), \langle x_0, 0\rangle) \to \pi_1(S^1) \to 1$, which splits because $\mathbb{Z}$ is free. Hence $\pi_1(M(\phi), \langle x_0, 0\rangle)$ is isomorphic to the semi-direct product $\pi_1(T) \rtimes \mathbb{Z}$. Further $c\alpha c^{-1} = \phi_\#(\alpha)$ because the class of the loop $s_0(t) = \langle x_0, t\rangle$ is projected by $p_\#$ in the generator c of $\pi_1(S^1)$ and the juxtaposed loop $s_0\langle\phi\circ\alpha, 1\rangle s_0^{-1}$ is homotopic to the loop $\langle\phi\circ\alpha, 0\rangle$. In the quotient space $M(\phi)$, this leads to $c\cdot\alpha = c\alpha c^{-1} = \phi_\#(\alpha)$, and the result follows. $\square$

Let us denote $\mathbf{0} = \left\langle \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], 0\right\rangle$ and $\mathbf{q} = \left\langle \left[\begin{smallmatrix}q\\q\end{smallmatrix}\right], 0\right\rangle$ elements of $M(\phi)$. Then we have the following

PROPOSITION 2.12. *The fundamental group* $\pi_1(M(\phi)\times_{S^1} M(\phi), (\mathbf{0}, \mathbf{q}))$ *is isomorphic to the semi-direct product* $\pi_1(T) \rtimes \pi_1(M(\phi), \mathbf{q})$*. Further, the action of* $\pi_1(M(\phi), \mathbf{q})$ *on* $\mathrm{Aut}(\pi_1(T))$*, which comes from the section* $s_1\colon \pi_1(M(\phi), \mathbf{q}) \to \pi_1(M(\phi)\times_{S^1} M(\phi), (\mathbf{0}, \mathbf{q}))$*, where* $s_1 = (s_0\circ p, 1_{M(\phi)})_\#$*, is given by* $\beta\cdot\alpha = \beta\alpha\beta^{-1} = p_\#(\beta)\cdot\alpha$*. The last action is the one which comes from the bundle* $p\colon (M(\phi), \mathbf{q}) \to S^1$*, i.e. the action is given by the following composition:*

$$\pi_1(M(\phi), \mathbf{q}) \xrightarrow{p_\#} \pi_1(S^1) \xrightarrow{\Gamma} \mathrm{Aut}(\pi_1(T))$$

where if we denote by c *the generator of* $\Pi_1(S^1)$ *then* $\Gamma(c) = \phi_\#$ *, so that if* $p_\#(\beta) = c^k$ *then* $p_\#(\beta)\cdot\alpha = \phi_\#^k(\alpha)$*.*

PROOF. We have that $\pi_1(M(\phi)\times_{S^1} M(\phi), (\mathbf{0}, \mathbf{q}))$ is isomorphic to the semi-direct product $\pi_1(T) \rtimes \pi_1(M(\phi), \mathbf{q})$, because the short exact sequence

$$1 \longrightarrow \pi_1(T) \xrightarrow{i_{1\#}} \pi_1(M(\phi)\times_{S^1} M(\phi), (\mathbf{0}, \mathbf{q})) \xrightarrow{p_{2\#}} \pi_1(M(\phi), \mathbf{q}) \longrightarrow 1$$

splits and the homomorphism $s_1 = (s_0\circ p, 1_{M(\phi)})_\#$ is a section and the isomorphism $\phi_1\colon \pi_1(M(\phi)\times_{S^1} M(\phi), (\mathbf{0}, \mathbf{q})) \to \pi_1(T) \rtimes \pi_1(M(\phi), \mathbf{q})$ is given by $\phi_1(\gamma) = (i_{1\#}^{-1}(\gamma(s_1\circ p_{2\#})(\gamma^{-1})), p_{2\#}(\gamma))$. $\square$

To calculate $\pi_1(M(\phi)\times_{S^1} M(\phi)\setminus\Delta)$, define a fibre bundle homeomorphism $h\colon M(\phi)\times_{S^1} M(\phi) \to M(\phi)\times_{S^1} M(\phi)$ over $M(\phi)$ by the formula

$$h\left(\left\langle\begin{bmatrix}x\\y\end{bmatrix}, t\right\rangle, \left\langle\begin{bmatrix}x'\\y'\end{bmatrix}, t\right\rangle\right) = \left(\left\langle\begin{bmatrix}x\\y\end{bmatrix}, t\right\rangle, \left\langle\begin{bmatrix}x'\\y'\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix}, t\right\rangle\right).$$

The range of the subspace $M(\phi)\times_{S^1} M(\phi)\setminus\Delta$ by the homeomorphism h is $M(\phi)\times_{S^1}(M(\phi)\setminus S^1)$. The last space is the total space of the pullback of $p\colon M(\phi)\to S^1$ by $p_{|_{M(\phi)\setminus S^1}}\colon M(\phi)\setminus S^1 \to S^1$, and the circle S^1 in $M(\phi)$ is the image of $c_0(t) = \left\langle \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], t \right\rangle$ for all $t \in [0,1]$. So, we have the commutative diagram:

(2.3)
$$\begin{tikzcd}
 & & & T\times T-\Delta \arrow[r] \arrow[dll] \arrow[dd] & & T\times(T\backslash 1) \arrow[dll] \arrow[dd] \\
T \arrow[r] \arrow[dd] & T\times T \arrow[rr] \arrow[dd] & & T\times T \arrow[dd] & & \\
 & & & M(\phi)\times_{S^1} M(\phi)\backslash\Delta \arrow[r] \arrow[dll, "\iota"'] & & M(\phi)\times_{S^1}(M(\phi)\backslash S^1) \arrow[dll, "(1,i)"] \\
M(\phi) \arrow[r, "(f,g)"'] \arrow[urrrrr, dashed, "\gamma"] & M(\phi)\times_{S^1} M(\phi) \arrow[rr, "h"'] & & M(\phi)\times_{S^1} M(\phi) & &
\end{tikzcd}$$

Therefore, the problem of existence of the section σ in the diagram (2.1) is equivalent to the existence of a lifting γ.

Let S^1 be the subset of $M(\phi)$ given by the elements $\left\langle \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], t \right\rangle$, $0 \leq t \leq 1$. So we have the fibre bundles $p_|\colon (M(\phi) \setminus S^1, \mathbf{q}) \to S^1$, where the fibre is $T - \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Another useful space is the pullback $(M(\phi)\times_{S^1}(M(\phi)\setminus S^1))$, which fundamental group is given by

PROPOSITION 2.13. *The fundamental group $\pi_1(M(\phi)\times_{S^1}(M(\phi)\setminus S^1), (\mathbf{0},\mathbf{q}))$ is isomorphic to the semi-direct product $\pi_1(T) \rtimes \pi_1(M(\phi) \setminus S^1, \mathbf{q})$. Further, the action of $\pi_1(M(\phi)\setminus S^1, \mathbf{q})$ on $\mathrm{Aut}(\pi_1(T))$ is given by $\beta\cdot\alpha = \beta\alpha\beta^{-1} = p_{\#}(\beta)\cdot\alpha$, where the last action is the one which comes from the bundle $p_|\colon M(\phi)\setminus S^1 \to S^1$ as in Proposition* 2.12.

PROOF. We proceed similarly to the proof of Proposition 2.12. In this situation the fibration provides the short exact sequence

$$0 \longrightarrow \pi_1(T) \xrightarrow{i_{1\#}} \pi_1(M(\phi)\times_{S^1}(M(\phi)\setminus S^1), (\mathbf{0},\mathbf{q})) \xrightarrow{p_{2\#}} \pi_1(M(\phi)\setminus S^1, \mathbf{q}) \longrightarrow 0$$

and the homomorphism $s_2 = (s_0 \circ p_| \,, 1_{M(\phi)\setminus S^1})_{\#}$ is a section, and we define an isomorphism

$$\Phi_2\colon \pi_1(M(\phi)\times_{S^1}(M(\phi)\setminus S^1), (\mathbf{0},\mathbf{q})) \to \pi_1(T) \rtimes \pi_1(M(\phi)\setminus S^1, \mathbf{q})$$

given by $\Phi_2(\gamma) = (i_{1\#}^{-1}(\gamma(s_2 \circ p_{2\#}(\gamma^{-1}))), p_{2\#}(\gamma))$. $\square$

The above proposition points out the relevancy of knowing $\pi_1(M(\phi)\setminus S^1)$.

PROPOSITION 2.14. *The existence of* γ *in the diagram* (2.3) *is equivalent to the existence of* Γ *in the diagram*

$$\begin{tikzcd} & 1 \arrow[d] \\ & \pi_1(\mathcal{F}(M(\phi)\setminus S^1)) \simeq \pi_2(T, T\setminus 1) \arrow[d] \\ & \pi_1(M(\phi)\setminus S^1) \arrow[d] \\ \pi_1(M(\phi)) \arrow[r] \arrow[ur, dashed, "\Gamma"] & \pi_1(M(\phi)) \arrow[d] \\ & 1 \end{tikzcd} \tag{2.4}$$

where the horizontal map is $(p_2 \circ h \circ (f,g))_{\#} \colon \pi_1(M(\phi)) \to \pi_1(M(\phi))$. *Here* $p_2 \colon M(\phi) \times_{S^1} M(\phi) \to M(\phi)$ *denotes the projection on the second factor.*

PROOF. By finding the same argument as in Proposition 2.10, the existence of γ is equivalent to find a certain homomorphism $\widetilde{\Gamma}$ at the level of the fundamental groups. Since $M(\phi) \times_{S^1} (M(\phi) \setminus S^1)$ is the total space of the pullback of the map p by $p_{|M(\phi)\setminus S^1}$, by the universal property of the pullback and using the isomorphisms ϕ_1 and ϕ_2 in Propositions 2.12 and 2.13 respectively, it is easy to show the equivalence of the existence of the lifting homomorphism in diagram (2.3) and the lifting homomorphism Γ that makes the diagram (2.4) commutative. In fact, given $\widetilde{\Gamma}$, it is sufficient to define $\Gamma(\delta) = p_{2\#}(\widetilde{\Gamma}(\delta))$ and given Γ defines $\widetilde{\Gamma}(\delta) = \Phi_2^{-1}(i_{1\#}^{-1}(p_{2\#}^{-1}(\Gamma(\delta))(s_2 \circ \Gamma(\delta^{-1}))), \Gamma(\delta))$. We observe that once $p_{2\#}^{-1}(\Gamma(\delta))$, $p_{2\#}(p_{2\#}^{-1}(\Gamma(\delta))(s_2 \circ \Gamma(\delta^{-1}))) = 1$ is choosen, then $i_{1\#}^{-1}(p_{2\#}^{-1}(\Gamma(\delta))(s_2 \circ \Gamma(\delta^{-1})))$ is uniquely determined. Thus $\widetilde{\Gamma}(\delta)$ is well defined. $\square$

3. Reductions on the torus

3.1. The generators of $\pi_1(T)$ and the Nielsen number of the pair (f,g), where $f,g \colon T \to T$. Let $T = S^1 \times S^1$ be the torus. Let us consider in $\mathbb{R}^2$ the equivalence relation $(x_1,y_1) \sim (x_2,y_2)$ if $x_1 \equiv x_2 \mod (\mathbb{Z})$ and $y_1 \equiv y_2 \mod (\mathbb{Z})$. The quotient space is T, the equivalence class of $(x,y) \in \mathbb{R}^2$ is denoted by $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T$ and the projection $p \colon \mathbb{R}^2 \to T$ is the universal covering.

We also know that $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ and two maps $f_1, f_2 \colon T \to T$ are homotopic if and only if $f_{1\#} = f_{2\#}$, i.e. the induced homomorphisms in $\pi_1(T)$ are equal.

Once we fix a point, say $1 \in T$, we have that $\pi_1(T,1)$ is a group with two generators a and b under the relation $aba^{-1}b^{-1} = 1$.

We remark that each element of $\pi_1(T,1)$ can be represented by a word $a^m b^n$ where m,n are in $\mathbb{Z}$, because we have the equality $ab=ba$.

Suppose that $f\colon T \to T$ is a continuous map. Then $f_\#\colon \pi_1(T) \to \pi_1(T)$ is a homomorphism of the form $f_\#(a)=a^m b^n$ and $f_\#(b)=a^p b^q$ and if we look at $\pi_1(T)=\mathbb{Z}\oplus\mathbb{Z}$ with generators $a=(1,0)$ and $b=(0,1)$, $f_\#\colon \mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}$ is a homomorphism whose matrix with respect to the canonic basis $\{(1,0),(0,1)\}$ of $\mathbb{Z}\oplus\mathbb{Z}$ is given by $\left(\begin{smallmatrix} m & p \\ n & q\end{smallmatrix}\right)$.

We denote $f_\#$ by $f(m,n,p,q)$ where m,n,p,q are the entries of the above matrix. With this notation, we have:

Theorem 3.1. *If $f_1,f_2\colon T\to T$ are continuous maps with*

$$f_{i\#}=f_i(m_i,n_i,p_i,q_i), \quad i=1,2,$$

then the Nielsen number

$$N(f_1,f_2)=\left|\det\begin{pmatrix} m_1-m_2 & p_1-p_2 \\ n_1-n_2 & q_1-q_2\end{pmatrix}\right|.$$

For the proof see [2, pp. 122–125]. We also have the following proposition:

Proposition 3.2. *If $f_1,f_2\colon T\to T$ are continuous maps such that $N(f_1,f_2)$ $=0$, then we can deform the pair (f_1,f_2) to a coincidence free pair (g_1,g_2).*

Proof. Suppose that $N(f_1,f_2)=0$. Then $\det\left(\begin{smallmatrix} m_1-m_2 & p_1-p_2 \\ n_1-n_2 & q_1-q_2\end{smallmatrix}\right)=0$. We consider maps $g_i\colon \mathbb{R}^2\to\mathbb{R}^2$, $i=1,2$, given by

$$g_i(x,y)=(m_i x+p_i y+\varepsilon_i, n_i x+q_i y)$$

where $\varepsilon_1=0$ and ε_2 is irrational. Then g_i induces a map $\overline{g}_i\colon T\to T$ such that $\overline{g}_i$ is homotopic to f_i because $\overline{g}_i$ induces in $\pi_1(T)$ the same homomorphism that f_i induces.

Now, we are going to calculate the number of coincidences of the pair $(\overline{g}_1,\overline{g}_2)$. For this, it is sufficient to solve the system

$$\begin{aligned} m_1x+p_1y &\equiv m_2x+p_2y+\varepsilon_2 && \mod (\mathbb{Z}),\\ n_1x+q_1y &\equiv n_2x+q_2y && \mod (\mathbb{Z}),\end{aligned}$$

or

$$\begin{aligned} (m_1-m_2)x+(p_1-p_2)y &= \varepsilon_2+k_1 && \text{for some } k_1\in\mathbb{Z},\\ (n_1-n_2)x+(q_1-q_2)y &= k_2 && \text{for some } k_2\in\mathbb{Z}.\end{aligned}$$

Since $N(f_1,f_2)=0$ we have that the rows of the matrix $\left(\begin{smallmatrix} m_1-m_2 & p_1-p_2 \\ n_1-n_2 & q_1-q_2\end{smallmatrix}\right)$ are proportional. Without loss generality, suppose that there exists r such that $r(n_1-n_2,q_1-q_2)=(m_1-m_2,p_1-p_2)$.

Certainly r is a rational number. So $r(n_1 - n_2)x + r(q_1 - q_2)y = rk_2$ and $\varepsilon_2 + k_1 = rk_2$, which is a contradiction because ε_2 is irrational. Therefore $\mathrm{Coin}(\overline{g}_1, \overline{g}_2) = \emptyset$. $\square$

3.2. Reduction of bundles $M(\phi)$ and of maps $M(\phi) \to M(\phi)$. We consider the question raised in Section 2: given fibre-preserving maps $f_i \colon M(\phi) \to M(\phi)$ over S^1, i.e. $p \circ f_i = p$, $i = 1, 2$, when the pair (f_1, f_2) can be deformed to a coincidence free pair, by a fibrewise homotopy over S^1?

From Section 2 we know that our problem is equivalent to finding a lifting to the following algebraic diagram:

$$\begin{tikzcd}
& \pi_1(\mathcal{F}) \simeq \pi_2(T, T \setminus 1) \arrow[d] \\
& \pi_1(E_{S^1}(M(\phi))) \simeq \pi_1(M(\phi) \times_{S^1} M(\phi) \setminus \Delta) \arrow[d, "q_\#"] \\
\pi_1(M(\phi)) \arrow[r, "(f_1, f_2)_\#"'] \arrow[ur, dashed, "\psi"] & \pi_1(M(\phi) \times_{S^1} M(\phi))
\end{tikzcd}$$

with suitable base points. The base point of the domain of f_i, $i = 1, 2$, is $\mathbf{0} = \left\langle \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], 0 \right\rangle$ and we can suppose that $f_1(\mathbf{0}) = \mathbf{0}$ and $f_2(\mathbf{0}) = \mathbf{q}$, where $\mathbf{q} = \left\langle \left[\begin{smallmatrix} q \\ q \end{smallmatrix}\right], 0 \right\rangle$. Otherwise we can replace the map f_2 by a map g_2 homotopic to f_2 which has the property above.

We denote by $B_i = \left(\begin{smallmatrix} m_i & p_i \\ n_i & q_i \end{smallmatrix}\right)$ the matrix of the induced homomorphisms of the restriction of the maps f_i to the fibre T, on the fundamental group.

Proposition 3.3. *Let $f_1, f_2 \colon M(\phi) \to M(\phi)$ be maps such that $(f_{i|_T})_\# = f_{i|_T}(m_i, n_i, p_i, q_i)$, $i = 1, 2$. We suppose that the pair (f_1, f_2) can be deformed, by fibrewise homotopy over S^1, to a pair of coincidence free maps (g_1, g_2). Then, the Nielsen number $N(f_{1|_T}, f_{2|_T})$ of f_1 and f_2 restricted to fibre T is zero and therefore the vectors $(m_1 - m_2, p_1 - p_2)$, $(n_1 - n_2, q_1 - q_2)$ are linearly dependent over $\mathbb{Q}$.*

Proof. If the pair (f_1, f_2) can be deformed, by fibrewise homotopy over S^1, to a pair of coincidence free maps (g_1, g_2), then the pair $(f_{1|_T}, f_{2|_T})$ can be deformed, by fibrewise homotopy over S^1, to a pair of coincidence free maps $(g_{1|_T}, g_{2|_T})$.

If the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is different of zero then $(g_{1|_T}, g_{2|_T})$ must have at least a coincidence point, once it is a deformation of $(f_{1|_T}, f_{2|_T})$. But this is a contradiction. Therefore we must have $N(f_{1|_T}, f_{2|_T}) = 0$.

By Theorem 3.1 we have

$$0 = N(f_{1|_T}, f_{2|_T}) = \left| \det \begin{pmatrix} m_1 - m_2 & p_1 - p_2 \\ n_1 - n_2 & q_1 - q_2 \end{pmatrix} \right|.$$

Therefore the rows of the matrix $\left(\begin{smallmatrix} m_1-m_2 & p_1-p_2 \\ n_1-n_2 & q_1-q_2 \end{smallmatrix}\right)$ are proportional. So, the vectors (m_1-m_2, p_1-p_2), (n_1-n_2, q_1-q_2) are linearly dependent over $\mathbb{Q}$. □

With the notation of the above proposition we observe that if the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is different of zero, then it is not possible to deform the pair (f_1, f_2) to a pair of coincidence free maps (g_1, g_2).

The next proposition provides a relationship between the matrices $\phi = \left(\begin{smallmatrix} a_1 & a_3 \\ a_2 & a_4 \end{smallmatrix}\right)$ and B_i for $i = 1, 2$.

PROPOSITION 3.4.

(a) $\pi_1(M(\phi), \mathbf{0}) = \langle a, b, c_0; [a, b] = 1, c_0 a c_0{}^{-1} = a^{a_1} b^{a_2}, c_0 b c_0{}^{-1} = a^{a_3} b^{a_4} \rangle$.

(b) *B_i commutes with ϕ.*

(c) *If $N(f_{1|T}, f_{2|T}) = 0$ then $B_2 - B_1$ has the eigenvalue 0 and eigenvector $v = (v_1, v_2) \in \mathbb{Z} \times \mathbb{Z}$ associated to 0, such that $\gcd(v_1, v_2) = 1$.*

(d) *If v is an eigenvector of $B_2 - B_1$ associated to 0 then $\phi(v)$ also is an eigenvector of $B_2 - B_1$ associated to 0.*

(e) *Let us denote $w = \phi(v)$. We take the pair v, w if it generates $\mathbb{Z} \times \mathbb{Z}$, otherwise let w be another vector such that v, w span $\mathbb{Z} \times \mathbb{Z}$. Define the linear operator $P\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $P(v) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $P(w) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Consider a homeomorphism of fibre bundles (also denoted by P) $P\colon M(\phi) \to M(\phi^1)$, given by*

$$P\left(\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, t \right\rangle\right) = \left\langle \left[P\begin{pmatrix} x \\ y \end{pmatrix} \right], t \right\rangle, \qquad \text{where } \phi^1 = P \circ \phi \circ P^{-1}.$$

Then we have for $B_i^1 = P \circ B_i \circ P^{-1}$ that:

(i) *If v and $w = \phi(v)$ span $\mathbb{Z} \times \mathbb{Z}$ then $B_i^1 = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ for $i = 1, 2$.*

(ii) *Otherwise, $\phi(v) = \lambda v$ with $\lambda \in \mathbb{Z}$. Then $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ n & q_i \end{smallmatrix}\right)$ for $i = 1, 2$ and $\phi^1 = \left(\begin{smallmatrix} \lambda & a_3 \\ 0 & a_4 \end{smallmatrix}\right)$ with $\lambda = \pm 1$ and $a_4 = \pm 1$.*

(iii) *B_i^1 commutes with ϕ^1 and the conditions* (i) *and* (ii) *determine the table below*

Case I	$\phi^1 = \left(\begin{smallmatrix} a_1 & a_3 \\ a_2 & a_4 \end{smallmatrix}\right)$ $B_i^1 = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ $\det \phi^1 = \pm 1$ *and* $\phi^1 B_i^1 = B_i^1 \phi^1$	Case IV	$\phi^1 = \left(\begin{smallmatrix} 1 & a_3 \\ 0 & -1 \end{smallmatrix}\right)$ $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ 0 & q_i \end{smallmatrix}\right)$ $a_3(q_i - m) = -2p_i$
Case II	$\phi^1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ n & q_i \end{smallmatrix}\right)$	Case V	$\phi^1 = \left(\begin{smallmatrix} -1 & a_3 \\ 0 & -1 \end{smallmatrix}\right)$ $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ n & q_i \end{smallmatrix}\right)$ $a_3(q_i - m) = 0$ *and* $a_3 n = 0$
Case III	$\phi^1 = \left(\begin{smallmatrix} 1 & a_3 \\ 0 & 1 \end{smallmatrix}\right)$ $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ 0 & m \end{smallmatrix}\right)$ $a_3 \neq 0$	Case VI	$\phi^1 = \left(\begin{smallmatrix} -1 & a_3 \\ 0 & 1 \end{smallmatrix}\right)$ $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ 0 & q_i \end{smallmatrix}\right)$ $a_3(q_i - m) = 2p_i$

PROOF. (a) Let us consider the following loops in $M(\phi)$ with base point $\mathbf{0}$:

$$a(t) = \left\langle \begin{bmatrix} t \\ 0 \end{bmatrix}, 0 \right\rangle, \quad b(t) = \left\langle \begin{bmatrix} 0 \\ t \end{bmatrix}, 0 \right\rangle \quad \text{and} \quad c_0(t) = \left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, t \right\rangle,$$

for $t \in [0,1]$. From Proposition 2.11 it follows that $\pi_1(M(\phi)) \cong \pi_1(T) \rtimes \mathbb{Z}$, and a presentation is $\pi_1(M(\phi),\mathbf{0}) = \langle a, b, c_0; [a,b] = 1, c_0 a c_0{}^{-1} = a^{a_1} b^{a_2}, c_0 b c_0{}^{-1} = a^{a_3} b^{a_4} \rangle$.

(b) Recall that $B_i = \left(\begin{smallmatrix} m_i & p_i \\ n_i & q_i \end{smallmatrix}\right)$ is the matrix of the induced homomorphism of the restriction of the maps f_i to the fibre T on the fundamental group of the fibre T, and f_i are maps over S^1. Then the induced homomorphisms $f_{i\#}$ on $\pi_1(M(\phi),\mathbf{0})$ are given by $f_{i\#}(a) = a^{m_i} b^{n_i}, f_{i\#}(b) = a^{p_i} b^{q_i}$ and $f_{i\#}(c_0) = a^{c_{1i}} b^{c_{2i}} c_0$. Since f_i are maps over S^1 it follows from Proposition 2.7 that B_i commutes with ϕ.

(c) If $N(f_{1|T}, f_{2|T}) = 0$ then we have that the rows of the matrix

$$\begin{pmatrix} m_2 - m_1 & p_2 - p_1 \\ n_2 - n_1 & q_2 - q_1 \end{pmatrix}$$

are proportional. Without loss generality, we shall to assume that there exists $r \in \mathbb{Q}$ such that $r(n_2 - n_1, q_2 - q_1) = (m_2 - m_1, p_2 - p_1)$. So

$$B_2 - B_1 = \begin{pmatrix} r(n_2 - n_1) & r(q_2 - q_1) \\ n_2 - n_1 & q_2 - q_1 \end{pmatrix}$$

and since 0 is a root of the characteristic polynomial $\det((B_2 - B_1) - \lambda I_2) = 0$, it follows that $\lambda = 0$ is an eigenvalue of $B_2 - B_1$ and $v = ((q_1 - q_2)/L, (n_2 - n_1)/L)$ where $L = \gcd(q_1 - q_2, n_2 - n_1)$ is an eigenvector associated to the eigenvalue 0 such that $\gcd((q_1 - q_2)/L, (n_2 - n_1)/L) = 1$. Now (d) follows from (b).

To prove (e) we observe that:

(i) if the pair $(v, w = \phi(v))$ generates $\mathbb{Z} \times \mathbb{Z}$, where v is an eigenvector of $B_2 - B_1$ associated to 0, then

$$B_1^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \circ B_1(v) = P \circ B_2(v) = B_2^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$B_1^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P \circ B_1(w) = P \circ B_2(w) = B_2^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So $B_i^1 = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$, for $i = 1, 2$.

(ii) Otherwise, if the pair (v, w) generates $\mathbb{Z} \times \mathbb{Z}$ with $w \neq \phi(v)$ and v an eigenvector of $B_2 - B_1$ associated to 0, then

$$B_1^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B_2^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$$

and

$$B_i^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P \circ B_i \circ P^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P \circ B_i(w) = \begin{pmatrix} p_i \\ q_i \end{pmatrix},$$

therefore $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ n & q_i \end{smallmatrix}\right)$, for $i = 1, 2$. In this case, since the pair $(v, \phi(v))$ does not generate $\mathbb{Z} \times \mathbb{Z}$ we have $(z_1, z_2) = \phi(v) = \lambda v$ with $\lambda \in \mathbb{Z}$.

Indeed since $v = (v_1, v_2)$ with gcd $(v_1, v_2) = 1$ there exists $(r, s) \in \mathbb{Z} \times \mathbb{Z}$ such that $rv_1 + sv_2 = 1$ and therefore $\lambda rv_1 + \lambda sv_2 = \lambda$ whence it follows that $rz_1 + sz_2 = \lambda \in \mathbb{Z}$. Then

$$\phi^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \circ \phi \circ P^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \circ \phi(v) = P(\lambda v) = \lambda P(v) = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

and $\phi^1 = \left(\begin{smallmatrix} \lambda & a_3 \\ 0 & a_4 \end{smallmatrix}\right)$ with $\lambda = \pm 1$ and $a_4 = \pm 1$.

(iii) Since $B_i^1 = P \circ B_i \circ P^{-1}$, $\phi^1 = P \circ \phi \circ P^{-1}$ and B_i commutes with ϕ it follows that B_i^1 commutes with ϕ^1. Now using the commutativity of B_i^1 with ϕ^1 the table follows. We remark that the Case I occurs when v and $\phi(v)$ span $\mathbb{Z} \times \mathbb{Z}$ and in this case we take

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi(v) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \phi^1 = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$$

and the remaining cases occur when the pair (v, w) generates $\mathbb{Z} \times \mathbb{Z}$ with $w \neq \phi(v)$ and v is an eigenvector of $B_2 - B_1$ associated to 0. $\square$

We will assume from now on that the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is zero. We denote by

$$f_{1\#} = f_1(m_1, n_1, p_1, q_1, c_{11}, c_{21})\colon \pi_1(M(\phi), \mathbf{0}) \to \pi_1(M(\phi), \mathbf{0}),$$
$$f_{2\#} = f_2(m_2, n_2, p_2, q_2, c_{12}, c_{22})\colon \pi_1(M(\phi), \mathbf{0}) \to \pi_1(M(\phi), \mathbf{q})$$

the homomorphisms that take $a \to a^{m_1} b^{n_1}$, $b \to a^{p_1} b^{q_1}$, $c_0 \to a^{c_{11}} b^{c_{21}} c_0$ and $a \to \overline{e}^{m_2} \overline{d}^{n_2}$, $b \to \overline{e}^{p_2} \overline{d}^{q_2}$, $c_0 \to \overline{e}^{c_{12}} \overline{d}^{c_{22}} \overline{c}$, respectively. We emphasize that $\overline{e}$, $\overline{d}$, $\overline{c}$ are defined in Subsection 3.3.

According to relations on $\pi_1(M(\phi), \mathbf{0})$, m_i, n_i, p_i, q_i, c_{1i}, c_{2i} must satisfy some equations. These equations are given by the

Proposition 3.5. *Let $f_i\colon M(\phi) \to M(\phi)$ be maps over S^1, where ϕ belongs to one of the cases from Proposition* 3.4. *If the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is zero, then*:

(a) *the vectors $(m_1 - m_2, p_1 - p_2)$, $(n_1 - n_2, q_1 - q_2)$ are proportional over $\mathbb{Q}$.*
(b) *$a_3 n_i = a_2 p_i$ and $a_2(m_i - q_i) = (a_1 - a_4) n_i$.*
(c) *$a_3(m_i - q_i) = (a_1 - a_4) p_i$.*

Conversely, given homomorphisms

$$f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})\colon \pi_1(M(\phi)) \to \pi_1(M(\phi)),$$

where $\phi = \left(\begin{smallmatrix} a_1 & a_3 \\ a_2 & a_4 \end{smallmatrix}\right)$ belongs to one of the cases from the Proposition 3.4, *with $(m_i, n_i, p_i, q_i, a_1, a_2, a_3, a_4)$, $i = 1, 2$, satisfying the above conditions* (a)–(c), *then there exist maps $f_i\colon (M(\phi), \mathbf{0}) \to (M(\phi), z_i)$ over S^1 such that*

$$f_{i\#} = f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i}), \quad i = 1, 2,$$

and the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is zero.

Proof. Since the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is zero, then (a) follows from Proposition 3.3. The equations (b) and (c) follow from the commutativity of ϕ and B_i.

Conversely, we consider the homomorphisms

$$f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})\colon \pi_1(M(\phi), \mathbf{0}) \to \pi_1(M(\phi), z_i)$$

where $\phi = \left(\begin{smallmatrix} a_1 & a_3 \\ a_2 & a_4 \end{smallmatrix}\right)$ belongs to one of the cases of Proposition 3.4, with $(m_i, n_i, p_i, q_i, a_1, a_2, a_3, a_4)$, $i = 1, 2$ satisfying the equations (a)–(c). From the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(T) \arrow[r, "i_\#"] \arrow[d] & \pi_1(M(\phi), \mathbf{0}) \arrow[r, "p_\#"] \arrow[d] & \pi_1(S^1) \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\ 1 \arrow[r] & \pi_1(T) \arrow[r, "i_\#"'] & \pi_1(M(\phi), z_i) \arrow[r, "p_\#"'] & \pi_1(S^1) \arrow[r] & 1 \end{tikzcd}$$

where the vertical arrows are deriving from $f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})$ we have that $p_\# \circ f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i}) = p_\#$.

Since all spaces are $K(\pi, 1)$, then by [17, V, Theorem (4.3)], there exists a bijection $[(M(\phi), \mathbf{0}; M(\phi), z_i)] \to \mathrm{Hom}(\pi_1(M(\phi), \mathbf{0}) \to \pi_1(M(\phi), z_i))$ with $z_1 = \mathbf{0}$ and $z_2 = \mathbf{q}$. So, for each $i = 1, 2$ there exists $g_i\colon (M(\phi), \mathbf{0}) \to (M(\phi), z_i)$. Similarly, from the bijection $[M(\phi), \mathbf{0}; S^1, \langle 0\rangle] \to \mathrm{Hom}(\pi_1(M(\phi), \mathbf{0}) \to \pi_1(S^1), \langle 0\rangle)$ there exist homotopies $H_i\colon (M(\phi) \times I, \mathbf{0} \times I) \to (S^1, \langle 0\rangle)$ such that

$$H_i\bigg(\bigg\langle\begin{bmatrix} x \\ y \end{bmatrix}, t\bigg\rangle, 0\bigg) = p \circ g_i\bigg\langle\begin{bmatrix} x \\ y \end{bmatrix}, t\bigg\rangle, \qquad H_i\bigg(\bigg\langle\begin{bmatrix} x \\ y \end{bmatrix}, t\bigg\rangle, 1\bigg) = p\bigg\langle\begin{bmatrix} x \\ y \end{bmatrix}, t\bigg\rangle$$

and $g_{i\#} = f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})$.

For each $i = 1, 2$, $G_i\colon (M(\phi)\times 0, \mathbf{0}\times I) \to (M(\phi), z_i)$ defined by $G_i(\mathbf{0}\times I) = z_i$, where $z_1 = \mathbf{0}$ and $z_2 = \mathbf{q}$, makes the diagram below commutative:

$$\begin{tikzcd} (M(\phi) \times 0, \mathbf{0} \times I) \arrow[r, "G_i"] \arrow[d, "j"'] & (M(\phi), z_i) \arrow[d, "p"] \\ (M(\phi) \times I, \mathbf{0} \times I) \arrow[r, "H_i"'] \arrow[ur, dotted, "L_i"] & (S^1, \langle 0\rangle = \langle 1\rangle) \end{tikzcd}$$

Since $p\colon (M(\phi), z_i) \to (S^1, \langle 0\rangle = \langle 1\rangle)$ is a fibration it follows that for each $i = 1, 2$ there exists $L_i\colon (M(\phi) \times I, \mathbf{0} \times I) \to (M(\phi), z_i)$ a lifting of H_i, i.e. $p \circ L_i = H_i$.

Remark that $f_i = L_i(\,\cdot\,, 1)\colon (M(\phi), \mathbf{0}) \to (M(\phi), z_i)$ is over S^1 and the induced homomorphism on the fundamental groups coincides with $f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})$ because $p \circ f_i = p \circ L_i(\,\cdot\,, 1) = H_i(\,\cdot\,, 1) = p$ and $f_{i\#} = L_i(\,\cdot\,, 1)_\# = g_{i\#} = f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})$.

Since $(m_i, n_i, p_i, q_i, a_1, a_2, a_3, a_4)$ satisfies the equations (a)–(c) and

$$f_{i\#} = f_i(m_i, n_i, p_i, q_i, c_{1i}, c_{2i})$$

then the Nielsen number $N(f_{1|_T}, f_{2|_T})$ is zero. □

To facilitate future computations, we will describe the homomorphisms

$$f_{i\#}\colon \pi_1(M(\phi), \mathbf{0}) \to \pi_1(M(\phi), z_i),$$

on the fundamental groups, where $z_1 = \mathbf{0}$, $z_2 = \mathbf{q}$, $f_{1\#}, f_{2\#}$ maps $a \to a^{m_1}b^{n_1}$, $b \to a^{p_1}b^{q_1}$, and $c_0 \to a^{c_{11}}b^{c_{21}}c_0$ and $a \to \overline{e}^{m_2}\overline{d}^{n_2}$, $b \to \overline{e}^{p_2}\overline{d}^{q_2}$ and $c_0 \to \overline{e}^{c_{12}}\overline{d}^{c_{22}}\overline{c}$, respectively.

THEOREM 3.6. *Let $f_1, f_2\colon M(\phi) \to M(\phi)$ be maps over S^1, where ϕ belongs to one of the cases of Proposition* 3.4. *If the Nielsen number $N(f_{1|_T}, f_{2|_T})$ of f_1 and f_2 restricted to the fibre T is zero then $f_{i\#}\colon \pi_1(M(\phi), \mathbf{0}) \to \pi_1(M(\phi), z_i)$ is given by the table:*

Case I	$f_i(m, n, p, q, c_{1i}, c_{2i})$
Case II	$f_i(m, n, p_i, q_i, c_{1i}, c_{2i})$, $(p_2 - p_1, q_2 - q_1) \neq (0, 0)$ *and* $p_1 - p_2, q_1 - q_2$ *are proportional over* $\mathbb{Q}$
Case III	$f_i(m, 0, p_i, m, c_{1i}, c_{2i})$, $p_1 \neq p_2$
Case IV	
$a_3 = 2r$	$f_i(m, 0, p_i, q_i, c_{1i}, c_{2i})$, $q_1 \neq q_2$, $(-r)(q_i - m) = p_i$
$a_3 = 2r + 1$	$f_i(m, 0, p_i, q_i, c_{1i}, c_{2i})$, $q_1 \neq q_2$, $q_i - m$ *is even and* $-(2r + 1)[(q_i - m)/2] = p_i$
Case V	$f_i(m, n, p_i, q_i, c_{1i}, c_{2i})$, $(p_2 - p_1, q_2 - q_1) \neq (0, 0)$ *and* $p_1 - p_2, q_1 - q_2$
$a_3 = 0$	*are proportional over* $\mathbb{Q}$
$a_3 \neq 0$	$f_i(m, 0, p_i, m, c_{1i}, c_{2i})$, $p_1 \neq p_2$
Case VI	
$a_3 = 2r$	$f_i(m, 0, p_i, q_i, c_{1i}, c_{2i})$, $q_1 \neq q_2$ *and* $r(q_i - m) = p_i$
$a_3 = 2r + 1$	$f_i(m, 0, p_i, q_i, c_{1i}, c_{2i})$, $q_1 \neq q_2$, $q_i - m$ *is even and* $(2r + 1)[(q_i - m)/2] = p_i$

PROOF. The proof of this theorem follows from the relations (a)–(c) given by Proposition 3.5. □

In the table above we have c_{1i}, c_{2i}, m, n, p, q, p_i, q_i, $r \in \mathbb{Z}$. The table is given for each $i = 1, 2$. So the pairs $(f_{1\#}, f_{2\#})$ are combinations in each one of the cases. By example, in the Case II we have 4 possibilities. Since $\mathrm{Coin}(f_1, f_2) = \mathrm{Coin}(f_2, f_1)$, then we can reduce the number of cases to be studied.

3.3. The generators from the fundamental groups in the diagram (2.4). The next theorem describes the groups and the homomorphisms of diagram (2.4). Let us consider the following loops in $M(\phi)$ with the base point $\mathbf{0}$ and loops in $M(\phi) \setminus S^1$ or $M(\phi)$ with the base point $\mathbf{q}$ with q small positive and $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}q\\q\end{smallmatrix}\right]$. For $t \in [0, 1]$ we consider the following loops:

$$a(t) = \left\langle \begin{bmatrix} t \\ 0 \end{bmatrix}, 0 \right\rangle, \qquad b(t) = \left\langle \begin{bmatrix} 0 \\ t \end{bmatrix}, 0 \right\rangle, \qquad c_0(t) = \left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, t \right\rangle,$$

$$e(t) = \left\langle \begin{bmatrix} q+t \\ q \end{bmatrix}, 0 \right\rangle, \qquad d(t) = \left\langle \begin{bmatrix} q \\ q+t \end{bmatrix}, 0 \right\rangle, \qquad c(t) = \langle [\gamma(t)], t \rangle,$$

where $\gamma(t)$ is $(1-t)\cdot \binom{q}{q} + t(\phi\binom{q}{q})$ if $\phi\binom{q}{q} \neq \binom{-q}{-q}$; otherwise, it is an arc which runs counterclockwise around the origin from $\binom{q}{q}$ to $\binom{-q}{-q}$. Here, we suppose that $\phi\binom{q}{q}$ belongs to the square centered at the origin $(0,0)$ with side two. Finally, let $W(t)$ be the circle around the origin having (q,q) as the base point and oriented counterclockwise.

In $\pi_1(M(\phi), \mathbf{0})$ we denote the homotopy classes of the loops $a(t)$, $b(t)$, $c_0(t)$ by a, b, c_0, respectively. In $\pi_1(M(\phi) \setminus S^1, \mathbf{q})$ we denote the homotopy classes of the loops $e(t)$, $d(t)$, $c(t)$ by e, d, c, respectively, and in $\pi_1(M(\phi), \mathbf{q})$ we denote the homotopy classes of the loops $e(t)$, $d(t)$, $c(t)$ by $\overline{e}$, $\overline{d}$, $\overline{c}$, respectively.

THEOREM 3.7. *Let ϕ and B_i be one of the six cases given by Proposition 3.4 and let $(f_1, f_2)_\#$ be the homomorphism induced by (f_1, f_2) on the fundamental groups. Then we have:*

(a) $\pi_1(M(\phi) \times_{S^1} M(\phi), (\mathbf{0}, \mathbf{q})) = \langle a, b, \overline{e}, \overline{d}, \widehat{c}; [a,b] = 1, [a, \overline{e}] = 1, [a, \overline{d}] = 1, [b, \overline{e}] = 1, [b, \overline{d}] = 1, [\overline{e}, \overline{d}] = 1, \widehat{c}\, a\, \widehat{c}^{-1} = a^{a_1} b^{a_2}, \widehat{c}\, b\, \widehat{c}^{-1} = a^{a_3} b^{a_4}, \widehat{c}\, \overline{e}\, \widehat{c}^{-1} = \overline{e}^{a_1} \overline{d}^{a_2}, \widehat{c}\, \overline{d}\, \widehat{c}^{-1} = \overline{e}^{a_3} \overline{d}^{a_4} \rangle$ *where $\widehat{c}$ is the homotopy class of the loop given by the pair of loops $(c_0(t), c(t))$, $\overline{d}$ is the homotopy class of the loop $(\mathbf{0}, d(t))$, $\overline{e}$ is the homotopy class of the loop $(\mathbf{0}, e(t))$, b is the homotopy class of the loop $(b(t), \mathbf{q})$ and a is the homotopy class of the loop $(a(t), \mathbf{q})$.*

(b)

		$\pi_1(M(\phi) \setminus S^1, \mathbf{q})$
Case II	$\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$B_i = \begin{pmatrix} m & p_i \\ n & q_i \end{pmatrix}$ *with* $p_1 \neq p_2$ *or* $q_1 \neq q_2$ $\langle e, d, c; cec^{-1} = e, cdc^{-1} = d \rangle$
Case III	$\phi = \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}$	$B_i = \begin{pmatrix} m & p_i \\ 0 & m \end{pmatrix}$ *with* $a_3 \neq 0$ *and* $p_1 \neq p_2$ $\langle e,d,c; cec^{-1}=e, cdc^{-1}=e^{a_3}d \rangle$
Case IV	$\phi = \begin{pmatrix} 1 & a_3 \\ 0 & -1 \end{pmatrix}$	$B_i = \begin{pmatrix} m & p_i \\ 0 & q_i \end{pmatrix}$ *with* $a_3(q_i - m) = -2p_i$ *and* $q_1 \neq q_2$ *if* $a_3 \geq -1$ *then* $\langle e, d, c; cec^{-1} = eW, cdc^{-1} = d^{-1}e^{a_3} \rangle$ *if* $a_3 \leq -2$ *then* $\langle e, d, c; cec^{-1} = We, cdc^{-1} = Wd^{-1}e^{a_3}W^{-1} \rangle$
Case V	$\phi = \begin{pmatrix} -1 & a_3 \\ 0 & -1 \end{pmatrix}$	$B_i = \begin{pmatrix} m & p_i \\ n & q_i \end{pmatrix}$ *with* $a_3(q_i - m) = 0$, $a_3 n = 0$ *and* $p_1 \neq p_2$ *or* $q_1 \neq q_2$ *if* $a_3 \geq 1$ *then* $\langle e, d, c; cec^{-1} = W^{-1}e^{-1}, cdc^{-1} = d^{-1}e^{a_3}W \rangle$ *if* $a_3 \leq 0$ *then* $\langle e, d, c; cec^{-1} = e^{-1}W^{-1}, cdc^{-1} = Wd^{-1}e^{a_3} \rangle$
Case VI	$\phi = \begin{pmatrix} -1 & a_3 \\ 0 & 1 \end{pmatrix}$	$B_i = \begin{pmatrix} m & p_i \\ 0 & q_i \end{pmatrix}$ *with* $a_3(q_i - m) = 2p_i$ *and* $q_1 \neq q_2$ $\langle e, d, c; cec^{-1} = e^{-1}, cdc^{-1} = e^{a_3}dW^{-1} \rangle$

where W is homotopic to the loop $e^{-1}d^{-1}ed$.

(c) *The homomorphism $(p_2 \circ h \circ (f_1, f_2))_\#$ is given by*

$$a \to \overline{e}^{\,m_2 - m_1} \overline{d}^{\,n_2 - n_1}, \qquad b \to \overline{e}^{\,p_2 - p_1} \overline{d}^{\,q_2 - q_1}, \qquad c_0 \to \overline{e}^{\,c_{12} - c_{11}}\, \overline{d}^{\,c_{22} - c_{21}}\, \overline{c}.$$

(d) *The homomorphism* $i_{\#}\colon \pi_1(M(\phi)\setminus S^1,\mathbf{q}) \to \pi_1(M(\phi),\mathbf{q})$ *is given by*

$$e \to \overline{e}, \qquad d \to \overline{d}, \qquad c \to \overline{c}.$$

(e) *The homomorphism* Γ *in the diagram* (2.4) *exists if and only if we can find elements* $Z_1, Z_2, Z_3 \in \ker(\pi_1(T\setminus 1) \to \pi_1(T))$ *such that* $\Gamma(a) = Z_1 e^{m_2-m_1} d^{n_2-n_1}$, $\Gamma(b) = Z_2 e^{p_2-p_1} d^{q_2-q_1}$, $\Gamma(c_0) = Z_3 e^{c_{12}-c_{11}} d^{c_{22}-c_{21}} c$ *and the following equations hold*:

$$\begin{cases} [\Gamma(a),\Gamma(b)] = 1, \\ \Gamma(c_0)\Gamma(a)\Gamma({c_0}^{-1}) = \Gamma(a^{a_1}b^{a_2}), \\ \Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(a^{a_3}b^{a_4}). \end{cases}$$

Proof. (a), (b) and (d) follows from [8, Theorem 2.2, p. 10].
(c) In fact,

$$\begin{aligned}
(p_2\circ h\circ(f_1,f_2))_{\#}(a) &= (p_2\circ h)_{\#}(f_1,f_2)_{\#}(a) = (p_2\circ h)_{\#}(f_{1\#}(a), f_{2\#}(a)) \\
&= (p_2\circ h)_{\#}((f_{1|_T})_{\#}(a), (f_{2|_T})_{\#}(a)) \\
&= (p_2\circ h)_{\#}([\,f_{1|_T}(a(t))\,], [\,f_{2|_T}(a(t))\,]) \\
&= (p_2\circ h)_{\#}\left(\left[f_{1|_T}\left(\left\langle \begin{bmatrix} t \\ 0 \end{bmatrix}, 0 \right\rangle\right)\right], \left[f_{2|_T}\left(\left\langle \begin{bmatrix} t \\ 0 \end{bmatrix}, 0 \right\rangle\right)\right]\right) \\
&= (p_2\circ h)_{\#}\left(\left[\left\langle \left[B_1\begin{pmatrix} t \\ 0 \end{pmatrix}\right], 0 \right\rangle\right], \left[\left\langle [B_2\begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} q \\ q \end{pmatrix}\right], 0 \right\rangle\right]\right) \\
&= \left[(p_2\circ h)\left(\left\langle \left[B_1\begin{pmatrix} t \\ 0 \end{pmatrix}\right], 0 \right\rangle, \left\langle \left[B_2\begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} q \\ q \end{pmatrix}\right], 0 \right\rangle\right)\right] \\
&= \left[(p_2)\left(\left\langle \left[B_1\begin{pmatrix} t \\ 0 \end{pmatrix}\right], 0 \right\rangle, \left\langle \left[B_2\begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} q \\ q \end{pmatrix} - B_1\begin{pmatrix} t \\ 0 \end{pmatrix}\right], 0 \right\rangle\right)\right] \\
&= \left[\left\langle \left[B_2\begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} q \\ q \end{pmatrix} - B_1\begin{pmatrix} t \\ 0 \end{pmatrix}\right], 0 \right\rangle\right] \\
&= \left[\left\langle \left[\begin{pmatrix} (m_2-m_1)t \\ (n_2-n_1)t \end{pmatrix} + \begin{pmatrix} q \\ q \end{pmatrix}\right], 0 \right\rangle\right] \\
&= \left[\left\langle \begin{bmatrix} q + (m_2-m_1)t \\ q \end{bmatrix}, 0 \right\rangle\right] \cdot \left[\langle[\begin{matrix} q \\ q + (n_2-n_1)t \end{matrix}\right], 0 \right\rangle\Big] \\
&= \overline{e}^{\,m_2-m_1}\overline{d}^{\,n_2-n_1},
\end{aligned}$$

$$\begin{aligned}
(p_2\circ h\circ(f_1,f_2))_{\#}(b) &= (p_2\circ h)_{\#}(f_1,f_2)_{\#}(b) \\
&= (p_2\circ h)_{\#}(f_{1\#}(b), f_{2\#}(b)) = (p_2\circ h)_{\#}((f_{1|_T})_{\#}(b), (f_{2|_T})_{\#}(b)) \\
&= (p_2\circ h)_{\#}([f_{1|_T}(b(t))], [f_{2|_T}(b(t))])
\end{aligned}$$

$$= (p_2 \circ h)_{\#}\Big(\Big[f_{1|_T}\Big(\Big\langle\begin{bmatrix}0\\t\end{bmatrix},0\Big\rangle\Big)\Big],\Big[f_{2|_T}\Big(\Big\langle\begin{bmatrix}0\\t\end{bmatrix},0\Big\rangle\Big)\Big]\Big)$$

$$= (p_2 \circ h)_{\#}\Big(\Big[\Big\langle\Big[B_1(\begin{matrix}0\\t\end{matrix}\Big)\Big],0\Big\rangle\Big],\Big[\Big\langle\Big[B_2\begin{pmatrix}0\\t\end{pmatrix}+\begin{pmatrix}q\\q\end{pmatrix}\Big],0\Big\rangle\Big]\Big)$$

$$= \Big[(p_2 \circ h)\Big(\Big\langle\Big[B_1\begin{pmatrix}0\\t\end{pmatrix}\Big],0\Big\rangle,\Big\langle\Big[B_2\begin{pmatrix}0\\t\end{pmatrix}+\begin{pmatrix}q\\q\end{pmatrix}\Big],0\Big\rangle\Big)\Big]$$

$$= \Big[(p_2)\Big(\Big\langle\Big[B_1\begin{pmatrix}0\\t\end{pmatrix}\Big],0\Big\rangle,\Big\langle\Big[B_2\begin{pmatrix}0\\t\end{pmatrix}+\begin{pmatrix}q\\q\end{pmatrix}-B_1\begin{pmatrix}0\\t\end{pmatrix}\Big],0\Big\rangle\Big)\Big]$$

$$= \Big[\Big\langle\Big[B_2\begin{pmatrix}0\\t\end{pmatrix}+\begin{pmatrix}q\\q\end{pmatrix}-B_1\begin{pmatrix}0\\t\end{pmatrix}\Big],0\Big\rangle\Big]$$

$$= \Big[\Big\langle\Big[\begin{pmatrix}(p_2-p_1)t\\(q_2-q_1)t\end{pmatrix}+\begin{pmatrix}q\\q\end{pmatrix}\Big],0\Big\rangle\Big]$$

$$= \Big[\Big\langle\begin{bmatrix}q+(p_2-p_1)t\\q\end{bmatrix},0\Big\rangle\Big]\cdot\Big[\Big\langle\begin{bmatrix}q\\q+(q_2-q_1)t\end{bmatrix},0\Big\rangle\Big] = \overline{e}^{\,p_2-p_1}\overline{d}^{\,q_2-q_1}$$

and

$$(p_2 \circ h \circ (f_1,f_2))_{\#}(c_0) = (p_2 \circ h)_{\#}(f_1,f_2)_{\#}(c_0)$$
$$= (p_2 \circ h)_{\#}(f_{1\#}(c_0), f_{2\#}(c_0)) = (p_2 \circ h)_{\#}(a^{c_{11}}b^{c_{21}}c_0, a^{c_{12}}b^{c_{22}}c_0)$$
$$= (p_2 \circ h)_{\#}\Big(\Big[\Big\langle\begin{bmatrix}c_{11}t\\c_{21}t\end{bmatrix},t\Big\rangle\Big],\Big[\Big\langle\begin{bmatrix}c_{12}t+q\\c_{22}t+q\end{bmatrix},t\Big\rangle\Big]\Big)$$
$$= [(p_2 \circ h)\Big(\Big[\Big\langle\begin{bmatrix}c_{11}t\\c_{21}t\end{bmatrix},t\Big\rangle\Big],\Big[\Big\langle\begin{bmatrix}c_{12}t+q\\c_{22}t+q\end{bmatrix},t\rangle\Big]\Big)\Big]$$
$$= \Big[p_2\Big(\Big\langle\begin{bmatrix}c_{11}t\\c_{21}t\end{bmatrix},t\Big\rangle,\Big\langle\begin{bmatrix}c_{12}t+q\\c_{22}t+q\end{bmatrix}-\begin{bmatrix}c_{11}t\\c_{21}t\end{bmatrix},t\Big\rangle\Big)\Big]$$
$$= \Big[\Big\langle\begin{bmatrix}c_{12}t+q\\c_{22}t+q\end{bmatrix}-\begin{bmatrix}c_{11}t\\c_{21}t\end{bmatrix},t\Big\rangle\Big] = \Big[\Big\langle\Big[\begin{pmatrix}(c_{12}-c_{11})t\\(c_{22}-c_{21})t\end{pmatrix}+\begin{pmatrix}q\\q\end{pmatrix}\Big],t\Big\rangle\Big]$$
$$= \Big[\Big\langle\begin{bmatrix}q+(c_{12}-c_{11})t\\q\end{bmatrix},0\Big\rangle\Big]\cdot\Big[\Big\langle\begin{bmatrix}q\\q+(c_{22}-c_{21})t\end{bmatrix},0\Big\rangle\Big]\cdot[\langle[\gamma(t)],t\rangle]$$
$$= \overline{e}^{\,c_{12}-c_{11}}\overline{d}^{\,c_{22}-c_{21}}\overline{c}.$$

(e) Initially we observe that if $\Gamma(a) = x$ then

$$i_{\#}(x) = i_{\#} \circ \Gamma(a) = (p_2 \circ h \circ (f,g))_{\#}(a) = \overline{e}^{m_2-m_1}\overline{d}^{n_2-n_1}.$$

On the other hand, $i_{\#}(e^{m_2-m_1}d^{n_2-n_1}) = \overline{e}^{m_2-m_1}\overline{d}^{n_2-n_1}$. Therefore

$$xd^{n_1-n_2}e^{m_1-m_2} = Z_1,$$

where $Z_1 \in \pi_1(\mathcal{F}(M(\phi) \setminus S^1)) \simeq \pi_2(T, T \setminus 1) =$ ker $(\pi_1(T \setminus 1) \to \pi_1(T))$ and $\Gamma(a) = Z_1 e^{m_2-m_1} d^{n_2-n_1}$. Similarly we prove for b and c_0. Now the equations

$$\begin{cases} [\Gamma(a), \Gamma(b)] = 1, \\ \Gamma(c_0)\Gamma(a)\Gamma({c_0}^{-1}) = \Gamma(a^{a_1} b^{a_2}), \\ \Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(a^{a_3} b^{a_4}), \end{cases}$$

follow from the relations on $\pi_1(M(\phi))$.

Conversely, it is easy to show that if we can find elements $Z_1, Z_2, Z_3 \in \ker(\pi_1(T \setminus 1) \to \pi_1(T))$ such that

$$\Gamma(a) = Z_1 e^{m_2-m_1} d^{n_2-n_1}, \; \Gamma(b) = Z_2 e^{p_2-p_1} d^{q_2-q_1}, \; \Gamma(c_0) = Z_3 e^{c_{12}-c_{11}} d^{c_{22}-c_{21}} c$$

and the following equations hold:

$$\begin{cases} [\Gamma(a), \Gamma(b)] = 1, \\ \Gamma(c_0)\Gamma(a)\Gamma({c_0}^{-1}) = \Gamma(a^{a_1} b^{a_2}), \\ \Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(a^{a_3} b^{a_4}). \end{cases}$$

then Γ makes the diagram (2.4) commutative. □

PROPOSITION 3.8 (Case I). *Let $f_i \colon M(\phi) \to M(\phi)$ be maps over S^1 where f_i corresponds to the case I of the table in Theorem* 3.6, *where $(p_2 - p_1, q_2 - q_1) = (0, 0)$. Then the pair (f_1, f_2) can always be deformed to a coincidence free pair (g_1, g_2) by a fibrewise homotopy over S^1.*

PROOF. Define a lifting for $(p_2 \circ h \circ (f_1, f_2))_\#$ by

$$\Gamma(a) = 1, \quad \Gamma(b) = 1, \quad \Gamma(c_0) = e^{c_{12}-c_{11}} d^{c_{22}-c_{21}} c$$

and the result follows. □

Now we derive a necessary algebraic condition for (f_1, f_2) be deformable to a pair of coincidence free maps.

PROPOSITION 3.9. *If $f_i \colon M(\phi) \to M(\phi)$ corresponds to the remaining cases other than I of the table in Theorem* 3.6, *where $(p_2 - p_1, q_2 - q_1) \neq (0, 0)$ and (f_1, f_2) can be deformed to a coincidence free pair (g_1, g_2) by a fibrewise homotopy over S^1, then we must have that $Z_1 = 1$. In this case in order to have the homomorphism Γ it is necessary and sufficient to solve the equation*:

$$\Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(a^{a_3} b^{a_4}) = \Gamma(a^{a_3})\Gamma(b^{a_4}) = \Gamma(b^{a_4}).$$

PROOF. In the remaining cases, since $(m_2 - m_1, n_2 - n_1) = (0, 0)$ we have that $\Gamma(a) = Z_1$. But by Theorem 3.7(e), $[Z_1, \Gamma(b)] = 1$. If $Z_1 \neq 1$ then we must have $Z_1 = u^\alpha$ and $\Gamma(b) = u^\beta$, where $u \in \pi_2(T, T \setminus 1)$ which is a free non abelian group. But this is impossible, since $i_\# \circ \Gamma(b) = (p_2 \circ h \circ (f_1, f_2))_\#(b) = \overline{e}^{p_2-p_1} \overline{d}^{q_2-q_1} \neq 1$, because $(p_2 - p_1, q_2 - q_1) \neq (0, 0)$. On the other hand $i_\# \circ$

$\Gamma(b) = i_{\#}(u^{\beta}) = 1$ since $u \in \pi_2(T, T \setminus 1)$. Therefore $Z_1 = 1$. So $\Gamma(a) = 1$ and, as in the remaining cases other than I, $a_2 = 0$, the equations $[\Gamma(a), \Gamma(b)] = 1$ and $\Gamma(c_0)\Gamma(a)\Gamma({c_0}^{-1}) = \Gamma(a^{a_1}b^{a_2})$ are always satisfied. Therefore by Theorem 3.7(e), in order to have the homomorphism Γ, it is necessary and sufficient to solve the equation $\Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(a^{a_3}b^{a_4}) = \Gamma(a^{a_3})\Gamma(b^{a_4}) = \Gamma(b^{a_4})$. $\square$

We will call the equation

$$\Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(a^{a_3}b^{a_4}) = \Gamma(a^{a_3})\Gamma(b^{a_4}) = \Gamma(b^{a_4})$$

the main equation.

4. Generalities and properties of the main equation

In this section we first write in a more explicit form the main equation given by Proposition 3.9, interpreted as an equation in the free non abelian group $\pi_2(T, T\setminus 1)$. We derive some general results which are useful to solve the equation. Then we study the main equation in the abelianized of $\pi_2(T, T \setminus 1)$. We derive a necessary condition in order to have a solution.

4.1. Main equation. Let us consider the equation given by Proposition 3.9. Let E, D be any elements such that $j_{\#}(E) = j_{\#}(e^{c_{12}-c_{11}}d^{c_{22}-c_{21}})$ and $j_{\#}(D) = j_{\#}(e^{p_2-p_1}\ d^{q_2-q_1})$, where $j_{\#}\colon \pi_1(T \setminus 1) \to \pi_1(T)$ is the induced homomorphism by the inclusion $j\colon T \setminus 1 \to T$. Then we have:

PROPOSITION 4.1. *Let E and D be as above. Then the main equation given by Proposition 3.9 can be written in one of the forms*:

(a) $X_3EcX_2Dc^{-1}E^{-1}X_3^{-1} = (X_2D)^{a_4}$,
(b) $X_3{\cdot}EcX_2c^{-1}E^{-1}{\cdot}E(cDc^{-1}D^{-a_4})E^{-1}{\cdot}[E, D^{a_4}]{\cdot}D^{a_4}X_3^{-1}D^{-a_4}{\cdot}D^{(a_4-1)/2}$ $X_2^{-a_4}D^{(1-a_4)/2} = 1$.

Furthermore, $cDc^{-1}D^{-a_4} \in \ker j_{\#}$ and the existence of a solution of the above equation is independent of the choices of E and D, for $X_2, X_3 \in \ker j_{\#}$.

PROOF. (a) Since

$$j_{\#}(E) = j_{\#}(e^{c_{12}-c_{11}}d^{c_{22}-c_{21}}) \quad \text{and} \quad j_{\#}(D) = j_{\#}(e^{p_2-p_1}d^{q_2-q_1})$$

then there exist $\alpha_E, \alpha_D \in \ker j_{\#}$ such that $e^{c_{12}-c_{11}}d^{c_{22}-c_{21}} = \alpha_E E$ and $e^{p_2-p_1}$ $d^{q_2-q_1} = \alpha_D D$. Now the equation $\Gamma(c_0)\Gamma(b)\Gamma({c_0}^{-1}) = \Gamma(b^{a_4})$ is

$$Z_3e^{c_{12}-c_{11}}d^{c_{22}-c_{21}}cZ_2e^{p_2-p_1}d^{q_2-q_1}c^{-1}d^{c_{21}-c_{22}}e^{c_{11}-c_{12}}Z_3^{-1} = (Z_2e^{p_2-p_1}d^{q_2-q_1})^{a_4}.$$

Substituting $e^{c_{12}-c_{11}}d^{c_{22}-c_{21}} = \alpha_E E$ and $e^{p_2-p_1}d^{q_2-q_1} = \alpha_D D$ we obtain

$$Z_3\alpha_E EcZ_2\alpha_D Dc^{-1}E^{-1}\alpha_E^{-1}{Z_3}^{-1} = (Z_2\alpha_D D)^{a_4}.$$

We denote $Z_3\alpha_E = X_3$ and $Z_2\alpha_D = X_2$ and so we obtain the equation:

$$X_3EcX_2Dc^{-1}E^{-1}X_3^{-1} = (X_2D)^{a_4}.$$

(b) The equation above is the same as

$$X_3EcX_2c^{-1}E^{-1}EcDc^{-1}D^{-a_4}E^{-1}ED^{a_4}E^{-1}D^{-a_4}D^{a_4}X_3^{-1}(X_2D)^{-a_4}=1$$

and therefore

$$X_3\cdot EcX_2c^{-1}E^{-1}\cdot E(cDc^{-1}D^{-a_4})E^{-1}\cdot[E,D^{a_4}]\cdot D^{a_4}X_3^{-1}(X_2D)^{-a_4}=1.$$

Now it is sufficient to observe that

$$(X_2D)^{-a_4}=D^{-a_4}\cdot D^{(a_4-1)/2}X_2^{-a_4}D^{(1-a_4)/2}$$

since $a_4=\pm1$.

To prove that $cDc^{-1}D^{-a_4}\in\ker j_\#$ it suffices to see that $j_\#(cDc^{-1}D^{-a_4})=cj_\#(D)c^{-1}j_\#(D^{-a_4})=0$, where the last equality is obtained using the action of c and the fact that $\pi_1(T)$ is abelian.

For the last part observe that any two choices of either $E's$ or $D's$ differ by elements of $\ker j_\#$. So, the equation given by Proposition 4.1 has a solution for one choice if and only if it has a solution for the other choice and the result follows. □

Motivated by the above proposition we define:

DEFINITION 4.2. An input data for the main equation given by Proposition 3.9 consists in a quadruple (ϕ,B_i,E,D) such that

$$j_\#(E)=j_\#(e^{c_{12}-c_{11}}d^{c_{22}-c_{21}})\quad\text{and}\quad j_\#(D)=j_\#(e^{p_2-p_1}d^{q_2-q_1}).$$

From Proposition 4.1 we see that the existence of a solution of the main equation depends only in the input. Also, observe that the input defines the maps f_i.

By $|\cdot|_e,|\cdot|_d\colon\pi_1(T\setminus1)\to\mathbb{Z}$ we denote the homomorphisms which map a word $w\in\pi_1(T\setminus1)$ to the sum of the powers of the generator e and the sum of the powers of d, respectively.

The next theorem shows, for a fixed group $\pi_1(M(\phi))$, that the existence of solution for one equation implies the existence of the solution for other equations. More precisely:

THEOREM 4.3. *Let ϕ and B_i be fixed.*

(a) *The equation given by Proposition* 4.1 *has a solution for a given E, D if and only if it has a solution for $wEcw^{-1}c^{-1}$, wDw^{-1}.*

(b) *The equations given in part* (a) *have $|E|_e=c_{12}-c_{11},|E|_d=c_{22}-c_{21}$ of the input related as follows*:

$$\begin{cases}|wEcw^{-1}c^{-1}|_e=|E|_e+|wcw^{-1}c^{-1}|_e,\\ |wEcw^{-1}c^{-1}|_d=|E|_d+|wcw^{-1}c^{-1}|_d.\end{cases}$$

For the proof see [8, Theorem 3.3, p. 21].

Corollary 4.4. *Let H be the image of the homomorphism $\pi_1(T\setminus 1)\to \mathbb{Z}\oplus\mathbb{Z}$ which maps α on the pair $(|\alpha c\alpha^{-1}c^{-1}|_e, |\alpha c\alpha^{-1}c^{-1}|_d)$. If two input data (ϕ, B_i, E, D) and (ϕ, B_i, E', D') have the property that the pairs $(c_{12}-c_{11}, c_{22}-c_{21})$ and $(c'_{12}-c'_{11}, c'_{22}-c'_{21})$ belong to the same equivalence class in $\mathbb{Z}\oplus\mathbb{Z}/H$, then there is a solution for the equation with one of the inputs if and only if there is a solution for the other.*

Proof. Suppose that for the input data (ϕ, B_i, E, D) there is a solution, and (ϕ, B_i, E', D') is another input data such that $(c_{12}-c_{11}, c_{22}-c_{21})$ and $(c'_{12}-c'_{11}, c'_{22}-c'_{21})$ belong to the same equivalence class in $\mathbb{Z}\oplus\mathbb{Z}/H$. Then there exists $w\in\pi_1(T\setminus 1)$ such that

$$(c'_{12}-c'_{11}, c'_{22}-c'_{21})-(c_{12}-c_{11}, c_{22}-c_{21}) = (|wcw^{-1}c^{-1}|_e, |wcw^{-1}c^{-1}|_d).$$

Let E be such that $(|E|_e, |E|_d)=(c_{12}-c_{11}, c_{22}-c_{21})$. Then the equation has a solution for a suitable D. Define $E' = wEcw^{-1}c^{-1}$. By Theorem 4.3(b),

$$(|E'|_e, |E'|_d) = (|wEcw^{-1}c^{-1}|_e, |wEcw^{-1}c^{-1}|_d).$$

Therefore, for the input $(\phi, B_i, E', D' = wDw^{-1})$ we also have a solution and the result follows. □

Remark 4.5. Let ϕ and B_i be given and $\mathcal{C}$ be a set of representatives of the equivalence classes of $\mathbb{Z}\oplus\mathbb{Z}/H$. In order to analyze all the cases it suffices to analyze the equation for the set of inputs (ϕ, B_i, E, D) such that $(c_{12}-c_{11}, c_{22}-c_{21})$ runs over the set $\mathcal{C}$.

4.2. Equation on the abelianized. Let $\pi_2 = \pi_2(T, T\setminus 1)$ denote the kernel of the map $j_\# : \langle e, d\rangle = \pi_1(T\setminus 1)\to\pi_1 T = \langle e, d; [e,d]=1\rangle$ and $B=[e,d]$. We will study the equation given by Proposition 3.9 on the abelianized $(\pi_2)_{ab} = \pi_2/[\pi_2,\pi_2]$ of π_2 and also on some quotient of this group. Whenever the equation in one of these quotient has no solution, we can infer that the original equation has no solution. The group π_2 is isomorphic to $\pi_1(\mathcal{F})$, where $\mathcal{F}\to E(T\setminus 1)\to T$ is the fibration obtained by making the inclusion $T\setminus 1\overset{j}{\hookrightarrow} T$ into a fibration. So the group $\pi_1(T)$ acts on $H_1(\mathcal{F}) = \pi_2/[\pi_2,\pi_2]$.

In [8, Proposition 3.5, p. 22] we proved the following proposition:

Proposition 4.6.

(a)

$$H_1(\mathcal{F})\cong \mathbb{Z}\pi_1(T)\cong \bigoplus_{x,y\in\mathbb{Z}} B_{e^xd^y},$$

where $B_{e^xd^y} = B_{(x,y)} = \mathcal{A}(e^xd^y[e,d]d^{-y}e^{-x})$ is a generator of a copy of $\mathbb{Z}$. Here $\mathcal{A}\colon \pi_2(T, T\setminus 1) = \pi_2\to\pi_2/[\pi_2,\pi_2] = \mathbb{Z}\pi_1(T)$ is the projection to the abelianized.

(b) *If, by means of this isomorphism, an element of $H_1(\mathcal{F})$ corresponds to the generator 1_w of the copy $\mathbb{Z}$, indexed by an element $w \in \pi_1(T)$, then the action of $\alpha \in \pi_1(T)$ on B_w is the generator of the copy of $\mathbb{Z}$ indexed by the product αw, namely $B_{\alpha w}$.*

We denote by $\mathcal{E}\colon \mathbb{Z}(\pi_1(T)) \to \mathbb{Z}$ the augmentation homomorphism, i.e.

$$\mathcal{E}(B_w) = 1 \in \mathbb{Z} \quad \text{for all } w \in \pi_1(T).$$

In [8, Theorem 3.6, p. 23] we proved the following theorem:

Theorem 4.7. *The homomorphism $\mathcal{E} \circ \mathcal{A}$ satisfies*:

(a) $\mathcal{E} \circ \mathcal{A}(\alpha Z \alpha^{-1}) = \mathcal{E} \circ \mathcal{A}(Z)$ *for all* $\alpha \in \pi_1(T \setminus 1)$ *and* $Z \in \pi_2(T, T \setminus 1)$.

(b) $\mathcal{E} \circ \mathcal{A}([e^{x_1} d^{y_1},\ e^{x_2} d^{y_2}]) = \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$.

(c) $\mathcal{E} \circ \mathcal{A}([Z e^{x_1} d^{y_1}, W e^{x_2} d^{y_2}]) = \mathcal{E} \circ \mathcal{A}([e^{x_1} d^{y_1}, e^{x_2} d^{y_2}])$ *for all* $Z, W \in \pi_2(T, T \setminus 1)$.

(d) *If* $c \in \pi_1(M(\phi) \setminus S^1)$ *as in diagram* (2.4) *then*

$$\mathcal{E} \circ \mathcal{A}(cZc^{-1}) = [\textit{sign of} \det(\phi)]\mathcal{E} \circ \mathcal{A}(Z), \quad \textit{for all } Z \in \pi_2(T, T \setminus 1).$$

Remark 4.8. $\mathcal{A}(\alpha Z \alpha^{-1}) = j_\#(\alpha).\mathcal{A}(Z)$, where $j_\#\colon \pi_1(T \setminus 1) \to \pi_1(T)$. So, if $j_\#(\alpha) = e^m b^n$ and $\mathcal{A}(Z) = B^{t_1}_{(x_1, y_1)}$ then the action $j_\#(\alpha).\mathcal{A}(Z) = B^{t_1}_{(x_1+m, y_1+n)}$.

Now we consider the equation given by Proposition 3.9. We will look at this equation in the abelianized of $\pi_2(T, T \setminus 1)$, which is $\mathbb{Z}\pi_1(T)$, and in a quotient of $\mathbb{Z}\pi_1(T)$, which is $\mathbb{Z}(\widetilde{H})$, where $\widetilde{H} = \mathbb{Z} \oplus \mathbb{Z}/\langle (c_{12} - c_{11}, c_{22} - c_{21}), (p_2 - p_1, q_2 - q_1) \rangle$. Denote by $\overline{\mathcal{A}}(x)$ the image of an element $x \in \pi_2(T, T \setminus 1)$ in $\mathbb{Z}(\widetilde{H})$. Then by [8, Proposition 3.7, p. 24] we have:

Proposition 4.9. *Let E and D as in Proposition* 4.1*. Then the main equation given in Proposition* 3.9 *is of the form*

(a)

$$\mathcal{A}(Z_3) \cdot \mathcal{A}(EcZ_2c^{-1}E^{-1}) \cdot \mathcal{A}(E(cDc^{-1}D^{-a_4})E^{-1}) \cdot \mathcal{A}([E, D^{a_4}]) \cdot \mathcal{A}(D^{a_4} Z_3^{-1} D^{-a_4}) \cdot \mathcal{A}(D^{(a_4-1)/2} Z_2^{-a_4} D^{(1-a_4)/2}) = 1$$

in the abelianized $\mathbb{Z}\pi_1(T)$, where $Z_2, Z_3 \in \pi_2(T, T \setminus 1)$, and

(b)

$$\overline{\mathcal{A}}(cZ_2c^{-1}) \cdot \overline{\mathcal{A}}(cDc^{-1}D^{-a_4}) \cdot \overline{\mathcal{A}}([E, D^{a_4}]) \cdot \overline{\mathcal{A}}(Z_2^{-a_4}) = 1$$

in $\mathbb{Z}\widetilde{H}$, where $Z_2 \in \pi_2(T, T \setminus 1)$.

By applying the homomorphism $\mathcal{E}$ to the left-hand side of the equation given in Proposition 4.9(a) we obtain:

COROLLARY 4.10.

$$[\textit{sign of} \det(\phi)]\mathcal{E}\circ\mathcal{A}(Z_2)+\mathcal{E}\circ\mathcal{A}(cDc^{-1}D^{-a_4})+\mathcal{E}\circ\mathcal{A}([E,D^{a_4}])+\mathcal{E}\circ\mathcal{A}(Z_2^{-a_4})=0.$$

5. The main result: solutions of the main equation

In this section we prove the main result of this work. The result is given as follows:

THEOREM 5.1. *Given fibre-preserving maps $f_i\colon M(\phi)\to M(\phi)$ over S^1 then the pair (f_1,f_2) can be deformed to a coincidence free pair (f_1',f_2') by a fibrewise homotopy over S^1 if and only if one of the cases below holds*:

(a) *$M(\phi)$ is as in the case I and f_i are arbitrary.*

(b) *$M(\phi)$ is as in one of the case* II, III, IV *and*

$$\det\begin{pmatrix} c_{12}-c_{11} & p_2-p_1\\ c_{22}-c_{21} & q_2-q_1\end{pmatrix}=0.$$

(c) *$M(\phi)$ is as in the case* V *and*

$$(p_1-p_2)[(c_{21}-c_{22})+(q_2-q_1)+1]+(q_2-q_1)[1+(c_{11}-c_{12})]\equiv 0 \bmod 2$$

except in the cases listed in the table below:

a_3	(p_2-p_1,q_2-q_1)	$(c_{12}-c_{11},c_{22}-c_{21})$	E	D
$2r>0$	$(2s,0)$	$\equiv(0,0)$	1	e^{2s}
$2r<0$	$(2s,0)$	$\equiv(0,0)$	$[d^{-1},e^{-1}]$	e^{2s}
$2r+1>0$	$(2s,0)$	$\equiv(0,0)$	1	e^{2s}
$2r+1<0$	$(2s,0)$	$\equiv(0,0)$	$[d^{-1},e^{-1}]$	e^{2s}
0	$(2s,2k)$	$\equiv(0,0)$	1	$d^ke^{2s}d^k$

(d) *$M(\phi)$ is as in the Case VI and we have either*

(I) *$a_3=2r$ is even,*

$$(q_2-q_1)[(c_{12}-c_{11})-1-(c_{22}-c_{21})r]\equiv 0 \bmod 2$$

except in the case where $q_2-q_1=2^{r_1}d_1$, where d_1 is odd, and $c_{22}-c_{21}=2^{r_2}d_2$, where d_2 is odd with $1<r_1\le r_2$ and $c_{12}-c_{11}-r(c_{22}-c_{21})\equiv 0 \bmod 2$ or

(II) *a_3 is odd and*

$$(t_2-t_1)(1+c_{22}-c_{21})\equiv 0 \bmod 2$$

except in the case $2(t_2-t_1)/L=2p+1$ and $c_{22}-c_{21}=2q$ where $L=\gcd(2(t_2-t_1),c_{22}-c_{21})$ is the greatest common divisor.

The rest of the section is devoted to the proof of this result. We briefly describe our approach to decide if an equation has a solution or not.

(1) First we compute the necessary condition given by Corollary 4.10 and the set of equivalence classes as defined by Corollary 4.4

(2) Then we choose a set $\mathcal{C}$ of representatives of the equivalence classes in $\mathbb{Z} \oplus \mathbb{Z}/H$ given by Corollary 4.4. For some $(c_{12} - c_{11}, c_{22} - c_{21}) \in \mathcal{C}$ we find elements E, D which correspond to the input data (ϕ, B_i, E, D) and satisfies the equation:

$$EcDc^{-1}E^{-1}D^{-a_4} = 1.$$

This tells us that the equation given by Proposition 4.9(a), with E, D chosen as above, admits the trivial solution $Z_2 = Z_3 = 1$, and allows us to write a sufficient condition, in terms of the data, which guarantees to have a solution.

(3) For some classes $(c_{12}-c_{11}, c_{22}-c_{21}) \in \mathcal{C}$ we show that there is no solution by the main equation in $\mathbb{Z}\pi_1(T)$, which is the abelianized of $\pi_2(T, T \setminus 1)$, or in $\mathbb{Z}\widetilde{H}$, where $\widetilde{H}$ is $\mathbb{Z} \oplus \mathbb{Z}/H$ such that H contains the subgroup $\langle (c_{12} - c_{11}, c_{22} - c_{21}), (p_2 - p_1, q_2 - q_1)\rangle$. Then we will use Corollary 4.4 and Proposition 4.9.

Case I. It was solved in Proposition 3.8.

Case II. $\phi^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B_i = \begin{pmatrix} m & p_i \\ n & q_i \end{pmatrix}$. The equation to be solved is

$$X_3 \cdot EcX_2c^{-1}E^{-1} \cdot E(cDc^{-1}D^{-1})E^{-1} \cdot [E, D] \cdot DX_3^{-1}D^{-1} \cdot X_2^{-1} = 1.$$

The condition given by Corollary 4.10, called the necessary condition, is

$$\mathcal{E} \circ \mathcal{A}(cDc^{-1}D^{-1}) + \mathcal{E} \circ \mathcal{A}([E, D]) = 0.$$

But for every D we have $cDc^{-1}D^{-1} = 1$, because c acts as identity. So

$$\mathcal{E} \circ \mathcal{A}([E, D]) = 0 = \det \begin{pmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & q_2 - q_1 \end{pmatrix}.$$

The sufficient condition is

$$\det \begin{pmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & q_2 - q_1 \end{pmatrix} = 0.$$

We consider $L = \gcd(p_2 - p_1, q_2 - q_1)$ and let (k_1, k_2) be such that $(p_2 - p_1, q_2 - q_1) = L(k_1, k_2)$. If the above determinant is 0 then there exists $t \in \mathbb{Z}$ that $(c_{12} - c_{11}, c_{22} - c_{21}) = t(k_1, k_2)$. We take $E = v^t$ and $D = v^L$, where $v = e^{k_1} d^{k_2}$, and it is easy to verify that $[E, D] = 1$ and $cDc^{-1}D^{-1} = 1$, so the equation admits the trivial solution $X_2 = X_3 = 1$.

Case III. $\phi^1 = \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}$ with $a_3 \neq 0$ and $B_i = \begin{pmatrix} m & p_i \\ 0 & m \end{pmatrix}$ with $p_1 \neq p_2$. The equation to be solved is

$$X_3EcX_2c^{-1}E^{-1}E(cDc^{-1}D^{-1})E^{-1}[E, D]DX_3{}^{-1}D^{-1}X_2{}^{-1} = 1.$$

The necessary condition is

$$\mathcal{E} \circ \mathcal{A}(cDc^{-1}D^{-1}) + \mathcal{E} \circ \mathcal{A}([E, D]) = 0.$$

To compute $\mathcal{E} \circ \mathcal{A}(cDc^{-1}D^{-1})$, we take $D = e^{p_2 - p_1}$ and for this D we have $cDc^{-1}D^{-1} = 1$. So, the above relation becomes:

$$\mathcal{E} \circ \mathcal{A}([E, D]) = 0 = \det \begin{pmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & 0 \end{pmatrix} = -(c_{22} - c_{21})(p_2 - p_1), \quad p_2 \neq p_1,$$

which implies $c_{22} = c_{21}$.

The sufficient condition is $c_{22} = c_{21}$. If this condition is satisfied we take $E = e^{c_{12} - c_{11}}$ and $D = e^{p_2 - p_1}$ and so $[E, D] = 1$. Therefore, the equation admits the trivial solution $X_2 = X_3 = 1$.

Case IV. $\phi^1 = \left(\begin{smallmatrix} 1 & a_3 \\ 0 & -1 \end{smallmatrix}\right)$, $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ 0 & q_i \end{smallmatrix}\right)$ and $a_3(q_i - m) = -2p_i$ with $q_1 \neq q_2$. The equation to be solved is

$$X_3 \cdot EcX_2c^{-1}E^{-1} \cdot E(cDc^{-1}D)E^{-1} \cdot [E, D^{-1}] \cdot D^{-1}X_3^{-1}D \cdot D^{-1}X_2D = 1.$$

The necessary condition is

$$\mathcal{E} \circ \mathcal{A}(cDc^{-1}D) + \mathcal{E} \circ \mathcal{A}([E, D^{-1}]) = 0.$$

In order to calculate this condition, first we consider $a_3 \geq -1$. Since $p_2 - p_1 = -a_3(q_2 - q_1)/2$ then 2 divides either a_3 or $q_2 - q_1$. If 2 divides a_3 consider $v = e^{-a_3/2}d$, otherwise it must divide $(q_2 - q_1)$ and consider $v = de^{-a_3}d$. From the presentation group for $a_3 \geq -1$ we have $cvc^{-1} = v^{-1}$.

Therefore, if either $D = v^{q_2 - q_1}$ or $D = v^{(q_2 - q_1)/2}$ then $cDc^{-1}D = 1$ and so $\mathcal{E} \circ \mathcal{A}(cDc^{-1}D) = 0$.

Let $a_3 \leq -2$ and consider the presentation which corresponds to this case. Denoting by $\beta = e^{-1}d^{-1}e$, then the presentation is given by

$$\langle e, d, c : cec^{-1} = \beta e \beta^{-1}, cdc^{-1} = \beta e^{a_3} d^{-1} \beta^{-1} \rangle.$$

Take v as in the previous case; similar calculation shows:

$$cvc^{-1} = \beta d v^{-1} d^{-1} \beta^{-1} = [e^{-1}, d^{-1}] v^{-1} [d^{-1}, e^{-1}].$$

So, if we take either $D = v^{q_2 - q_1}$ or $D = v^{(q_2 - q_1)/2}$, then $cDc^{-1}D = [[e^{-1}, d^{-1}], D]$ and therefore $\mathcal{E} \circ \mathcal{A}(cDc^{-1}D) = 0$. Since $[E, D^{-1}]D^{-1}[E, D]D = 1$ it follows that

$$\mathcal{E} \circ \mathcal{A}([E, D^{-1}]) = -\mathcal{E} \circ \mathcal{A}([E, D]) = -\det \begin{pmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & q_2 - q_1 \end{pmatrix}.$$

So, the necessary condition is

$$\mathcal{E} \circ \mathcal{A}([E, D^{-1}]) = 0 = \det \begin{pmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & q_2 - q_1 \end{pmatrix}.$$

To prove that

$$\mathcal{E} \circ \mathcal{A}([E, D^{-1}]) = 0 = \det\begin{pmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & q_2 - q_1 \end{pmatrix}$$

is also a sufficient condition, first we will reduce the cases of E using the Corollary 4.4 and finally we will find the solution of the equation for each reduced case.

The image of the map $\pi_1(T \setminus 1) = \langle e, d\rangle \to \mathbb{Z} \oplus \mathbb{Z}$ given by

$$\alpha \to (|\alpha c \alpha^{-1} c^{-1}|_e, |\alpha c \alpha^{-1} c^{-1}|_d)$$

is denoted by $\operatorname{im}(|\cdot|_e, |\cdot|_d)$. We have $e \mapsto (0,0)$ $d \mapsto (-a_3, 2)$, so

$$\frac{\mathbb{Z} \oplus \mathbb{Z}}{\operatorname{im}(|\cdot|_e, |\cdot|_d)} = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (a_3, -2)\rangle}.$$

This quotient is

$$\frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (a_3, -2)\rangle} = \begin{cases} \dfrac{\langle (a_3, -2), (-r, 1)\rangle}{\langle (a_3, -2)\rangle} \simeq \mathbb{Z} & \text{if } a_3 = 2r + 1, \\ \dfrac{\langle (-1, 0), (r, -1)\rangle}{\langle (2r, -2)\rangle} \simeq \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } a_3 = 2r. \end{cases}$$

If $a_3 = 2r + 1$ then a complete set of representatives of E is given by elements of the form $(0, y)$, where these are the coordinates relative to the basis $((a_3, -2), (-r, 1))$ and so $(c_{12} - c_{11}, c_{22} - c_{21}) = (-yr, y)$. These values must satisfy the necessary condition:

$$\det\begin{pmatrix} -yr & p_2 - p_1 \\ y & q_2 - q_1 \end{pmatrix} = 0.$$

So $y[r(q_2 - q_1) + (p_2 - p_1)] = 0$. From the condition $(2r+1)(q_2 - q_1) = -2(p_2 - p_1)$ it follows that:

(a) If $q_2 - q_1 = 0$ then $p_2 - p_1 = 0$ and the result follows from the case I.

(b) If $q_2 - q_1 \neq 0$, follows that $r(q_2 - q_1) + (p_2 - p_1) \neq 0$ and therefore $y = 0$.

For $a_3 \geq -1$, take $E = 1$ and $D = v^{(q_2 - q_1)/2}$, where $v = de^{-a_3}d$, and for $a_3 \leq -2$ take $E = [d^{-1}, e^{-1}]$ and $D = v^{(q_2 - q_1)/2}$ where $v = de^{-a_3}d$, so in each case we have $EcDc^{-1}E^{-1}D = 1$ and therefore the equation admits the trivial solution.

If $a_3 = 2r$ then a complete set of representatives for E is given by elements of the form $(x, \overline{y})$, where (x, y) are the coordinates relative to the basis $\{(-1, 0), (r, -1)\}$ and $\overline{y}$ means y module 2. So a set of representatives are the elements of the form either $(c_{12} - c_{11}, c_{22} - c_{21}) = (-x, 0)$ or $(c_{12} - c_{11}, c_{22} - c_{21}) = (-x + r, -1)$.

From the necessary condition $\det\left(\begin{smallmatrix} c_{12} - c_{11} & p_2 - p_1 \\ c_{22} - c_{21} & q_2 - q_1 \end{smallmatrix}\right) = 0$ and since $2r(q_2 - q_1) = -2(p_2 - p_1)$ we have:

(a) If $q_2 = q_1$ then $p_1 = p_2$ and the result follows from Case I.

(b) If $q_1 \neq q_2$, substituting the representatives in the necessary condition we conclude that $x = 0$.

So, for E and D below, we have $EcDc^{-1}E^{-1}D = 1$ and therefore the equation admits the trivial solution.

a_3	$(c_{12} - c_{11}, c_{22} - c_{21})$	E	D
$2r \geq -1$	$(0, 0)$	1	$(e^{-r}d)^{q_2 - q_1}$
$2r \geq -1$	$(-r, 1)$	$e^{-r}d$	$(de^{-r})^{q_2 - q_1}$
$2r \leq -2$	$(0, 0)$	$[d^{-1}, e^{-1}]$	$(e^{-r}d)^{q_2 - q_1}$
$2r \leq -2$	$(-r, 1)$	$e^{-r}e^{-1}de$	$(de^{-r})^{q_2 - q_1}$

For the table above we observe that $(-r, 1)$ and $(r, -1)$ are in the same class in $\langle (-1, 0), (r, -1) \rangle / \langle (2r, -2) \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

Case V. $\phi^1 = \left(\begin{smallmatrix} -1 & a_3 \\ 0 & -1 \end{smallmatrix}\right)$, $B_i^1 = \left(\begin{smallmatrix} m & p_i \\ n & q_i \end{smallmatrix}\right)$ with $a_3(q_i - m) = 0$, $a_3 n = 0$ and $(p_2 - p_1, q_2 - q_1) \neq (0, 0)$. The equation to be solved is

$$X_3 \cdot EcX_2c^{-1}E^{-1} \cdot E(cDc^{-1}D)E^{-1} \cdot [E, D^{-1}] \cdot D^{-1}X_3^{-1}D \cdot D^{-1}X_2D = 1.$$

The necessary condition is

$$2\mathcal{E} \circ \mathcal{A}(X_2) + \mathcal{E} \circ \mathcal{A}(cDc^{-1}D) + \mathcal{E} \circ \mathcal{A}([E, D^{-1}]) = 0.$$

Observe that $\mathcal{E} \circ \mathcal{A}(cDc^{-1}D)$ depends on the choice of D, but $\mathcal{E} \circ \mathcal{A}([E, D^{-1}])$ does not, since if $D_1 = \alpha D$ and $E_1 = \beta E$ with $\alpha, \beta \in \pi_2(T, T \setminus 1)$, then

$$\begin{aligned} \mathcal{E} \circ \mathcal{A}(cD_1c^{-1}D_1) &= 2\mathcal{E} \circ \mathcal{A}(\alpha) + \mathcal{E} \circ \mathcal{A}(cDc^{-1}D), \\ \mathcal{E} \circ \mathcal{A}([E_1, D_1^{-1}] &= \mathcal{E} \circ \mathcal{A}([E, D^{-1}]). \end{aligned}$$

From the above we conclude that the augmentation mod 2, denoted by $(\mathcal{E} \circ \mathcal{A})_2$, is independent of D and in order to calculate this condition module 2, we can take $D = e^{p_2 - p_1}d^{q_2 - q_1}$ and so $(\mathcal{E} \circ \mathcal{A})_2(cDc^{-1}D) + (\mathcal{E} \circ \mathcal{A})_2([E, D^{-1}]) = 0$ or

$$(p_1 - p_2)[(c_{21} - c_{22}) + (q_2 - q_1) + 1] + (q_2 - q_1)[1 + (c_{11} - c_{12})] \equiv 0 \bmod 2.$$

Since $[E, D^{-1}]D^{-1}[E, D]D = 1$ follows that

$$\mathcal{E} \circ \mathcal{A}([E, D^{-1}]) = \mathcal{E} \circ \mathcal{A}([D, E]) = \det \begin{pmatrix} p_2 - p_1 & c_{12} - c_{11} \\ q_2 - q_1 & c_{22} - c_{21} \end{pmatrix}.$$

If $a_3 \neq 0$,

$$cDc^{-1}D = \begin{cases} e^{-1}[d^{-1}, e^{p_1 - p_2}]e & \text{if } a_3 < 0, \\ [d^{-1}, e^{p_1 - p_2}] & \text{if } a_3 \geq 1. \end{cases}$$

If $a_3 = 0$,

$$cDc^{-1}D = e^{-1}[d^{-1}, e^{p_1 - p_2}]e\ e^{p_1 - p_2}[e^{-1}, d^{q_1 - q_2}]e^{p_2 - p_1}[e^{p_1 - p_2}, d^{q_1 - q_2}]$$

and therefore

$$\mathcal{E}\circ\mathcal{A}(cDc^{-1}D)=\begin{cases}\det\begin{pmatrix}0 & p_1-p_2\\ -1 & 0\end{pmatrix} & \text{if } a_3\neq 0,\\ \det\begin{pmatrix}0 & p_1-p_2\\ -1 & 0\end{pmatrix}+\det\begin{pmatrix}-1 & 0\\ 0 & q_1-q_2\end{pmatrix} & \\ \qquad +\det\begin{pmatrix}p_1-p_2 & 0\\ 0 & q_1-q_2\end{pmatrix} & \text{if } a_3=0.\end{cases}$$

So

$$\mathcal{E}\circ\mathcal{A}(cDc^{-1}D)=\begin{cases}(p_1-p_2) & \text{if } a_3\neq 0,\\ (p_1-p_2)[1+(q_1-q_2)]+(q_2-q_1) & \text{if } a_3=0.\end{cases}$$

Therefore $\mathcal{E}\circ\mathcal{A}(cDc^{-1}D)=(p_1-p_2)[1+(q_1-q_2)]+(q_2-q_1)$ for all a_3 (note that $a_3\neq 0$ implies $q_2-q_1=0$). So $(\mathcal{E}\circ\mathcal{A})_2(cDc^{-1}D)+(\mathcal{E}\circ\mathcal{A})_2([E,D^{-1}])=0$ is equivalent to

$$(p_1-p_2)[(c_{21}-c_{22})+(q_2-q_1)+1]+(q_2-q_1)[1+(c_{11}-c_{12})]\equiv 0 \bmod 2.$$

The image of the map $\Pi_1(T\setminus 1)=\langle e,d\rangle\to\mathbb{Z}\oplus\mathbb{Z}$ given by

$$\alpha\to(|\alpha c\alpha^{-1}c^{-1}|_e,|\alpha c\alpha^{-1}c^{-1}|_d)$$

is denoted by $\mathrm{im}(|\cdot|_e,|\cdot|_d)$. We have $e\to(2,0)$, $d\to(-a_3,2)$.

If $a_3=2r$, where $r\geq 0$, then

$$\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{im}(|\cdot|_e,|\cdot|_d)}=\frac{\langle(1,0),(-r,1)\rangle}{\langle 2(1,0),2(-r,1)\rangle}\simeq\mathbb{Z}_2\oplus\mathbb{Z}_2.$$

Let $(|E|_e,|E|_d)=(c_{12}-c_{11},c_{22}-c_{21})$. If $c_{22}-c_{21}$ is odd, a set of representatives of E is given by $\{(-r,1),(-r+1,1)\}$ and if $c_{22}-c_{21}$ is even, a set of representatives of E is either $\{(-2r+1,2),(0,0)\}$ or $\{(1,0),(0,0)\}$.

If $a_3=2r+1$ then

$$\frac{\mathbb{Z}\oplus\mathbb{Z}}{im(|\cdot|_e,|\cdot|_d)}=\frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle 2(1,0),(-2r-1,2)\rangle}=\frac{\langle(1,2),(0,1)\rangle}{\langle(1,2),(0,4)\rangle}\simeq 0\oplus\mathbb{Z}_4.$$

Let $(|E|_e,|E|_d)=(c_{12}-c_{11},c_{22}-c_{21})$. If $c_{22}-c_{21}$ is odd, a set of representatives of E is either $\{(0,3),(0,1)\}$ or $\{(1,1),(0,1)\}$ and if $c_{22}-c_{21}$ is even, a set of representatives of E is given by $\{(0,0),(0,2)\}$.

For $a_3=2r$, where $r\in\mathbb{Z}$, the sufficient condition is $(c_{12}-c_{11},c_{22}-c_{21})\not\equiv(0,0)$, that is $c_{12}-c_{11}\not\equiv 0 \bmod 2$ or $c_{22}-c_{21}\not\equiv 0 \bmod 2$. For $a_3=2r+1$, where $r\in\mathbb{Z}$, it is $(c_{12}-c_{11},c_{22}-c_{21})\not\equiv(0,0)$, that is $c_{22}-c_{21}\not\equiv 0 \bmod 4$.

For E and D specified below we have $EcDc^{-1}E^{-1}D = 1$, so the main equation admits trivial solution $X_2 = X_3 = 1$:

$a_3 = 2r > 0$	$c_{22} - c_{21}$	$(p_2 - p_1, q_2 - q_1)$	$(c_{12} - c_{11}, c_{22} - c_{21})$	E	D
r even	odd	$(p_2 - p_1, 0)$	$(-r, 1) \equiv (0, 1)$	d	$e^{p_2 - p_1}$
r even	odd	$(p_2 - p_1, 0)$	$(-r + 1, 1) \equiv (1, 1)$	ed	$e^{p_2 - p_1}$
r odd	odd	$(p_2 - p_1, 0)$	$(-r + 1, 1) \equiv (0, 1)$	d	$e^{p_2 - p_1}$
r odd	odd	$(p_2 - p_1, 0)$	$(-r, 1) \equiv (1, 1)$	ed	$e^{p_2 - p_1}$
	even	$(2s, 0)$	$(-2r + 1, 2) \equiv (1, 0)$	$de^{-2r+1}d$	$e^s de^s d^{-1}$
$a_3 = 2r < 0$	$c_{22} - c_{21}$	$(p_2 - p_1, q_2 - q_1)$	$(c_{12} - c_{11}, c_{22} - c_{21})$	E	D
r even	odd	$(p_2 - p_1, 0)$	$(-r, 1) \equiv (0, 1)$	$e^{-1}de$	$e^{p_2 - p_1}$
r even	odd	$(p_2 - p_1, 0)$	$(-r + 1, 1) \equiv (1, 1)$	de	$e^{p_2 - p_1}$
r odd	odd	$(p_2 - p_1, 0)$	$(-r + 1, 1) \equiv (0, 1)$	$e^{-1}de$	$e^{p_2 - p_1}$
r odd	odd	$(p_2 - p_1, 0)$	$(-r, 1) \equiv (1, 1)$	de	$e^{p_2 - p_1}$
	even	$(2s, 0)$	$(-2r + 1, 2) \equiv (1, 0)$	$de^{-2r}de$	$e^s de^s d^{-1}$

$a_3 = 2r+1 > 0$	$c_{22} - c_{21}$	$(p_2 - p_1, q_2 - q_1)$	$(c_{12} - c_{11}, c_{22} - c_{21})$	E	D
	odd	$(p_2 - p_1, 0)$	$(0, 3) \equiv (1, 1)$	ed	$e^{p_2 - p_1}$
	odd	$(p_2 - p_1, 0)$	$(0, 1)$	d	$e^{p_2 - p_1}$
	even	$(2s, 0)$	$(0, 2) \equiv (-2r, 2)$	$de^{-2r}d$	$e^s de^s d^{-1}$
$a_3 = 2r+1 < 0$	$c_{22} - c_{21}$	$(p_2 - p_1, q_2 - q_1)$	$(c_{12} - c_{11}, c_{22} - c_{21})$	E	D
	odd	$(p_2 - p_1, 0)$	$(0, 3) \equiv (1, 1)$	de	$e^{p_2 - p_1}$
	odd	$(p_2 - p_1, 0)$	$(0, 1)$	$e^{-1}de$	$e^{p_2 - p_1}$
	even	$(2s, 0)$	$(0, 2) \equiv (-2r, 2)$	$de^{-2r-1}de$	$e^s de^s d^{-1}$

$a_3 = 0$	$(c_{12} - c_{11}, c_{22} - c_{21})$	$(p_2 - p_1, q_2 - q_1)$	E	D
	$\equiv (0, 1)$	$p_2 - p_1$ odd, or	$e^{p_2 - p_1 - 2}de$	$e^{p_2 - p_1 - 1}d^{q_2 - q_1}e$
		$q_2 - q_1 = 2k$ and $p_2 - p_1$ even	$e^{p_1 - p_2 + 1}de$	$d^k ed^k e^{p_2 - p_1 - 1}$
	$\equiv (1, 0)$	$q_2 - q_1 = 2k + 1$, or	$d^{2k}e$	$d^{2k}e^{p_2 - p_1}d$
		$q_2 - q_1 = 2k$ and $p_2 - p_1 = 2s$	$d^{q_1 - q_2 + 2}e$	$e^s de^s d^{2k-1}$
	$(1, 1)$	$p_2 - p_1 = 2s$ or	de	$e^s d^{q_2 - q_1} e^s$
		$q_2 - q_1 = 2k$	de	$d^k e^{p_2 - p_1} d^k$

There is no solution for the remaining cases. To see this, let us consider E, D as given below:

a_3	$(p_2 - p_1, q_2 - q_1)$	$(c_{12} - c_{11}, c_{22} - c_{21})$	E	D
$2r > 0$	$(2s, 0)$	$\equiv (0, 0)$	1	e^{2s}
$2r < 0$	$(2s, 0)$	$\equiv (0, 0)$	$[d^{-1}, e^{-1}]$	e^{2s}
$2r + 1 > 0$	$(2s, 0)$	$\equiv (0, 0)$	1	e^{2s}
$2r + 1 < 0$	$(2s, 0)$	$\equiv (0, 0)$	$[d^{-1}, e^{-1}]$	e^{2s}
0	$(2s, 2k)$	$\equiv (0, 0)$	1	$d^k e^{2s} d^k$

In order to prove that there is no solution for the cases above, we write the term

$$EcDc^{-1}DE^{-1}[E, D^{-1}] = EcDc^{-1}E^{-1}D$$

on the generators $B_{(x,y)}$ of the abelianized group $\mathbb{Z}\pi_1(T)$.

For $a_3 \neq 0$ we have $EcDc^{-1}E^{-1}D = [d^{-1}, e^{-2s}]$, so

$$\mathcal{A}(EcDc^{-1}E^{-1}D) = \mathcal{A}([d^{-1}, (e^{-1})^{2s}]) = B^{-1}_{(-1,-1)} B^{-1}_{(-2,-1)} B^{-1}_{(-3,-1)} \cdots B^{-1}_{(-2s,-1)}$$

If $\mathcal{A}(X_2)$ contains $B^n_{(x,y)}$ as a summand then

$$\mathcal{A}(EcX_2c^{-1}E^{-1}) \quad \text{contains} \quad B^n_{(-x+a_3y+a_3-2,-y-2)}$$

and

$$\mathcal{A}(D^{-1}X_2D) \quad \text{contains} \quad B^n_{(x-2s,y)}.$$

Let $H = \langle(2s,0)\rangle$ be the subgroup of $\pi = \pi_1(T) \simeq \mathbb{Z}\oplus\mathbb{Z}$. Now look the equation in $\mathbb{Z}(\pi/H)$. In $\mathbb{Z}(\pi/H)$ it reduces to:

$$\overline{\mathcal{A}}(EcX_2c^{-1}E^{-1})\overline{\mathcal{A}}(EcDc^{-1}E^{-1}D)\overline{\mathcal{A}}(D^{-1}X_2D) = 1$$

and

$$\overline{\mathcal{A}}(EcDc^{-1}E^{-1}D) \;=\; \overline{\mathcal{A}}([d^{-1},(e^{-1})^{2s}]) \;=\; B^{-1}_{(2s-1,-1)}B^{-1}_{(2s-2,-1)}\cdots B^{-1}_{(0,-1)}$$

In order to cancel the term $B^{-1}_{(s-1,-1)}$ or $B^{-1}_{(2s-1,-1)}$, we must have $B_{(s-1,-1)}$ or $B_{(2s-1,-1)}$ as a summand in $\overline{\mathcal{A}}(X_2)$. Then

$$\begin{aligned}\overline{\mathcal{A}}(EcX_2c^{-1}E^{-1}) &\quad \text{contains} \quad B_{(s-1,-1)} \quad \text{or} \quad B_{(2s-1,-1)},\\ \overline{\mathcal{A}}(D^{-1}X_2D) &\quad \text{contains} \quad B_{(s-1,-1)} \quad \text{or} \quad B_{(2s-1,-1)}.\end{aligned}$$

So, the total exponent in $B_{(s-1,-1)}$ or $B_{(2s-1,-1)}$ is even, therefore it is impossible to cancel with $B^{-1}_{(s-1,-1)}$ or $B^{-1}_{(2s-1,-1)}$.

For $a_3 = 0$, we have $EcDc^{-1}E^{-1}D = cDc^{-1}D = [e^{-1}d^{-1}, d^{-k}e^{-2s}d^{-k}]$, so

$$\begin{aligned}(cDc^{-1}D)^{-1} &= [d^{-k}e^{-2s}d^{-k}, e^{-1}d^{-1}]\\ &= [d^{-k}e^{-2s}d^{-k}, e^{-1}]e^{-1}[d^{-k}e^{-2s}d^{-k}, d^{-1}]e,\end{aligned}$$

and hence

$$\begin{aligned}cDc^{-1}D &= e^{-1}[d^{-1}, d^{-k}e^{-2s}d^{-k}]e[e^{-1}, d^{-k}e^{-2s}d^{-k}]\\ &= e^{-1}d^{-k}[d^{-1}, e^{-2s}]d^ke[e^{-1}, d^{-k}e^{-2s}]d^{-k}e^{-2s}[e^{-1}, d^{-k}]e^{2s}d^k\\ &= e^{-1}d^{-k}[d^{-1}, e^{-2s}]d^ke[e^{-1}, d^{-k}]d^{-k}e^{-2s}[e^{-1}, d^{-k}]e^{2s}d^k.\end{aligned}$$

Using the formula $[x,y^n] = [x,y]y[x,y]y^{-1}\dots y^{n-1}[x,y]y^{-n+1}$, for the commutators $[d^{-1},{(e^{-1})}^{2s}]$ and $[e^{-1},{(d^{-1})}^{k}]$ (here we suppose that $s>0$ and $k>0$) we have

$$[d^{-1},{(e^{-1})}^{2s}] = [d^{-1},e^{-1}]e^{-1}[d^{-1},e^{-1}]e\dots e^{-2s+1}[d^{-1},e^{-1}]e^{2s-1}.$$

In $\mathbb{Z}\pi$ we have $\mathcal{A}([d^{-1},e^{-1}]) = \mathcal{A}(d^{-1}e^{-1}ded^{-1}e^{-1}ed) = B^{-1}_{(-1,-1)}$ and therefore,

$$\mathcal{A}([d^{-1},(e^{-1})^{2s}]) = B^{-1}_{(-1,-1)}B^{-1}_{(-2,-1)}B^{-1}_{(-3,-1)}\cdots B^{-1}_{(-2s,-1)}.$$

Next,

$$[e^{-1},{(d^{-1})}^{k}] = [e^{-1},d^{-1}]d^{-1}[e^{-1},d^{-1}]dd^{-2}[e^{-1},d^{-1}]d^2\dots d^{-k+1}[e^{-1},d^{-1}]d^{k-1}.$$

In $\mathbb{Z}\pi$ we have

$$\mathcal{A}([e^{-1},d^{-1}]) = \mathcal{A}(e^{-1}d^{-1}ede^{-1}d^{-1}de) = B_{(-1,-1)}$$

and therefore,

$$\mathcal{A}([e^{-1},(d^{-1})^k]) = B_{(-1,-1)}B_{(-1,-2)}B_{(-1,-3)}\cdots B_{(-1,-k)}.$$

Finally,

$$\begin{aligned}\mathcal{A}(cDc^{-1}D) = &(B^{-1}_{(-2,-1-k)}B^{-1}_{(-3,-1-k)}\cdots B^{-1}_{(-2s-1,-1-k)})\\ &\cdot(B_{(-1,-1)}B_{(-1,-2)}B_{(-1,-3)}\cdots B_{(-1,-k)})\\ &\cdot(B_{(-1-2s,-1-k)}B_{(-1-2s,-2-k)}B_{(-1-2s,-3-k)}\cdots B_{(-1-2s,-k-k)}).\end{aligned}$$

Let $H=\langle(2s,0),(0,2k)\rangle$ be the subgroup of $\pi\simeq\mathbb{Z}\oplus\mathbb{Z}$. Now look the equation in $\mathbb{Z}(\pi/H)$. The equivalence classes admit $B_{(x,y)}$ as a set of representatives for $0\le x\le 2s-1$ and $0\le y\le 2k-1$. After projecting it on $\mathbb{Z}(\pi/H)$, we get

$$\begin{aligned}\overline{\mathcal{A}}(cDc^{-1}D) = &(B^{-1}_{(2s-2,k-1)}B^{-1}_{(2s-3,k-1)}\cdots B^{-1}_{(0,k-1)}B^{-1}_{(2s-1,k-1)})\\ &\cdot(B_{(2s-1,2k-1)}B_{(2s-1,2k-2)}B_{(2s-1,2k-3)}\cdots B_{(2s-1,2k-k)})\\ &\cdot(B_{(2s-1,k-1)}B_{(2s-1,k-2)}B_{(2s-1,k-3)}\cdots B_{(2s-1,0)}).\end{aligned}$$

The term $B^{-1}_{(s-1,k-1)}B_{(2s-1,2k-1)}$ is different from 1 since in this case $s\neq 0$ or $k\neq 0$ and it is impossible to be cancelled using the variable X_2 (certainly also using the variable X_3). In $\mathbb{Z}(\pi/H)$ it reduces to

$$\overline{\mathcal{A}}(cX_2c^{-1})\overline{\mathcal{A}}(cDc^{-1}D)\overline{\mathcal{A}}(D^{-1}X_2D)=1.$$

If $\mathcal{A}(X_2)$ has the term $B_{(s-1,k-1)}B^{-1}_{(2s-1,2k-1)}$ then

$$\overline{\mathcal{A}}(cX_2c^{-1}) \text{ has the term } B_{(-s-1,-k-1)}B^{-1}_{(-2s-1,-2k-1)}.$$

In $\mathbb{Z}(\pi/H)$: $\overline{\mathcal{A}}(cX_2c^{-1})$ has the term $B_{(s-1,k-1)}B^{-1}_{(2s-1,2k-1)}$ and

$$\overline{\mathcal{A}}(D^{-1}X_2D) \text{ has the term } B_{(-s-1,-k-1)}B^{-1}_{(-1,-1)} = B_{(s-1,k-1)}B^{-1}_{(2s-1,2k-1)},$$

which shows that we cannot make powers of $B_{(s-1,k-1)}$ and $B_{(2s-1,2k-1)}$ to be zero.

CASE VI. $\phi=\left(\begin{smallmatrix}-1 & a_3\\ 0 & 1\end{smallmatrix}\right)$, $B_i=\left(\begin{smallmatrix}m & p_i\\ 0 & q_i\end{smallmatrix}\right)$ and $a_3(q_i-m)=2p_i$ with $q_1\neq q_2$. The equation to be solved is

$$X_3\cdot EcX_2c^{-1}E^{-1}\cdot E(cDc^{-1}D^{-1})E^{-1}\cdot[E,D]\cdot DX_3^{-1}D^{-1}\cdot X_2^{-1}=1.$$

The augmentation homomorphism $\mathcal{E}\circ\mathcal{A}$ on the equation provides the condition

$$-2\mathcal{E}\circ\mathcal{A}(X_2)+\mathcal{E}\circ\mathcal{A}(cDc^{-1}D^{-1})+\mathcal{E}\circ\mathcal{A}([E,D])=0.$$

This condition module 2 is $(\mathcal{E}\circ\mathcal{A})_2(cDc^{-1}D^{-1})+(\mathcal{E}\circ\mathcal{A})_2([E,D])=0 \bmod 2$.

We divide in two subcases: a_3 even and a_3 odd.

Subcase $a_3=2r$. Then $2r(q_2-q_1)=2(p_2-p_1)$, $\phi=\left(\begin{smallmatrix}-1 & 2r\\ 0 & 1\end{smallmatrix}\right)$, $B_i=\left(\begin{smallmatrix}m & r(q_i-m)\\ 0 & q_i\end{smallmatrix}\right)$ with $q_2-q_1\neq 0$.

We summarize the data of this case by:

$$(\phi, B_i, |E|_e, |E|_d) = \left(\begin{pmatrix} -1 & 2r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & r(q_i - m) \\ 0 & q_i \end{pmatrix}, c_{12} - c_{11}, c_{22} - c_{21}\right).$$

To compute the necessary condition module 2, take $v = e^r d$ and so $cvc^{-1} = e^{-1}ve$. Now, if $D = v^{q_2 - q_1}$, then $cDc^{-1}D^{-1} = [e^{-1}, v^{q_2-q_1}] = [e^{-1}, D]$ and therefore

$$\begin{aligned}&(\mathcal{E} \circ \mathcal{A})_2(cDc^{-1}D^{-1}) + (\mathcal{E} \circ \mathcal{A})_2([E, D]) \\ &\qquad = \det\begin{pmatrix} -1 & r(q_2 - q_1) \\ 0 & q_2 - q_1 \end{pmatrix} + \det\begin{pmatrix} c_{12} - c_{11} & r(q_2 - q_1) \\ c_{22} - c_{21} & q_2 - q_1 \end{pmatrix} \equiv 0 \bmod 2\end{aligned}$$

and so

$$(q_2 - q_1)[(c_{12} - c_{11}) - 1 - (c_{22} - c_{21})r] \equiv 0 \bmod 2.$$

To solve the equation for the input data

$$(\phi, B_i, |E|_e, |E|_d) = \left(\begin{pmatrix} -1 & 2r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & r(q_i - m) \\ 0 & q_i \end{pmatrix}, c_{12} - c_{11}, c_{22} - c_{21}\right)$$

is equivalent to solve for the input data

$$\begin{aligned}&(\phi', B'_i, |E'|_e, |E'|_d) \\ &\qquad = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & q_i \end{pmatrix}, (c_{12} - c_{11}) - r(c_{22} - c_{21}), c_{22} - c_{21}\right).\end{aligned}$$

To see this, it is sufficient to consider the isomorphism $\varphi\colon G_1 \to G_2$ given by

$$e \to e, \quad d \to e^{-r}d, \quad c \to c,$$

where $G_1 = \langle e, d, c : cec^{-1} = e^{-1}, cdc^{-1} = e^{-1}e^{2r}de\rangle$ is the group for the first data and $G_2 = \langle e, d, c : cec^{-1} = e^{-1}, cdc^{-1} = e^{-1}de\rangle$ is the group for the second data.

Now we consider the input data

$$(\phi', B'_i, |E'|_e, |E'|_d) = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & q_i \end{pmatrix}, (c_{12} - c_{11}) - r(c_{22} - c_{21}), c_{22} - c_{21}\right)$$

satisfying the necessary condition

$$(q_2 - q_1)[(c_{12} - c_{11}) - 1 - (c_{22} - c_{21})r] \equiv 0 \bmod 2.$$

In this case the map $\pi_1(T\setminus 1)) \to \mathbb{Z}\oplus\mathbb{Z}$ given by $\alpha \to (|\alpha c\alpha^{-1}c^{-1}|_e, |\alpha c\alpha^{-1}c^{-1}|_d)$ maps $e \mapsto (2, 0)$ and $d \mapsto (0, 0)$.

We consider the quotient group

$$\mathbb{Z} \oplus \mathbb{Z}/\langle(2, 0)\rangle = \frac{\langle(1, 0), (0, 1)\rangle}{\langle(2, 0)\rangle} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}.$$

So, it suffices to take E' such that $(|E'|_e, |E'|_d) = (0, c_{22} - c_{21})$, $(|E'|_e, |E'|_d) = (1, c_{22} - c_{21})$.

If $|E'|_e = (c_{12}-c_{11})-r(c_{22}-c_{21}) \equiv 1 \bmod 2$, then the problem has a solution. Take $D' = \varphi(D) = d^{(q_2-q_1)}$ and $E' = d^{c_{22}-c_{21}}$, so $E'cD'c^{-1}E'^{-1}D'^{-1} = 1$ and the result follows.

If $|E'|_e = (c_{12}-c_{11})-r(c_{22}-c_{21}) \equiv 0 \bmod 2$, it follows from the necessary condition that q_2-q_1 is even, i.e. $q_2-q_1 = 2k$. Let $L = \gcd(q_2-q_1, c_{22}-c_{21})$ be the greatest common divisor. If $(q_2-q_1)/L = 2p$, then $c_{22}-c_{21}/L = 2q-1$ and in this case the equation has solution. Take $v = cdc^{-1} = e^{-1}de$ and note that $cvc^{-1} = d$. Now, if $D' = (d^Lv^L)^p$ and $E' = v^{-L}(v^Ld^L)^q$ then

$$E'cD'c^{-1}E'^{-1}D'^{-1} = E'(v^Ld^L)^pE'^{-1}D'^{-1} = v^{-L}(v^Ld^L)^pv^LD'^{-1} = 1$$

and the result follows.

Now suppose $q_2-q_1 = 2^{r_1}d_1$, where d_1 is odd and $c_{22}-c_{21} = 2^{r_2}d_2$, where d_2 is odd with $1 < r_1 \le r_2$ and $(c_{12}-c_{11})-r(c_{22}-c_{21}) \equiv 0 \bmod 2$ where the last condition is equivalent to $c_{12}-c_{11} \equiv 0 \bmod 2$, since $c_{22}-c_{21}$ is even. Let us show that in this case we have no solution.

Let $D' = d^{q_2-q_1}$ and $E' = d^{c_{22}-c_{21}}$. Then

$$\begin{aligned}
&E'(cD'c^{-1}D'^{-1})E'^{-1}[E',D'] \\
&\qquad = E'cD'c^{-1}E'^{-1}D'^{-1} = d^{c_{22}-c_{21}}cd^{q_2-q_1}c^{-1}d^{c_{21}-c_{22}}d^{q_1-q_2} \\
&\qquad = d^{c_{22}-c_{21}}e^{-1}d^{q_2-q_1}ed^{c_{21}-c_{22}}d^{q_1-q_2} = d^{c_{22}-c_{21}}[e^{-1},d^{q_2-q_1}]d^{c_{21}-c_{22}}.
\end{aligned}$$

But, since $\mathcal{A}([e^{-1},d]) = B^{-1}_{(-1,0)}$ and $\mathcal{A}([e^{-1},d^{-1}]) = B_{(-1,-1)}$ we have

$$\mathcal{A}([e^{-1},d^{q_2-q_1}]) = \begin{cases} B^{-1}_{(-1,0)}B^{-1}_{(-1,1)}B^{-1}_{(-1,2)}\dots B^{-1}_{(-1,q_2-q_1-1)} & \text{if } q_2-q_1 \ge 1, \\ B_{(-1,-1)}B_{(-1,-2)}B_{(-1,-3)}\dots B_{(-1,q_1-q_2)} & \text{if } q_2-q_1 \le -1. \end{cases}$$

Therefore,

$$\begin{aligned}
&\mathcal{A}(E'cD'c^{-1}E'^{-1}D'^{-1}) \\
&= \begin{cases} B^{-1}_{(-1,c_{22}-c_{21})}B^{-1}_{(-1,1+c_{22}-c_{21})}\dots B^{-1}_{(-1,q_2-q_1-1+c_{22}-c_{21})} & \text{if } q_2-q_1 \ge 1, \\ B_{(-1,-1+c_{22}-c_{21})}B_{(-1,-2+c_{22}-c_{21})}\dots B_{(-1,q_1-q_2+c_{22}-c_{21})} & \text{if } q_2-q_1 \le -1. \end{cases}
\end{aligned}$$

The equation is

$$\mathcal{A}(X_3)\mathcal{A}(E'cX_2c^{-1}E'^{-1})\mathcal{A}(E'cD'c^{-1}E'^{-1}D'^{-1})\mathcal{A}(D'X_3^{-1}D'^{-1})\mathcal{A}(X_2^{-1}) = 1.$$

If we denote $\mathcal{A}(X_2) = B^n_{(x,y)}$ and $\mathcal{A}(X_3) = B^m_{(z,w)}$ we have $\mathcal{A}(cX_2c^{-1}) = B^{-n}_{(-x-2,y)}$, since $cB_{(0,0)}c^{-1} = B^{-1}_{(-2,0)}$. So, $\mathcal{A}(E'cX_2c^{-1}E'^{-1}) = B^{-n}_{(-x-2,y+c_{22}-c_{21})}$ and $\mathcal{A}(D'X_3^{-1}D'^{-1}) = B^{-m}_{(z,w+q_2-q_1)}$.

In fact, consider the subgroup $H = \langle(0,L)\rangle$ of $\mathbb{Z}\oplus\mathbb{Z}$ where $L = \gcd(q_2-q_1, c_{22}-c_{21})$ and $(q_2-q_1)/L = 2u+1$. Now, we look at the equation in $\mathbb{Z}(\pi/H)$. In $\mathbb{Z}\pi$ the equation

$$\mathcal{A}(X_3)\mathcal{A}(E'cX_2c^{-1}E'^{-1})\mathcal{A}(E'cD'c^{-1}E'^{-1}D'^{-1})\mathcal{A}(D'X_3^{-1}D'^{-1})\mathcal{A}(X_2^{-1}) = 1$$

is given by

$$1 = B^m_{(z,w)} B^{-n}_{(-x-2,y+c_{22}-c_{21})} B^{-1}_{(-1,c_{22}-c_{21})} B^{-1}_{(-1,1+c_{22}-c_{21})} B^{-1}_{(-1,2+c_{22}-c_{21})} \cdots \\ B^{-1}_{(-1,q_2-q_1-1+c_{22}-c_{21})} B^{-m}_{(z,w+q_2-q_1)} B^{-n}_{(x,y)}$$

After projecting it on $\mathbb{Z}(\pi/H)$ we get

$$B^{-n}_{(-x-2,y)} B^{-2u-1}_{(-1,0)} B^{-2u-1}_{(-1,1)} \dots B^{-2u-1}_{(-1,L-1)} B^{-n}_{(x,y)} = 1.$$

In $\mathbb{Z}(\pi/H)$ we have that $\mathcal{A}(E'cX_2c^{-1}E'^{-1}) = B^{-n}_{(-x-2,y)}$. Therefore, the sum of the powers of all $B_{(-1,i)}$, $i = 0, \dots, L-1$, which appears in $\mathcal{A}(E'cX_2c^{-1}E'^{-1}X_2^{-1})$ is even. On the other hand, this sum is $-2u-1$ which is odd. So, there is no solution.

Subcase $a_3 = 2r+1$. Then $q_2 - q_1 = 2(t_2 - t_1) \neq 0$ and therefore

$$p_2 - p_1 = (2r+1)(t_2 - t_1), \quad \phi = \begin{pmatrix} -1 & 2r+1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} m & (2r+1)t_i \\ 0 & 2t_i + m \end{pmatrix}.$$

We summarize the input data of this case by

$$(\phi, B_i, |E|_e, |E|_d) = \left(\begin{pmatrix} -1 & 2r+1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & (2r+1)t_i \\ 0 & 2t_i + m \end{pmatrix}, c_{12} - c_{11}, c_{22} - c_{21} \right).$$

To compute the necessary condition module 2, take $v = de^{2r+1}d$ and so $cvc^{-1} = e^{-1}d^{-1}vde$. Now, if $\overline{D} = v^{t_2-t_1}$ then $c\overline{D}c^{-1}\overline{D}^{-1} = [e^{-1}d^{-1}, v^{t_2-t_1}] = [e^{-1}d^{-1}, \overline{D}]$ and therefore

$$\begin{aligned} &(\mathcal{E}\circ\mathcal{A})_2(c\overline{D}c^{-1}\overline{D}^{-1}) + (\mathcal{E}\circ\mathcal{A})_2([E,\overline{D}]) \\ &= \det\begin{pmatrix} -1 & (2r+1)(t_2-t_1) \\ -1 & 2(t_2-t_1) \end{pmatrix} + \det\begin{pmatrix} c_{12}-c_{11} & (2r+1)(t_2-t_1) \\ c_{22}-c_{21} & 2(t_2-t_1) \end{pmatrix} \\ &\equiv 0 \bmod 2 \end{aligned}$$

and so $(t_2 - t_1)[1 + (c_{22} - c_{21})] \equiv 0 \bmod 2$.

To solve the equation for the input data

$$(\phi, B_i, |E|_e, |E|_d) = \left(\begin{pmatrix} -1 & 2r+1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & (2r+1)t_i \\ 0 & 2t_i + m \end{pmatrix}, c_{12} - c_{11}, c_{22} - c_{21} \right)$$

is equivalent to solve for the input data

$$\begin{aligned} &(\phi', B'_i, |E'|_e, |E'|_d) \\ &\quad = \left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & t_i \\ 0 & 2t_i + m \end{pmatrix}, (c_{12} - c_{11}) - r(c_{22} - c_{21}), c_{22} - c_{21} \right). \end{aligned}$$

To see this it is sufficient to consider the isomorphism $\varphi\colon G_1 \to G_2$ given by

$$e \to e, \quad d \to e^{-r}d, \quad c \to c,$$

where $G_1 = \langle e, d, c; , cec^{-1} = e^{-1}, cdc^{-1} = e^{-1}e^{2r+1}de\rangle$ is the group for the first data and $G_2 = \langle e, d, c; cec^{-1} = e^{-1}, cdc^{-1} = e^{-1}ede\rangle$ is the group for the second data.

Now, we consider the input data

$$(\phi', B'_i, |E'|_e, |E'|_d) \\ = \left(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} m & t_i \\ 0 & 2t_i + m \end{pmatrix}, (c_{12} - c_{11}) - r(c_{22} - c_{21}), c_{22} - c_{21}\right)$$

satisfying the necessary condition $(t_2 - t_1)[1 + (c_{22} - c_{21})] \equiv 0 \bmod 2$. In this case the image of the map $\pi_1(T \setminus 1)) \to \mathbb{Z} \oplus \mathbb{Z}$ given by

$$\alpha \to (|\alpha c \alpha^{-1} c^{-1}|_e, |\alpha c \alpha^{-1} c^{-1}|_d)$$

maps $e \mapsto (2, 0)$ and $d \mapsto (-1, 0)$.

We consider the quotient group

$$\frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (2, 0), (-1, 0)\rangle} = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (1, 0)\rangle} = \frac{\langle (1, 0), (0, 1)\rangle}{\langle (1, 0)\rangle} \simeq 0 \oplus \mathbb{Z}.$$

So, it suffices to take E' such that $(|E'|_e, |E'|_d) = (a, c_{22} - c_{21})$, where $a \in \mathbb{Z}$ is fixed and $c_{22} - c_{21} \in \mathbb{Z}$.

If $|E'|_d = c_{22} - c_{21} \equiv 1 \bmod 2$, i.e. $c_{22} - c_{21} = 2u - 1$, then the problem has a solution. Take $D' = (ded)^{t_2 - t_1}$ and $E' = (ded)^u d^{-1}$ and so $E'cD'c^{-1}E'^{-1}D'^{-1} = 1$ and the result follows.

If $|E'|_d = c_{22} - c_{21} \equiv 0 \bmod 2$, i.e., $c_{22} - c_{21} = 2u$ it follows from the necessary condition that $t_2 - t_1$ is even. Let $L = \gcd(2(t_2 - t_1), c_{22} - c_{21})$ be the greatest common divisor. Since $c_{22} - c_{21} = 2u$ then L is even. If $2(t_2 - t_1)/L = 2d_1$ then $(c_{22} - c_{21})/L = 2l + 1$, and in this case the equation has a solution.

Indeed, first we observe that if $w_1 = ded$ and $w_2 = cw_1c^{-1} = dde = dw_1d^{-1}$ then $cw_2c^{-1} = w_1$ and so $c(w_1w_2)^x c^{-1} = (w_2w_1)^x$.

Now, if $D' = (w_1^{L/2} w_2^{L/2})^{d_1}$ and $E' = w_1^{L/2}(w_2^{L/2} w_1^{L/2})^l$ then

$$E'cD'c^{-1}E'^{-1}D'^{-1} = 1$$

and the result follows.

If $(q_2 - q_1)/L = 2u+1$ e $c_{22} - c_{21} = 2l$ we are going to prove that the equation has no solution.

It is sufficient to prove that it has no solution for $D' = w_1^{t_2 - t_1}$ and $E' = w_2^l$, where $w_1 = ded$ and $w_2 = dde$. We have

$$E'cD'c^{-1}E'^{-1}D'^{-1} = w_2^l w_2^{t_2 - t_1}(w_2^l)^{-1} w_1^{t_1 - t_2} = [d, (ded)^{t_2 - t_1}],$$

where

$$\mathcal{A}([d, (ded)^{t_2 - t_1}]) = \begin{cases} B^{-1}_{(0,1)} B^{-1}_{(1,3)} \cdots B^{-1}_{(t_2 - t_1 - 1, 2(t_2 - t_1) - 1)} & \text{for } t_2 - t_1 \geq 1, \\ B^{-1}_{(-1,-1)} B^{-1}_{(-2,-3)} \cdots B^{-1}_{(t_2 - t_1, 2(t_2 - t_1) + 1)} & \text{for } t_2 - t_1 \leq -1. \end{cases}$$

The equation to be solved is

$$\mathcal{A}(X_3)\mathcal{A}(E'cX_2c^{-1}E'^{-1})\mathcal{A}(EcDc^{-1}E^{-1}D^{-1})\mathcal{A}(DX_3^{-1}D^{-1})\mathcal{A}(X_2^{-1}) = 1.$$

If $\mathcal{A}(X_2) = B^n_{(x,y)}$ and $\mathcal{A}(X_3) = B^m_{(z,w)}$ we obtain the following calculation for the terms of the above equation:

$\mathcal{A}(X_3)$	$B^m_{(z,w)}$
$\mathcal{A}((E'cX_2c^{-1}E'^{-1})$	$B^{-n}_{(-x-1+y+(c_{22}-c_{21})/2,y+c_{22}-c_{21})}$
$\mathcal{A}(EcDc^{-1}E^{-1}D^{-1})$	$\mathcal{A}([d,(ded)^{(t_2-t_1)}])$
$\mathcal{A}(DX_3^{-1}D^{-1})$	$B^{-m}_{(z+(t_2-t_1),w+2(t_2-t_1))}$
$\mathcal{A}(X_2^{-1})$	$B^{-n}_{(x,y)}$

We consider the subgroup $H = \langle(1,0),(0,L)\rangle$ of $\mathbb{Z}\oplus\mathbb{Z}$, where $L = \gcd(q_2 - q_1, c_{22} - c_{21}) = \gcd(2(t_2 - t_1), 2l)$ and $2(t_2 - t_1)/L = (q_2 - q_1)/L = 2u + 1$. Now, we look at the equation in $\mathbb{Z}(\pi/H)$. So, for $(t_2 - t_1) \geq 1$, in terms of representatives classes, the equation in $\mathbb{Z}\pi$ is

$$B^m_{(z,w)}B^{-n}_{(-x-1+y+(c_{22}-c_{21})/2,y+c_{22}-c_{21})}B^{-1}_{(0,1)}B^{-1}_{(1,3)}\dots B^{-1}_{((t_2-t_1)-1,2(t_2-t_1)-1)} B^{-m}_{(z+(t_2-t_1),w+2(t_2-t_1))}B^{-n}_{(x,y)} = 1.$$

After projecting it on $\mathbb{Z}(\pi/H)$ we get

$$B^{-n}_{(x,y)}B^{-(2u+1)}_{(0,1)}B^{-(2u+1)}_{(1,3)}\dots B^{-(2u+1)}_{(L/2-1,2L/2-1)}B^{-n}_{(x,y)} = 1.$$

In $\mathbb{Z}(\pi/H)$ we have that $E'cX_2c^{-1}E'^{-1} = B^{-n}_{(x,y)}$. Therefore, the sum of the powers of all $B_{(i-1,2i-1)}$, for $i = 1,\dots,L/2$, which appears in $E'cX_2c^{-1}E'^{-1}X_2^{-1}$, is $-2u$, which is even. On the other hand, this sum is $-(2u+1)$, which is odd. So, there is no solution.

We note that, if $t_2 - t_1 \geq 1$ then, for all $i = 1,2,\dots L/2$, the $B_{(i-1,2i-1)}$ are different classes in $\mathbb{Z}(\pi/H)$. In fact, if $1 \leq i < j \leq L/2$ then $(j-1,2j-1)-(i-1,2i-1) = \alpha(1,0)+\beta(0,L)$ and we have no solution in $\mathbb{Z}$ because $1 \leq j-i < L/2$ and so $L \not| \, 2(j-i)$.

If $t_2 - t_1 \leq -1$, the computation is the same, because in $\mathbb{Z}(\pi/H)$ we have

$$B^{-1}_{(-1,-1)}B^{-1}_{(-2,-3)}\dots B^{-1}_{(t_2-t_1,2(t_2-t_1)+1)} = B^{-1}_{(0,1)}B^{-1}_{(1,3)}\dots B^{-1}_{(t_1-t_2-1,2(t_1-t_2)-1)}.$$

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