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p-REGULAR NONLINEAR DYNAMICS

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Abstract. In this paper we generalize the notion of p-factor operator which is the basic notion of the so-called p-regularity theory for nonlinear and degenerated operators. We prove a theorem related to a new construction of p-factor operator. The obtained results are illustrated by an example concerning nonlinear dynamical system.

1. Introduction

There are many situations where classical regularity conditions are not satisfied. We call such situations degenerate and we often meet them in the field of nonlinear differential equations or systems of equations. The paper concerns the problem of solving of a nonlinear system of differential equations of the form

$$\dot{x} = f(\mu, x), \qquad x(0) = x(\tau), \tag{1.1}$$

where $f \in \mathbb{C}^p(\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n$ and $f(\mu, 0) = 0$. We will consider an equivalent equation

$$F(\mu, x) = \dot{x} - f(\mu, x) = 0 \tag{1.2}$$

and apply to it the p-regularity theory.

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In the literature there are only a few applications of p-regularity theory to dynamical systems. In paper [3], M. Buchner, J. Marsden, S. Schechter applied the so-called Lyapunov–Schmidt procedure (which is similar to the construction of p-factor operator) to some bifurcation problem of the form (1.1). The authors proved a theorem about Andronov-Hopf bifurcation, which refers to periodic solutions of system (1.1) with period near 2π. They consider the case $p = 2$. For $p > 2$, our application of p-regularity theory to dynamical system are quite new. To this purpose we define the modified p-factor operator and search the periodic solutions with period 2π. This is more general result for the solutions existence of the mappings which are not 2-regular.

Regular problems are usually given in the form

$$F(x) = 0$$

where F is a sufficiently smooth map between Banach spaces X and Y. If a solution x^* of this equation is regular, i.e. the operator $F'(x^*)$ is surjective, then the above equation describes a regular submanifold of X near the point x^*.

The p-regularity theory [4], [6]–[11] deals with irregular cases. In [11], it was shown that the notions of nonlinearity and irregularity are strongly connected. The main idea of our p-regularity construction is to replace the operator $F'(x^*)$ (which is not surjective) with another linear operator (constructed by means of the first and higher order derivatives) which is surjective. The latter operator is denoted by $\Psi_p(x^*, h)$. Here the vector h belongs to the tangent cone to the set $\{x \in X : F(x) = 0\}$ at x^* and p is taken so large (if ever exists) that the operator $\Psi_p(x^*, h)$ is turned out to be surjective (the so-called p-regularity condition). In the next section, we will recall the main concepts of p-regularity theory.

We begin with some notation. Suppose X and Y are Banach spaces and denote the space of all continuous linear operators from X to Y by $\mathcal{L}(X, Y)$. Let p be a natural number and let $B\colon X \times \ldots \times X$ (p-copies of X)$\to Y$ be a continuous symmetric p-multilinear mapping. The p-form associated to B is the map $B[\,\cdot\,]^p\colon X \to Y$ defined by $B[x]^p = B(x, \ldots, x)$ for $x \in X$. Moreover,

$$(B[x]^q)[y]^{p-q} = B(\underbrace{x, \ldots, x}_{q}, \underbrace{y, \ldots, y}_{p-q}),$$

for $q < p$ (see [1]). Alternatively we may simply view $B[\,\cdot\,]^p$ as homogeneous polynomial $Q\colon X \to Y$ of degree p, i.e. $Q(\alpha x) = \alpha^p Q(x)$. Throughout this paper we assume that the mapping $F\colon X \to Y$ is continuously p-times Fréchet differentiable on X and its pth order derivative at $x \in X$ will be denoted as $F^{(p)}(x)$ (a symmetric multilinear map of p copies of X to Y) and the associated p-form, also called the pth order mapping, is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)[h, \ldots, h].$$

We also use the notation

$$\operatorname{Ker}^p F^{(p)}(x) = \{h \in X : F^{(p)}(x)[h]^p = 0\}$$

and refer to it as the p-kernel of the pth order mapping. Note that this set is a (non convex) closed cone.

The set $M = M(x^*) = \{x \in X : F(x) = F(x^*) = 0\}$ is called *the solution set* for the mapping F. We call h *a tangent vector* to the set $M \subseteq X$ at $x^* \in M$ if there exist $\varepsilon > 0$ and a function $r\colon [0,\varepsilon] \to X$ with the property that for $t \in [0,\varepsilon]$ we have $x^* + th + r(t) \in M$ and $\|r(t)\| = o(t)$. The set of all tangent vectors at x^* is called *the tangent cone* to M at x^* and is denoted by $T_{x^*}M$ (see [6]). In the regular case, the tangent cone to the solution set coincides with the kernel of the first derivative of the map F. Recall the following theorem:

Theorem 1.1 (Classical Lusternik Theorem). *Let X and Y be the Banach spaces and let the map $F\colon X \to Y$ be regular at $x^* \in X$. Then*

$$T_{x^*}M = \operatorname{Ker} F'(x^*).$$

The notion of regularity is generalized to the notion of so called p-regularity.

2. Elements of p-regularity theory

Assume that $x^* \in U \subseteq X$, U is a neighbourhood of the element x^*. Let a map $F\colon U \to Y$ be p-times Frechet differentiable in U and $\operatorname{Im} F'(x^*) \neq Y$ (the regularity condition does not hold). To define the notion of p-regularity, let us first define the so called p-factor operator (see [6]). Assume that the space Y is decomposed into a direct sum

$$Y = Y_1 \oplus \ldots \oplus Y_p, \tag{2.1}$$

where $Y_1 = \operatorname{cl}(\operatorname{Im} F'(x^*))$ (the closure of the image of the first derivative of F evaluated at x^*) and the next spaces are defined as follows. Let Z_2 be a closed complementary subspace to Y_1, that is $Y = Y_1 \oplus Z_2$ (we assume that such a closed complement exists) and let $P_{Z_2}\colon Y \to Z_2$ be the projection operator onto Z_2 along Y_1. Let $Y_2 = \operatorname{cl}(\operatorname{span}\operatorname{Im} P_{Z_2}F''(x^*)[\,\cdot\,]^2) \subseteq Z_2$ (the closed linear span of the image of the quadratic map $P_{Z_2}F''(x^*)[\,\cdot\,]^2$). More generally, define

$$Y_i = \operatorname{cl}(\operatorname{span}\operatorname{Im} P_{Z_i}F^{(i)}(x^*)[\,\cdot\,]^i) \subseteq Z_i, \quad i = 2,\ldots,p-1, \tag{2.2}$$

where Z_i is a closed complementary subspace to $Y_1 \oplus \ldots \oplus Y_{i-1}$, $i = 2,\ldots,p$ with respect to Y, and $P_{Z_i}\colon Y \to Z_i$ is the projection operator onto Z_i along $Y_1 \oplus \ldots \oplus Y_{i-1}$, $i = 2,\ldots,p$ with respect to Y. Finally $Y_p = Z_p$.

Now, let us define the following mappings

$$f_i\colon U \to Y_i, \qquad f_i(x) = \Pi_i F(x), \quad i = 1,\ldots,p,$$

where $\Pi_i\colon Y \to Y_i$ is the projection operator along $Y_1 \oplus \ldots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \ldots \oplus Y_p$. Then the mapping F can be represented as

$$F(x) = f_1(x) + \ldots + f_p(x), \tag{2.3}$$

or equivalently

$$F(x) = (f_1(x), \ldots, f_p(x)).$$

Let us recall some important definitions of p-regularity theory for the further considerations.

Definition 2.1. The linear operator mapping X to Y

$$\Psi_p(h) = \Psi_p(x^*, h) = f_1'(x^*) + \ldots + f_p^{(p)}(x^*)[h]^{p-1}$$

such that

$$\Psi_p(h)x = \Psi_p(x^*, h)x = f_1'(x^*)x + \ldots + f_p^{(p)}(x^*)[h]^{p-1}x, \quad x \in X,$$

is called *p-factor operator*.

Sometimes it is convenient to use the following equivalent definition of p-factor operator:

$$\begin{aligned}\Psi_p(h) = \Psi_p(x^*, h) &= (f_1'(x^*), \ldots, f_p^{(p)}(x^*)[h]^{p-1}) \\ &= (\Pi_1 F'(x^*), \ldots, \Pi_p F^{(p)}(x^*)[h]^{p-1})\end{aligned}$$

for $h \in X$.

We say that F is *completely degenerate* at x^* up to the order p if $F^{(i)}(x^*) = 0$, $i = 1, \ldots, p-1$.

Remark 2.2. In the completely degenerate case the p-factor operator reduces to $F^{(p)}(x^*)[h]^{p-1}$.

Remark 2.3. For each mapping f_i, $i = 2, \ldots, p$ we have ([6, p. 145]):

$$f_i^{(k)}(x^*) = 0, \quad k = 1, \ldots, i-1, \text{ for all } i = 2, \ldots, p. \tag{2.4}$$

Remark 2.4. According to the Remark 2.3 the expressions

$$f_i^{(i)}(x^*)[h]^{i-1} = \Pi_i F^{(i)}(x^*)[h]^{i-1}, \quad i = 2, \ldots, p$$

are i-factor operators corresponding to completely degenerate mappings f_i. So the general degeneration of F can be reduced to the study of completely degenerated mappings f_i.

Definition 2.5. The *p-kernel* of the operator $\Psi_p(h)$ is a set

$$\begin{aligned}H_p(x^*) = \operatorname{Ker}^p \Psi_p(h) &= \{h \in X : \Psi_p(h)[h] = 0\} \\ &= \{h \in X : f_1'(x^*)[h] + \ldots + f_p^{(p)}(x^*)[h]^p = 0\}.\end{aligned}$$

Note that the following relation holds

$$\operatorname{Ker}^p \Psi_p(h) = \left\{ \bigcap_{i=1}^{p} \operatorname{Ker}^i f_i^{(i)}(x^*) \right\}.$$

Again, this set is a non convex closed cone. Furthermore, p-kernel of the operator $F^{(p)}(x^*)$ in the completely degenerate case is a set

$$\operatorname{Ker}^p F^{(p)}(x^*) = \{h \in X : F^{(p)}(x^*)[h]^p = 0\}.$$

Definition 2.6. A mapping F is called *p-regular at x^* along h* if $\operatorname{Im} \Psi_p(h) = Y$ (i.e. the operator $\Psi_p(h)$ is surjective).

Definition 2.7. A mapping F is called *p-regular* at x^* if either it is p-regular along every $h \in H_p(x^*) \setminus \{0\}$ or $H_p(x^*) = \{0\}$.

The following theorem gives the description of the tangent cone to the solution set M in the degenerate case.

Theorem 2.8 (Generalized Lusternik theorem, [6]). *Let X and Y be the Banach spaces and let the mapping $F \in C^p(X,Y)$ be p-regular at $x^* \in M$. Then*

$$T_{x^*}M = H_p(x^*).$$

We conclude this section with lemmas which will be used later.

Lemma 2.9. *Let $A_1, \dots, A_p \in \mathcal{L}(X,Y)$, $Y = Y_1 \oplus \dots \oplus Y_p$. Let $\operatorname{Im} \Pi_k A_k = Y_k$, where $\Pi_k \colon Y \to Y_k$ is a projection operator from the space Y onto Y_k along $Y_1 \oplus \dots \oplus Y_{k-1} \oplus Y_{k+1} \oplus \dots \oplus Y_p$, $k = 1, \dots, p$ and $\Pi_1 A_1 = A_1$. Then*

$$(\Pi_1 A_1 + \dots + \Pi_p A_p)X = Y \ \Leftrightarrow\ (\Pi_p A_p)\left(\bigcap_{i=1}^{p-1} \operatorname{Ker} \Pi_i A_i \right) = Y_p.$$

From this lemma we obtain:

Corollary 2.10. *Under assumptions of Lemma* 2.9*, the following relations hold*:

$$Y_1 \oplus \dots \oplus Y_k = (\Pi_1 A_1 + \dots + \Pi_k A_k)X \ \Leftrightarrow\ (\Pi_k A_k)\left(\bigcap_{i=1}^{k-1} \operatorname{Ker} \Pi_i A_i \right) = Y_k,$$

for $k = 2, \dots, p$.

Let X and Y be the Banach spaces. By the mapping $\Phi \colon X \to 2^Y$ we mean a multivalued mapping (multimapping) from X to the set of all subsets of a space Y. Let $\rho(x,y) = \|x - y\|$ be the distance between elements x and y in a Banach space and let $\rho(x,M) = \inf\{\|x - z\| : z \in M\}$ be the distance from element x to subset M in this space. By $\operatorname{dist}_H(A_1, A_2) = \max\{\sup\{\rho(x, A_2) : x \in A_1\}, \sup\{\rho(x, A_1) : x \in A_2\}\}$ we denote the Hausdorff distance between sets A_1 and A_2.

Lemma 2.11 (Multimapping contraction principle, [5]). *Let Z be a Banach space. Assume that a multimapping*

$$\Phi\colon U_\varepsilon(z_0) \to 2^Z$$

on a ball $U_\varepsilon(z_0) = \{z : \rho(z, z_0) < \varepsilon\} \subset Z$, $(\varepsilon > 0)$, where the sets $\Phi(z)$ are non-empty and closed for any $z \in U_\varepsilon(z_0)$ is given. Further, assume that there exists a number θ, $0 < \theta < 1$, such that

(a) $\operatorname{dist}_H(\Phi(z_1), \Phi(z_2)) \le \theta\rho(z_1, z_2)$ *for any* $z_1, z_2 \in U_\varepsilon(z_0)$,
(b) $\rho(z_0, \Phi(z_0)) < (1-\theta)\varepsilon$.

Then, for every number ε_1 satisfying the inequality

$$\rho(z_0, \Phi(z_0)) < \varepsilon_1 < (1-\theta)\varepsilon,$$

there exists $z \in B_{\varepsilon_1/(1-\theta)} = \{\omega : \rho(\omega, z_0) \le \varepsilon_1/(1-\theta)\}$ such that

$$z \in \Phi(z). \tag{2.5}$$

Moreover, among the points satisfying (2.5), *there exists a point z such that*

$$\rho(z, z_0) \le \frac{2}{1-\theta}\rho(z_0, \Phi(z_0)).$$

For a linear operator $\Lambda\colon X \to Y$ we denote by Λ^{-1} its right inverse, that is $\Lambda^{-1}\colon Y \to 2^X$ which maps any element $y \in Y$ on its complete inverse image of the mapping Λ, $\Lambda^{-1}y = \{x \in X : \Lambda x = y\}$, and of course $\Lambda\Lambda^{-1} = I_Y$.

By the "norm" of such right inverse operator we mean the number

$$\|\Lambda^{-1}\| = \sup_{\|y\|=1} \inf\{\|x\| : \Lambda x = y, x \in X\}. \tag{2.6}$$

Note, that if Λ is one-to-one, than $\|\Lambda^{-1}\|$ can be considered as the usual norm of the element Λ^{-1}. In our considerations, by Λ^{-1} we shall mean just right inverse multivalued operator with the norm defined by (2.6).

Lemma 2.12. *Let $F\colon X \to Y$, $y = y_1 + \ldots + y_p \in Y$, $Y = Y_1 \oplus \ldots \oplus Y_p$, $y_i \in Y_i$, $i = 1, \ldots, p$, $\|h\| = 1$ and*

$$\|\{F'(x^*) + \Pi_2 F''(x^*)[h] + \ldots + \Pi_p F^{(p)}(x^*)[h]^{p-1}\}^{-1}\| = c < \infty.$$

Then

$$\begin{aligned}\|\{\alpha_1 F'(x^*) + \ldots + \alpha_p \Pi_p F^{(p)}(x^*)[th]^{p-1}\}^{-1}(y_1 + \ldots + y_p)\| \\ \le c\biggl(\frac{1}{\alpha_1}\|y_1\| + \ldots + \frac{1}{\alpha_p t^{p-1}}\|y_p\|\biggr),\end{aligned}$$

where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $i = 1, \ldots, p$, $t \neq 0$.

Lemma 2.13 (Mean value theorem, [5]). *Let X, Y be the Banach spaces, U an open subset of the space X, $[x, x+\Delta]$ a closed segment in U. If $F\colon U \to Y$ and $F \in \mathcal{C}^1([x, x+\Delta])$, then for any $\Lambda \in \mathcal{L}(X,Y)$ we have:*

$$\|F(x+\Delta) - F(x) - \Lambda\Delta\| \le \sup_{\theta\in[0,1]} \|F'(x+\theta\Delta) - \Lambda\| \cdot \|\Delta\|. \tag{2.7}$$

Lemma 2.14. *Let X, Y be the vector spaces, $B[\,\cdot\,]^p\colon X \to Y$ be the homogeneous p-form defined on the space X associated to continuous, symmetric, p-multilinear mapping $B\colon X \times \ldots \times X$ (p copies of X) $\to Y$ and $h \in X$. Then the p-th derivative $B^{(p)}$ of the mapping $B[\,\cdot\,]^p$ is equal*

$$B^{(p)}[h]^p = p!B[h]^p, \tag{2.8}$$

and

$$B^{(p)}[h]^{p-1} = (p-1)!(B[h]^p)'. \tag{2.9}$$

The proof of this lemma follows from properties of homogeneous p-form.

3. Modification of p-factor operator and generalization of theorem on tangent cone

3.1. The case of complete degeneration up to the order $p-1$ and modified p-factor operator. We will prove the following theorem:

Theorem 3.1. *Let $F\colon U \subset X \longrightarrow Y$, $x^* \in U$, $F(x^*) = 0$, $F^{(i)}(x^*) = 0$ for $i = 1, \ldots, p-2$, $F^{(p-1)}(x^*) \neq 0$, where $p \ge 3$. Assume that there exists $h \neq 0$, such that*

$$\begin{aligned} F^{(p-1)}(x^*)[h]^{p-1} &= 0, & \Pi_{p-1} F^{(p)}(x^*)[h]^p &= 0,\\ \Pi_p F^{(p)}(x^*)[h]^p &= 0, & \Pi_p F^{(p+1)}(x^*)[h]^{p+1} &= 0,\\ \Pi_{p-1}\colon Y &\to \widetilde{Y}_{p-1} & \Pi_p\colon Y &\to \widetilde{Y}_{p-1}^{\perp}, \end{aligned}$$

where $Y = \widetilde{Y}_{p-1} \oplus \widetilde{Y}_{p-1}^{\perp}$ and

$$\widetilde{Y}_{p-1} = \operatorname{Im} F^{(p-1)}(x^*)[h]^{p-2} \subset Y_{p-1} = \operatorname{cl}(\operatorname{span} \operatorname{Im} F^{(p-1)}(x^*)[\,\cdot\,]^{p-1}).$$

Let for such settled h there exists a number $t \neq 0$ such that the operator

$$\overline{\Psi}_p(th)\colon X \to Y,$$

$$\overline{\Psi}_p(th) = \overline{\Psi}_p(x^*, th) = \frac{1}{(p-2)!} F^{(p-1)}(x^*)[th]^{p-2} + \frac{1}{(p-1)!} \Pi_p F^{(p)}(x^*)[th]^{p-1},$$

is surjection. Then

$$h \in T_{x^*} M. \tag{3.1}$$

Remark 3.2. The surjectivity of p-factor operator $\overline{\Psi}_p(th)$ does not depend on nonzero coefficients at its components.

Before we prove the above theorem, let us illustrate it on a simple example.

Example 3.3. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$F(x) = x_1x_2^2 + x_1^3x_2. \tag{3.2}$$

Note that for $x^* = 0$ we have $F(0) = 0$ and

$$\begin{aligned} F'(x) &= (x_2^2 + 3x_1^2x_2, 2x_1x_2 + x_1^3), & F'(0) &= (0,0),\\ F''(x) &= ((6x_1x_2, 2x_2 + 3x_1^2), (2x_2 + 3x_1^2, 2x_1)), & F''(0) &= ((0,0),(0,0)),\\ F^{(3)}(x) &= (((6x_2, 6x_1), (6x_1, 2)), ((6x_1, 2), (2, 0))), \\ F^{(3)}(0) &= (((0,0),(0,2)),((0,2),(2,0))). \end{aligned}$$

Let us note that here and in the sequel we write matrices and tensors as simple vectors of appropriate dimension.

We will point out 3-kernel $\operatorname{Ker}^3 F^{(3)}(0)$. Let $h = (h_1, h_2)$. Then

$$\begin{aligned} F^{(3)}(0)[h] &= ((0, 2h_2), (2h_2, 2h_1)),\\ F^{(3)}(0)[h]^2 &= (2h_2^2, 4h_2h_1),\\ F^{(3)}(0)[h]^3 &= 6h_1h_2^2, \end{aligned}$$

and hence

$$\operatorname{Ker}^3 F^{(3)}(0) = \operatorname{span}\{(1,0)\} \cup \operatorname{span}\{(0,1)\}. \tag{3.3}$$

Let us note, that 3-factor operator of the form

$$\Psi_3(h) = F'''(0)[h]^2$$

considered on the element $\overline{h}_1 = (1,0)$ is not a surjection, because for every element $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\Psi_3((1,0))(x_1,x_2) = F'''(0)(1,0)^2(x_1,x_2) = (0,0)(x_1,x_2) = 0,$$

i.e. $\operatorname{Im}\Psi_3((1,0)) = \{0\}$. For element $\overline{h}_2 = (0,1)$ we obtain:

$$\Psi_3(0,1)(x_1,x_2) = F'''(0)(0,1)^2(x_1,x_2) = (2,0)(x_1,x_2) = 2x_1,$$

and $\operatorname{Im}\Psi_3(0,1) = \mathbb{R}$.

Therefore the map F is 3-regular on the element $\overline{h}_2 = (0,1)$. Then it is not 3-regular on the element $\overline{h}_1$ and thus we can not guarantee, that $\overline{h}_1$ belongs to the tangent cone T_0M ([1]).

Now, we proceed to the verification of the conditions of Theorem 3.1 for mapping (3.2). Here we have $p = 4$, $F(0) = 0$, $F'(0) = 0$, $F''(0) = 0$, $F'''(0) \neq 0$

([1]) Directly from the definition of map (3.2), we can see that the element $\overline{h}_1$ belongs to the tangent cone T_0M, because $F(t\overline{h}_1) = 0$, $r(t) = 0$. Generally it can be difficult to prove that if F is not p-regular on element h, then h belongs to the tangent cone. This is an open problem.

and for the vector $\overline{h} = \overline{h}_1 = (1,0)$ the result is $\widetilde{Y}_3 = \mathrm{Im} F'''(0)[\overline{h}]^2 = \{0\}$, $\widetilde{Y}_3^{\perp} = \mathbb{R}$. Using this facts we define the following projections:

$$\Pi_3 \colon \mathbb{R}^2 \to \{0\}, \quad \Pi_3 = 0, \qquad \Pi_4 \colon \mathbb{R}^2 \to \mathbb{R}, \quad \Pi_4 = 1.$$

Now let us find:

$$\begin{aligned} F^{(4)}(x) &= ((((0,6),(6,0)),((6,0),(0,0))),(((6,0),(0,0)),((0,0),(0,0)))) \\ &= F^{(4)}(0), \\ F^{(5)}(x) &= 0 = F^{(5)}(0), \\ F^{(4)}(0)[h] &= (((6h_2,6h_1),(6h_1,0)),((6h_1,0),(0,0))), \\ F^{(4)}(0)[h]^2 &= ((12h_1h_2,6h_1^2),(6h_1^2,0)), \\ F^{(4)}(0)[h]^3 &= (18h_1^2h_2,6h_1^3), \\ F^{(4)}(0)[h]^4 &= 24h_1^3h_2. \end{aligned}$$

Using the forms this and the forms of projections defined above, we see that:

$$\Pi_3 F^{(4)}(0)[\overline{h}]^4 = 0, \qquad \Pi_4 F^{(4)}(0)[\overline{h}]^4 = F^{(4)}(0)[\overline{h}]^4 = 0, \qquad \Pi_4 F^{(5)}(0)[\overline{h}]^5 = 0.$$

We are now ready to describe the form of the modified 4-factor operator and we will prove that it is a surjection on the vector $\overline{h} = (1,0)$. Note that

$$F^3(0)[\overline{h}]^2 = (0,0), \qquad \Pi_4 F^{(4)}(0)[\overline{h}]^3 = (0,6).$$

Hence the modified 4-factor operator considered on the vector $\overline{h} = (1,0)$ has the form:

$$\overline{\Psi}_4(\overline{h}) = \frac{1}{3!}\Pi_4 F^{(4)}(0)[\overline{h}]^3 = \frac{1}{6}(0,6) = (0,1) \neq (0,0),$$

therefore, it is a surjection. Since all the assumptions of Theorem 3.1 are satisfied, we have $\overline{h} = (1,0) \in T_0M$.

Additionally we will prove that $T_0M = \mathrm{span}\{(1,0)\} \cup \mathrm{span}\{(0,1)\}$. Note that

$$h \in T_0M \Leftrightarrow 0 + th + r(t) \in M \Leftrightarrow F(th + r(t)) = 0, \quad \|r(t)\| = o(t).$$

Hence we have

$$\begin{aligned} &F(th_1 + r_1(t), th_2 + r_2(t)) \\ &\qquad = (th_1 + r_1(t))(th_2 + r_2(t))^2 + (th_1 + r_1(t))^3(th_2 + r_2(t)) = 0. \end{aligned}$$

The latter is equivalent to the identity

$$(th_1 + r_1(t))(th_2 + r_2(t))(th_2 + r_2(t) + (th_1 + r_1(t))^2) = 0$$

Since $\|r_1(t)\| = o(t)$ and $\|r_2(t)\| = o(t)$ we have $h_1 = 0$ and $h_2 = 1$ or $h_2 = 0$ and $h_1 = 1$ or

$$th_2 + r_2(t) = -(t^2h_1^2 + 2th_1r_1(t) + r_1^2(t)).$$

Dividing the each sides of the last identity by t and letting $t \to 0$ we obtain $h_2 = 0$, h_1 – arbitrary. Then $T_0M = \operatorname{span}\{(1,0)\} \cup \operatorname{span}\{(0,1)\}$, that is, in the presented example, 3-kernel of the map F (see (3.3)) coincide with the tangent cone at the point $x^* = 0$.

3.2. Proof of Theorem 3.1.

Proof. Let $t \neq 0$. Define a multivalued mapping $\Phi\colon U(0,\varepsilon) \to 2^Y$ as follows

$$\Phi(x) = x - \{\overline{\Psi}_p(th)\}^{-1} F(x^* + th + x) \quad \text{for all } x \in U(0,\varepsilon). \tag{3.4}$$

First we will prove that the mapping Φ satisfies on $U(0,\varepsilon)$ the conditions of multimapping contraction principle for $\varepsilon = \overline{c}t^3$ where $\overline{c}$ will be described later.

Let us recall that $\|\Phi(0)\| \leq \overline{c}t^3$, that is $\|\Phi(0)\| = O(t^3)$. It follows that:

$$\|\Phi(0)\| = \| - \{\overline{\Psi}_p(th)\}^{-1}(\Pi_{p-1}F(x^* + th) + \Pi_p F(x^* + th))\|.$$

This and Lemma 2.12 imply:

$$\|\Phi(0)\| \leq \frac{(p-2)!c_1}{t^{p-2}} \|\Pi_{p-1}F(x^* + th)\| + \frac{(p-1)!c_1}{t^{p-1}} \|\Pi_p F(x^* + th)\|.$$

Expanding the expressions $\Pi_{p-1}F(x^* + th)$ and $\Pi_p F(x^* + th)$ in Taylor's series and by assumption we have:

$$\|\Phi(0)\| \leq \frac{(p-2)!c_1}{t^{p-2}}\, t^{p+1} + \frac{(p-1)!c_1}{t^{p-1}}\, t^{p+2} = ct^3 \tag{3.5}$$

where $c = ((p-2)! + (p-1)!)c_1$ and

$$\|\Phi(0)\| \leq 4ct^3 = \overline{c}t^3 \tag{3.6}$$

for $\overline{c} = 4c$. Therefore $\varepsilon = \overline{c}t^3$, which completes the proof that $\|\Phi(0)\| = O(t^3)$.

We next show that for all $x_1, x_2 \in U(0, \overline{c}t^3)$ the following estimate holds

$$\operatorname{dist}_H(\Phi(x_1), \Phi(x_2)) \leq \theta \|x_1 - x_2\|, \tag{3.7}$$

where $0 < \theta < 1$, $\theta = c_3 t$, $c_3 > 0$ is a constant independent of t.

First let us note that $\overline{\Psi}_p(th)\Phi(x_i) = \overline{\Psi}_p(th)x_i - F(x^* + th + x_i)$ for $i = 1, 2$. Let $z_1 \in \Phi(x_1)$, $z_2 \in \Phi(x_2)$. Then we have:

$$\begin{aligned}
\operatorname{dist}_H(\Phi(x_1), \Phi(x_2)) &= \inf\{\|z_1 - z_2\| : z_i \in \Phi(x_i), i = 1, 2\} \\
&= \inf \bigg\{ \bigg\| \overline{\Psi}_p(th)^{-1} \bigg\{ \bigg[\frac{1}{(p-2)!} F^{(p-1)}(x^*)[th]^{p-2}(x_1 - x_2) \\
&\qquad - \Pi_{p-1}(F(x^* + th + x_1) - F(x^* + th + x_2)) \bigg] \\
&\qquad + \bigg[\frac{1}{(p-1)!} \Pi_p F^{(p)}(x^*)[th]^{p-1}(x_1 - x_2) \\
&\qquad - \Pi_p(F(x^* + th + x_1) - F(x^* + th + x_2)) \bigg] \bigg\} \bigg\| \bigg\}.
\end{aligned}$$

Further, by the Lemma 2.12, we can give the following estimate

$$\begin{aligned}\mathrm{dist}_H(\Phi(x_1),\Phi(x_2)) \le{}& \frac{(p-2)!c}{t^{p-2}}\|\Pi_{p-1}(F(x^*+th+x_1)-F(x^*+th+x_2))\\ &-\frac{1}{(p-2)!}\,F^{(p-1)}(x^*)[th]^{p-2}(x_1-x_2)\|\\ &+\frac{(p-1)!c}{t^{p-1}}\,\|\Pi_p(F(x^*+th+x_1)-F(x^*+th+x_2))\\ &-\frac{1}{(p-1)!}\,\Pi_pF^{(p)}(x^*)[th]^{p-1}(x_1-x_2)\| = A+B,\end{aligned}$$

where A and B stand for the first and the second components of the above sum of the norms multiplied by scalars.

For components A and B we apply the mean value theorem, the Taylor's formula and Remark 2.3. Finally since $\|x_1\| \le \overline{c}t^3$ the result is

$$\begin{aligned}A &\le \left(\frac{c_1}{t^{p-2}}\,t^{p-2-1+3}+\frac{c_2}{t^{p-2}}\,t^{p-1}\right)\|x_1-x_2\|\\ &\le (c_1t^2+c_2t)\|x_1-x_2\| \le 2d_1t\|x_1-x_2\| = \theta_1\|x_1-x_2\|, \quad \theta_1=2d_1t\end{aligned}$$

and

$$\begin{aligned}B &\le \left(\frac{c_3}{t^{p-1}}\,t^{p-2-1+3}+\frac{c_4}{t^{p-1}}\,t^{p+1}\right)\|x_1-x_2\|\\ &\le (c_3t+c_4t^2)\|x_1-x_2\| \le 2d_2t\|x_1-x_2\| = \theta_2\|x_1-x_2\|, \quad \theta_2=2d_2t,\end{aligned}$$

because $t\in(0,\delta)$, where $\delta>0$ is sufficiently small.

Finally, taking $\theta=\theta_1+\theta_2$, we get: $\mathrm{dist}_H(\Phi(x_1),\Phi(x_2)) \le A+B \le \theta\|x_1-x_2\|$, where $0<\theta<1$ and $\theta=d_3t$. Therefore the mapping Φ is a contraction on $U(0,\overline{c}t^3)$.

According to multivalued contraction principle (Lemma 2.11) we will next prove, that $\varrho(0,\Phi(0)) = \|\Phi(0)\| < (1-\theta)\varepsilon$, where $\theta=d_3t$, $\varepsilon=\overline{c}t^3$, $\overline{c}=4c$ for t sufficiently small.

We can take $\theta=d_3t<1/2$. This inequality is equivalent to $1<2(1-d_3t)$. This and the inequality $\|\Phi(0)\| \le ct^3 < 4ct^3$ (see (3.5), (3.6)) imply:

$$\|\Phi(0)\| \le ct^3 \le 2(1-d_3t)ct^3 < (1-d_3t)4ct^3 = (1-\theta)\varepsilon.$$

For $z_0=0$, the multivalued contraction principle implies that there exists an element $z=r(t)$ such that $\|r(t)\| \le (2/(1-\theta))\|\Phi(0)\| \le 4\|\Phi(0)\| = 16ct^3$ or $\|r(t)\|=o(t)$ and $r(t)\in\Phi(r(t))$. Then $r(t)$ is a fixed point of the mapping Φ. Hence $0\in\{-\{\overline{\Psi}_p(th)\}^{-1}F(x^*+th+r(t))\}$. Thus we get $F(x^*+th+r(t))=0$ and $\|r(t)\|=o(t)$ or $h\in T_{x^*}M$ and this finishes the proof ([2]). □

([2]) Under the assumptions of theorem 3.1 we obtained t^3 (precisely $O(t^3)$) as the rank of tangency. This result is much finer then $o(t)$. Probably the assumptions of theorem could be weakened but then we obtain a rougher rank of tangency $o(t)$.

3.3. The case of general degeneration. Now we generalize the Theorem 3.1 on the case of general degeneration. Assume that the space Y is the following direct sum:

$$Y = Y_1 \oplus \underbrace{Y_{21} \oplus Y_{22}}_{Y_2} \oplus \underbrace{Y_{31} \oplus Y_{32}}_{Y_3} \oplus \underbrace{Y_{41} \oplus Y_{42}}_{Y_4} \ldots \oplus \underbrace{Y_{p-1,1} \oplus Y_{p-1,2}}_{Y_{p-1}} \oplus Y_p,$$

where, for some h,

$$\begin{aligned}
Y_1 &= \mathrm{Im} F'(x^*), \\
Y_{21} &= \mathrm{Im} P_{(Y_1)^\perp} F''(x^*)[h], & Y_{21} \oplus Y_{22} &= Y_2, \\
Y_{22} \oplus Y_{31} &= \mathrm{Im}\, P_{(Y_1 \oplus Y_{21})^\perp} F'''(x^*)[h]^2, & Y_{31} \oplus Y_{32} &= Y_3, \\
Y_{32} \oplus Y_{41} &= \mathrm{Im}\, P_{(Y_1 \oplus \ldots \oplus Y_{31})^\perp} F^{(4)}(x^*)[h]^3, & Y_{41} \oplus Y_{42} &= Y_4, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
Y_{p-2,2} \oplus Y_{p-1,1} &= \mathrm{Im}\, P_{(Y_1 \oplus \ldots \oplus Y_{p-2,1})^\perp} F^{(p-1)}(x^*)[h]^{p-2}, & Y_{p-1,1} \oplus Y_{p-1,2} &= Y_{p-1}, \\
Y_{p-1,2} \oplus Y_p &= \mathrm{Im} P_{(Y_1 \oplus \ldots \oplus Y_{p-1,1})^\perp} F^{(p)}(x^*)[h]^{p-1}
\end{aligned}$$

and Y_i is defined in (2.2) for $i = 2, \ldots, p$.

Define the corresponding projection operators as follows:

$$\begin{aligned}
&\Pi_1 \colon Y \to Y_1, \\
&\Pi_2 = \Pi_{21} \colon Y \to Y_{21}, \\
&\Pi_3 = \Pi_{22} \oplus \Pi_{31} \colon Y \to Y_{22} \oplus Y_{31}, \qquad \Pi_{22} \colon Y \to Y_{22}, \quad \Pi_{31} \colon Y \to Y_{31}, \\
&\Pi_4 = \Pi_{32} \oplus \Pi_{41} \colon Y \to Y_{32} \oplus Y_{41}, \qquad \Pi_{32} \colon Y \to Y_{32}, \quad \Pi_{41} \colon Y \to Y_{41}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\Pi_{p-1} = \Pi_{p-2,2} \oplus \Pi_{p-1,1} \colon Y \to Y_{p-2,2} \oplus Y_{p-1,1}, \\
&\qquad \Pi_{p-2,2} \colon Y \to Y_{p-2,2}, \qquad \Pi_{p-1,1} \colon Y \to Y_{p-1,1}, \\
&\Pi_p = \Pi_{p-1,2} \oplus \Pi_{p,1} \colon Y \to Y_{p-1,2} \oplus Y_p, \\
&\qquad \Pi_{p-1,2} \colon Y \to Y_{p-1,2}, \qquad \Pi_{p,1} \colon Y \to Y_p.
\end{aligned}$$

Under the above assumptions, we formulate the next theorem:

Theorem 3.4. *Let $F \in C^{p+1}(X, Y)$, $F(x^*) = 0$, $p \geq 3$ and assume that there exists $h \neq 0$ such that*

$$\begin{aligned}
F'(x^*)[h] &= 0, & \Pi_1 F''(x^*)[h]^2 &= 0 \\
\Pi_2 F''(x^*)[h]^2 &= 0, & \Pi_2 F'''(x^*)[h]^3 &= 0, \\
\Pi_3 F'''(x^*)[h]^3 &= 0, & \Pi_3 F^{(4)}(x^*)[h]^4 &= 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Pi_{p-1} F^{(p-1)}(x^*)[h]^{p-1} &= 0, & \Pi_{p-1} F^{(p)}(x^*)[h]^p &= 0, \\
\Pi_p F^{(p)}(x^*)[h]^p &= 0, & \Pi_p F^{(p+1)}(x^*)[h]^{p+1} &= 0.
\end{aligned} \tag{3.8}$$

Let, for the above settled h and some $t \neq 0$, the modified p-factor operator $\overline{\Psi}_p(th)\colon X \to Y$,

$$\begin{aligned}\overline{\Psi}_p(th) =&\overline{\Psi}_p(x^*, th) = F'(x^*) + \Pi_2 F''(x^*)[th] + \frac{1}{2}\Pi_3 F'''(x^*)[th]^2\\ &+ \frac{1}{3!}\Pi_4 F^{(4)}(x^*)[th]^3 + \ldots + \frac{1}{(p-2)!}\Pi_{p-1}F^{(p-1)}(x^*)[th]^{p-2}\\ &+ \frac{1}{(p-1)!}\Pi_p F^{(p)}(x^*)[th]^{p-1},\end{aligned}$$

be surjection. Then

$$h \in T_{x^*}M. \tag{3.9}$$

The proof of Theorem 3.4 is similar to the proof of Theorem 3.1.

3.4. Applications of theorems on modified p-factor operator to nonlinear dynamics.

Example 3.5. Consider the following nonlinear dynamical system of ordinary differential equations:

$$\begin{cases}\dot{x}_1 - x_2 + \mu x_1^2 + (\sqrt{10}/3)\mu x_1^3 - \mu^5 x_1 = 0,\\ \dot{x}_2 + x_1 + \mu x_2^2 - \mu^3 x_2 - \mu x_2^5 = 0,\end{cases} \tag{3.10}$$

subject to the conditions $x_1(0) = x_1(2\pi)$, $x_2(0) = x_2(2\pi)$, where $\mu \in \mathbb{R}$ is the parameter. Therefore we are looking for periodic solutions with settled period.

We will show that the assumptions of theorem 3.4 are fulfilled for the vector

$$\overline{h} = \left[\pm\frac{\sqrt[4]{10}}{2}, \cos t, -\sin t\right]. \tag{3.11}$$

for this problem. We first consider a reduced form of the system (3.10), that is:

$$\begin{cases}\dot{x}_1 - x_2 + \mu x_1^2 + (\sqrt{10}/3)\mu x_1^3 = 0,\\ \dot{x}_2 + x_1 + \mu x_2^2 - \mu^3 x_2 = 0,\end{cases} \tag{3.12}$$

subject to the above conditions $x_1(0) = x_1(2\pi)$, $x_2(0) = x_2(2\pi)$.

To analyze the structure of solutions of system (3.12), we write it in the form

$$F(\mu, x_1, x_2) = (\dot{x}_1 - x_2 + \mu x_1^2 + (\sqrt{10}/3)\mu x_1^3, \qquad \dot{x}_2 + x_1 + \mu x_2^2 - \mu^3 x_2) = 0, \tag{3.13}$$

where $F\colon \mathbb{R} \times \mathcal{C}^2(\mathbb{R}, \mathbb{R}^2) \to \mathcal{C}(\mathbb{R}, \mathbb{R}^2)$ and $x_1(0) = x_1(2\pi)$, $x_2(0) = x_2(2\pi)$.

Note that $x^* = (0, 0, 0)$ is a trivial solution. Let us describe the kernel of first derivative

$$\operatorname{Ker} F'(0,0,0) = \left\{(\mu, x_1, x_2) \in \mathbb{R} \times \mathcal{C}^2_{2\pi}(\mathbb{R}, \mathbb{R}^2) : \frac{dx_1}{dt} - x_2 = 0,\ \frac{dx_2}{dt} + x_1 = 0\right\},$$

where $\mathcal{C}^2_{2\pi}(\mathbb{R},\mathbb{R}^2)$ denotes the space of 2π-periodic functions from $\mathbb{R}$ to $\mathbb{R}^2$ of the class $\mathcal{C}^2$. In other words, we must solve a system of equations, which we can write as

$$\left(\frac{d}{dt}+L_0\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0, \quad \text{where } L_0=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with settled above boundary conditions. Thus we have

$$\operatorname{Ker} F'(0,0,0)=\mathbb{R}\times\operatorname{Ker}\left(\frac{d}{dt}+L_0\right)=\mathbb{R}\times\operatorname{span}(\phi_1,\phi_2), \tag{3.14}$$

where

$$\phi_1=\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \phi_2=\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

Observe also that $\phi_1'=-\phi_2$ and $\phi_2'=\phi_1$.

Now we define the space $Y_1=\operatorname{Im}(d/dt+L_0)$. Define the adjoint operator

$$\left(\frac{d}{dt}+L_0\right)^*=-\frac{d}{dt}+L_0^*=-\frac{d}{dt}+L_0^\top.$$

Let us choose the basis $\{\psi_1,\psi_2\}$ of the space $\operatorname{Ker}(d/dt+L_0)^*$ using the conditions $\psi_1'=-\psi_2$, $\psi_2'=\psi_1$ and $\langle\psi_i,\phi_j\rangle=\delta_{ij}$, where

$$\langle g,h\rangle=\int_0^{2\pi}(g(\tau),h(\tau))\,d\tau$$

and $(g(\tau),h(\tau))$ is a standard scalar vector product in $\mathbb{R}^2$. Since

$$\int_0^{2\pi}\cos^2\tau\,d\tau=\int_0^{2\pi}\sin^2\tau\,d\tau=\pi, \qquad \int_0^{2\pi}\sin\tau\cos\tau\,d\tau=0,$$

we put $\psi_1=\phi_1/(2\pi)$, $\psi_2=\phi_2/(2\pi)$ and

$$\operatorname{Ker}\left(\frac{d}{dt}+L_0\right)^*=\operatorname{span}(\psi_1,\psi_2)=\operatorname{span}(\phi_1,\phi_2)=\operatorname{Ker}\left(\frac{d}{dt}+L_0\right). \tag{3.15}$$

Finally,

$$\begin{aligned} Y_1=\operatorname{Im}\left(\frac{d}{dt}+L_0\right)&=\left(\operatorname{Ker}\left(\frac{d}{dt}+L_0\right)^*\right)^\perp \\ &=\left\{g\in\mathcal{C}_{2\pi}(\mathbb{R},\mathbb{R}^2):\int_0^{2\pi}(g(\tau),\phi_i)\,d\tau=0,\ i=1,2\right\}, \end{aligned}$$

where $\mathcal{C}_{2\pi}(\mathbb{R},\mathbb{R}^2)$ denotes the space of 2π-periodic, continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$.

Let $h\in\operatorname{Ker} F'(0,0,0)$. Hence $F'(0,0,0)[h]=0$ and next $F''(0,0,0)=0$, $F''(0,0,0)[h]=0$, $F''(0,0,0)[h]^2=0$.

Let us observe that for the projections $\Pi_1\colon Y\to Y_1$ and $\Pi_2=\Pi_{21}\colon Y\to Y_{21}$ the relations

$$\Pi_1F''(0,0,0)[h]^2=0, \qquad \Pi_2F''(0,0,0)[h]^2=0$$

evidently hold. Moreover, let us note that $Y_{21} = \operatorname{Im} P_{(Y_1)^\perp} F''(0,0,0)[h] = \{0\}$, $Y_{21} \oplus Y_{22} = \{0\} \oplus Y_{22} = Y_2$. This implies that $\Pi_2 = 0$. We have also:

$$Y_{22} \oplus Y_{31} = Y_2 \oplus Y_{31} = \operatorname{Im} P_{(Y_1 \oplus Y_{21})^\perp} F'''(0,0,0)[h]^2 = \operatorname{Im} P_{(Y_1)^\perp} F'''(0,0,0)[h]^2.$$

In the next step, it must be $Y_{31} \oplus Y_{32} = Y_3$. Note that for $h = [h_\mu, h_{x_1}, h_{x_2}]$ we have

$$\begin{aligned} F'''(0,0,0)[h]^2 &= ((2h_{x_1}^2, 4h_\mu h_{x_1}, 0), (2h_{x_2}^2, 0, 4h_\mu h_{x_2})), \\ F'''(0,0,0)[h]^3 &= (6h_\mu h_{x_1}^2, 6h_\mu h_{x_2}^2). \end{aligned}$$

Now, let us compute $\operatorname{Im} P_{(Y_1)^\perp} F'''(0,0,0)[h]^2$. To this end according to Corollary 2.10 for elements $[\varepsilon, u_1, u_2] \in \operatorname{Ker} F'(0,0,0)$, we obtain:

$$\begin{aligned} &P_{(Y_1)^\perp} F'''(0,0,0)[h]^2[\varepsilon, u_1, u_2] \\ &\quad = \phi_1 \int_0^{2\pi} [(2h_{x_1}^2 \varepsilon + 4h_\mu h_{x_1} u_1) \cos\tau + (2h_{x_2}^2 \varepsilon + 4h_\mu h_{x_2} u_2)(-\sin\tau)]\, d\tau \\ &\quad + \phi_2 \int_0^{2\pi} [(2h_{x_1}^2 \varepsilon + 4h_\mu h_{x_1} u_1) \sin\tau + (2h_{x_2}^2 \varepsilon + 4h_\mu h_{x_2} u_2)(\cos\tau)]\, d\tau. \end{aligned} \tag{3.16}$$

Now substitution $h_{x_1} = a\cos\tau + b\sin\tau$, $h_{x_2} = -a\sin\tau + b\cos\tau$ and $u_1 = c\cos\tau + d\sin\tau$, $u_2 = -c\sin\tau + d\cos\tau$ give the following form of formula (3.16):

$$P_{(Y_1)^\perp} F'''(0,0,0)[h]^2[\varepsilon, u_1, u_2] = 0. \tag{3.17}$$

Then $Y_{22} \oplus Y_{31} = Y_2 \oplus Y_{31} = \operatorname{Im} P_{(Y_1)^\perp} F'''(0,0,0)[h]^2 = \{0\}$. Since $\Pi_2 = \Pi_{21} \colon Y \to Y_{21} = \{0\}$ we have $\Pi_2 F'''(0,0,0)[h]^3 = 0$ and also note that

$$\Pi_3 = \Pi_{22} \oplus \Pi_{31} \colon Y \to Y_{22} \oplus Y_{31} = Y_2 \oplus Y_{31} = \{0\}.$$

Thus we obtain $\Pi_3 = 0$ and $\Pi_3 F'''(0,0,0)[h]^3 = 0$. We also have

$$\operatorname{Im} \Pi_3 F'''(0,0,0)[h]^2 = \{0\} \quad \text{and} \quad \Pi_3 F^{(4)}(0,0,0)[h]^4 = 0.$$

Let us compute the fourth derivative of the mapping F and consequently we obtain $F^{(4)}(0,0,0) = F^{(4)}(\mu, x_1, x_2)$ and

$$\begin{aligned} F^{(4)}(0,0,0)[h]^3 &= ((2\sqrt{10}h_{x_1}^3, 6\sqrt{10}h_\mu h_{x_1}^2, 0), (-18h_\mu^2 h_{x_2}, 0, -6h_\mu^3)), \\ &= 3!\Bigg(\bigg(\frac{\sqrt{10}}{3} h_{x_1}^3, \sqrt{10} h_\mu h_{x_1}^2, 0\bigg), (-3h_\mu^2 h_{x_2}, 0, -h_\mu^3)\Bigg), \end{aligned}$$

$$F^{(4)}(0,0,0)[h]^4 = (8\sqrt{10}h_\mu h_{x_1}^3, -24h_\mu^3 h_{x_2}) = 4!\bigg(\frac{\sqrt{10}}{3} h_\mu h_{x_1}^3, -h_\mu^3 h_{x_2}\bigg). \tag{3.18}$$

Note that

$$\Pi_4 = \Pi_{32} \oplus \Pi_{41} \colon Y \to Y_{32} \oplus Y_4, \tag{3.19}$$

but $Y_{32} \oplus Y_4 = \operatorname{Im} P_{(Y_1 \oplus Y_{21} \oplus Y_{22} \oplus Y_{31})^\perp} F^{(4)}(0,0,0)[h]^3 = \operatorname{Im} P_{(Y_1)^\perp} F^{(4)}(0,0,0)[h]^3$, because $Y_{21} = \{0\}$ and $Y_{22} \oplus Y_{31} = \{0\}$. Since $F^{(5)}(0,0,0) = 0$ we have $\Pi_4 = P_{(Y_1)^\perp}$. Hence

$$\begin{aligned}
&\Pi_4 F^{(4)}(0,0,0)[h]^4 = P_{(Y_1)^\perp} F^{(4)}(0,0,0)[h]^4 \\
&= \phi_1 \int_0^{2\pi} [8\sqrt{10} h_\mu (a\cos\tau + b\sin\tau)^3 \cos\tau - 24 h_\mu^3 (-a\sin\tau + b\cos\tau)(-\sin\tau)]\, d\tau \\
&\quad + \phi_2 \int_0^{2\pi} [8\sqrt{10} h_\mu (a\cos\tau + b\sin\tau)^3 \sin\tau - 24 h_\mu^3 (-a\sin\tau + b\cos\tau)\cos\tau]\, d\tau.
\end{aligned}$$

Now from the condition $\Pi_4 F^{(4)}(0,0,0)[h]^4 = 0$ (see (3.8)) we obtain the equation $8\sqrt{10} h_\mu 3\pi/4 - 24 h_\mu^3 \pi = 0$ for $a = 1$ and $b = 0$. From this we conclude that $h_\mu = \pm\sqrt[4]{10}/2$ and the vector h can be chosen as (3.11).

Then we check if the mapping $\Pi_4 F^{(4)}(0,0,0)[h]^3$ is a surjection on a settled element $\overline{h}$. The Lemma 2.9 shows that for the element $[\varepsilon, u_1, u_2] \in \operatorname{Ker} F'(0,0,0)$ and for the element $h = [h_\mu, h_{x_1}, h_{x_2}]$ we have:

$$\begin{aligned}
&\Pi_4 F^{(4)}(0,0,0)[h]^3[\varepsilon, u_1, u_2] = P_{(Y_1)^\perp} F^{(4)}(0,0,0)[h]^3[\varepsilon, u_1, u_2] \\
&= \phi_1 \int_0^{2\pi} [(2\sqrt{10}\varepsilon h_{x_1}^3 + 6\sqrt{10} h_\mu h_{x_1}^2 u_1)\cos\tau + (-18 h_\mu^2 h_{x_2}\varepsilon - 6 h_\mu^3 u_2)(-\sin\tau)]\, d\tau \\
&\quad + \phi_2 \int_0^{2\pi} [(2\sqrt{10}\varepsilon h_{x_1}^3 + 6\sqrt{10} h_\mu h_{x_1}^2 u_1)\sin\tau + (-18 h_\mu^2 h_{x_2}\varepsilon - 6 h_\mu^3 u_2)\cos\tau]\, d\tau.
\end{aligned}$$

It is sufficient to substitute the element $\overline{h}$ (3.11) and ε, $u_1 = c\cos\tau + d\sin\tau$, $u_2 = -c\sin\tau + d\cos\tau$ into above equation for receiving a dependence

$$\Pi_4 F^{(4)}(0,0,0)[\overline{h}]^3[\varepsilon, u_1, u_2] = 3\pi\sqrt{10}\bigg(-\varepsilon \pm \frac{c\sqrt[4]{10}}{2}\bigg)\phi_1 = p\phi_1, \quad p \in \mathbb{R}.$$

Then the mapping $\Pi_4 F^{(4)}(0,0,0)[h]^3$ on the given element $\overline{h}$ is not a surjection and the projection $\Pi_4 = P_{Y_1^\perp} = \Pi_{\operatorname{span}\{\phi_1\}}$. Moreover, the modified 4-regularity does not hold for the system (3.12). However above calculations considerably make easier the next ones.

Therefore let us come back to the system (3.10), and write it as follows:

$$\begin{aligned}
F(\mu, x_1, x_2) = (\dot{x}_1 - x_2 + \mu x_1^2 + (\sqrt{10}/3)\mu x_1^3 \\
- \mu^5 x_1, \dot{x}_2 + x_1 + \mu x_2^2 - \mu^3 x_2 - \mu x_2^5) = 0,
\end{aligned}$$

where $F\colon \mathbb{R} \times \mathcal{C}^2(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ and $x_1(0) = x_1(2\pi)$, $x_2(0) = x_2(2\pi)$.

Continuing in the same way (see (3.19)), we conclude that $\Pi_4 = \Pi_{32} \oplus \Pi_{41}\colon Y \to Y_{32} \oplus Y_{41}$ and

$$\begin{aligned}
Y_{32} \oplus Y_{41} &= \operatorname{Im} P_{(Y_1 \oplus Y_{21} \oplus Y_{22} \oplus Y_{31})^\perp} F^{(4)}(0,0,0)[h]^3 \\
&= \operatorname{Im} P_{(Y_1)^\perp} F^{(4)}(0,0,0)[h]^3 = \operatorname{span}\{\phi_1\},
\end{aligned}$$

because $Y_{21} = \{0\}$ and $Y_{22} \oplus Y_{31} = \{0\}$.

Moreover, $F^{(5)}(0,0,0)=0$, $F^{(5)}(0,0,0)[h]^5=0$ and

$$\begin{aligned}Y_{42}\oplus Y_{51}&=\operatorname{Im}P_{(Y_1\oplus Y_{21}\oplus Y_{22}\oplus Y_{31}\oplus Y_{32}\oplus Y_{41})^\perp}F^{(5)}(0,0,0)[h]^4\\&=\operatorname{Im}P_{(Y_1\oplus \mathrm{span}\{\phi_1\})^\perp}F^{(5)}(0,0,0)[h]^4=\{0\}.\end{aligned}$$

Since $\Pi_5\colon Y\to Y_{42}\oplus Y_{51}=\{0\}$ we have $\Pi_5=0$ and $\Pi_5F^{(5)}(0,0,0)[h]^5=0$, $\Pi_5F^{(6)}(0,0,0)[h]^6=0$. Then

$$\begin{aligned}Y_{52}\oplus Y_6&=\operatorname{Im}P_{(Y_1\oplus Y_{21}\oplus Y_{22}\oplus Y_{31}\oplus Y_{32}\oplus Y_{41}\oplus Y_{42}\oplus Y_{51})^\perp}F^{(6)}(0,0,0)[h]^5\\&=\operatorname{Im}P_{(Y_1\oplus \mathrm{span}\{\phi_1\})^\perp}F^{(6)}(0,0,0)[h]^5\\&=\operatorname{Im}P_{\mathrm{span}\{\phi_2\}}F^{(6)}(0,0,0)[h]^5.\end{aligned}$$

Continuing, we could find a form of $F^{(6)}(0,0,0)[h]^5$. Let us write

$$F^{(6)}(0,0,0)[h]^6=6![-h_\mu^5h_{x_1},-h_\mu h_{x_2}^5],$$

where $h=[h_\mu,h_{x_1},h_{x_2}]$ and $B[h]^6=[-h_\mu^5h_{x_1},-h_\mu h_{x_2}^5]$. Note that (Lemma 2.14)

$$\begin{aligned}B^{(6)}[h]^5&=F^{(6)}(0,0,0)[h]^5=5!(B[h]^6)'\\&=5!((-5h_\mu^4h_{x_1},-h_\mu^5,0),(-h_{x_2}^5,0,-5h_\mu h_{x_2}^4)).\end{aligned}$$

From this we obtain

$$F^{(6)}(0,0,0)[h]^5[\varepsilon,u_1,u_2]=5!(-5h_\mu^4h_{x_1}\varepsilon-h_\mu^5u_1,-h_{x_2}^5\varepsilon-5h_\mu h_{x_2}^4u_2)$$

and

$$\begin{aligned}&P_{\mathrm{span}\{\phi_2\}}F^{(6)}(0,0,0)[h]^5[\varepsilon,u_1,u_2]\\&\quad=5!\phi_2\int_0^{2\pi}[(-5h_\mu^4h_{x_1}\varepsilon-h_\mu^5u_1)\sin\tau+(-h_{x_2}^5\varepsilon-5h_\mu h_{x_2}^4u_2)\cos\tau]\,d\tau.\end{aligned}$$

Next we substitute the element $\overline{h}$ (3.11), ε, $u_1=c\cos\tau+d\sin\tau$ and $u_2=-c\sin\tau+d\cos\tau$ to the above equation and this implies a following result:

$$P_{\mathrm{span}\{\phi_2\}}F^{(6)}(0,0,0)[\overline{h}]^5[\varepsilon,u_1,u_2]=q\phi_2,\quad q\in\mathbb{R}.$$

This shows that $Y_{52}\oplus Y_6=\mathrm{span}\{\phi_2\}$ and a projection $\Pi_6\colon Y\to Y_{52}\oplus Y_6$ is described by $\Pi_6=\Pi_{\mathrm{span}\{\phi_2\}}=P_{\mathrm{span}\{\phi_2\}}$. Then $\Pi_{\mathrm{span}\{\phi_2\}}F^{(6)}(0,0,0)[h]^5=P_{\mathrm{span}\{\phi_2\}}F^{(6)}(0)[h]^5$ is a surjection along the vector $\overline{h}$.

REMARK 3.6. Note that the modified 6-factor operator has the form:

$$\begin{aligned}\overline{\Psi}_6(\overline{h})&=\overline{\Psi}_6((0,0,0),\overline{h})\\&=F'(0,0,0)+\frac{1}{3!}\Pi_4F^{(4)}(0,0,0)[\overline{h}]^3+\frac{1}{5!}\Pi_6F^{(6)}(0,0,0)[\overline{h}]^5\\&=F'(0,0,0)+\frac{1}{3!}\Pi_{\mathrm{span}\{\phi_1\}}F^{(4)}(0,0,0)[\overline{h}]^3+\frac{1}{5!}\Pi_{\mathrm{span}\{\phi_2\}}F^{(6)}(0,0,0)[\overline{h}]^5.\end{aligned}$$

If $[\varepsilon, u_1, u_2] \in \operatorname{Ker} F'(0,0,0)$ then

$$\begin{aligned}\overline{\Psi}_6(\overline{h})[\varepsilon, u_1, u_2] &= F'(0,0,0)[\varepsilon, u_1, u_2] + \frac{1}{3!}\Pi_4 F^{(4)}(0,0,0)[\overline{h}]^3[\varepsilon, u_1, u_2] \\ &+ \frac{1}{5!}\Pi_6 F^{(6)}(0,0,0)[\overline{h}]^5[\varepsilon, u_1, u_2] = \frac{\pi}{2}\sqrt{10}(-\varepsilon \pm \frac{c\sqrt[4]{10}}{2})\phi_1 \\ &- \frac{5d\pi}{4}(\pm\frac{\sqrt[4]{10}}{2})\phi_2 = \overline{p}\phi_1 + \overline{q}\phi_2, \quad \overline{p}, \overline{q} \in \mathbb{R}.\end{aligned}$$

i.e. the modified 6-factor operator is a surjection along the vector $\overline{h}$ onto the space $Y_1^{\perp}$ and consequently, by the Lemma 2.9, onto the space Y.

Let us remark also that $\Pi_{\operatorname{span}\{\phi_2\}} F^{(6)}(0,0,0)[\overline{h}]^6 = 0$. Of course

$$\Pi_6 F^{(7)}(0,0,0)[h]^7 = \Pi_{\operatorname{span}\{\phi_2\}} F^{(7)}(0,0,0)[h]^7 = 0,$$

which is clear from $F^{(7)}(0,0,0) = 0$.

We verified all assumptions of the theorem on modified p-factor operator. Hence the element $\overline{h} = [\pm\sqrt[4]{10}/2, \cos t, -\sin t]$ belongs to the tangent cone $T_{(0,0,0)}M$. Therefore the theorem guarantees the existence of the solutions of system (3.10) for each $\mu \neq 0$. The following theorem allows to find these solutions.

Theorem 3.7. *For sufficiently small $\mu \in (-\varepsilon, \varepsilon)$ the system* (3.10) *has two* (*trivial and nontrivial*) *solutions of the form*:

$$\overline{x}(\mu) = (\mu, 0, 0), \qquad \overline{\overline{x}}(\mu) = \left(\pm\frac{\sqrt[4]{10}}{2}\mu, \mu\cos t, -\mu\sin t\right) + r(\mu),$$

where $\|r(\mu)\| = o(\mu)$.

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