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FLAT FACES IN PUNCTURED TORUS GROUPS

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1. Introduction. One of the intriguing constructions which arises from the use of the Minkowski space model for hyperbolic space is that of the *canonical triangulation* of a one cusped manifold. We briefly recall its definition in the dimensions relevant to this paper, namely, $n = 2, 3$ (a somewhat more complete description is included below). It follows from results of Epstein and Penner [**1**] that Jorgensen's lemma implies that the action of a finite covolume group on a lightlike vector corresponding to the cusp forms a discrete subset of the light cone. One can take the convex hull in Minkowski space of this set and project into hyperbolic space to obtain an equivariant cellulation (generically a triangulation) which descends to the manifold. In the case that the hyperbolic manifold in question only has one cusp, the only ambiguity is a possibly scaling in the initial choice of lightlike vector representing this cusp. Two scaled convex hulls project to the same triangulation.

Despite the elegant nature of this construction, it is not very well understood, and several authors have attempted to identify this triangulation in special cases, see Weeks [**11**], Sakuma [**7**], Lackenby [**2**]. This paper attempts to understand the convex hull in a somewhat simpler setting, namely in the case of punctured torus groups acting on $\mathbf{H}^2$. Even here the answers turn out to be rather subtle. We begin by proving a condition (originally sketched by Thurston) which guarantees that once we have taken the hull of enough points, this will indeed form part of the canonical convex hull. As an initial step, we show:

Theorem 3.1.2. *Local boundary convexity of a polyhedron implies boundary convexity.*

With this theorem in hand, we can begin to analyze the convex hulls of punctured torus groups. Here we use a description first introduced in [**3**] of punctured torus groups as $\Delta(u^2, 2\tau)$; this is explained in subsection 3.1. While not a parametrization (different values may give

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rise to conjugate groups), this description is conveniently geometric for our purposes. Somewhat surprisingly, we find that there is one real parameter family of nonconjugate punctured torus groups $\Delta(u^2, 4)$ which have the property that their convex hulls are tiled, not by triangulations, but are actually *cellulations* by ideal rectangles; that is to say, the canonical construction gives rise to a canonical ideal rectangle which is therefore a canonical fundamental domain for the punctured torus. We call these *flat face groups.*

In general, determining the commensurability of groups is a challenging endeavor. Flat face groups demonstrate properties which ease this task considerably. With the aid of a theorem of Margulis, we first show that the minimal element in the commensurability class of the nonarithmetic $\Delta(u^2, 4)$ groups is the quotient of $\mathbf{H}^2$ by the group formed by adjoining the hyper-elliptic involution to the flat face group. By appealing to the cross-ratio, we then prove

Theorem 4.1.5. *Let $0 < u^2 < 1/2$. No two nonarithmetic groups in $\Delta(u^2, 4)$ are commensurable.*

The remainder of the paper is devoted to examining various properties of flat face groups, in particular the $\Delta(u^2, 4)$ groups. The results of this section focus on the geometric properties of the convex hulls of flat face groups, the main result being

Theorem 3.3.7. *Let C be the convex hull of a punctured torus group in Minkowski space. If ∂C is composed of rectangles, then the axes of the geodesics which identify opposite sides of these rectangles meet at right angles.*

2. Preliminaries.

2.1 The Minkowski hyperboloid model. We recall some basic material about the Minkowski hyperboloid model. Let $v, w \in \mathbf{R}^{n+1}$, $v = t_1, x_1, x_2, \dots x_n$, $w = t_2, y_1, y_2, \dots, y_n$. If we equip Euclidean $n+1$-space with the nondegenerate quadratic form of signature $(n, 1)$

$$\langle v, w\rangle = -t_1t_2 + x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

the resulting vector space equipped with the corresponding (path) metric

$$ds^2 = -dt_1^2 + dx_1^2 + dx_2^2 + \cdots + dx_n^2$$

is *Minkowski n-space*, M^n, also called *Lorentzian n-space.*

This metric gives rise to three types of vectors in M^n. If $v \in M^n$, then $||v||^2 = \langle v, v\rangle$ is either positive, negative or zero. If $||v||^2$ is negative, we say v is *time-like.* If $||v||^2$ is positive, we say v is *space-like.* Finally, if $||v||^2$ is zero, we say v is *light-like.*

Definition. The set of all light-like vectors

$$\{v \in M^n | \langle v, v\rangle = 0\} = \{v \in M^n | -t_1^2 + x_1^2 + x_2 + \cdots + x_n^2 = 0\}$$

is called the *light-cone* and denoted L. We will be interested in the component of $L \setminus \{0\}$ in which $t_1 > 0$.

Every ray from the origin in the light-cone corresponds to a point in the sphere at infinity of the Poincaré model. If we think of a ray in the light-cone as encoding the center of a family of horospheres, the heights along the ray encode the sizes of the horospheres centered on that ray.

The horoball corresponding to $v_0 \in L^+$ is

$$\{w \in \mathbf{H}^n \mid \langle w, v_0\rangle = -1\}.$$

Further, the horoball corresponding to the horosphere v_0 is given by

$$\{w \in \mathbf{H}^n \mid \langle w, v_0\rangle \geq -1\}.$$

Points of L^+ closer to the origin correspond to larger horospheres, and conversely, as we choose points in L^+ whose t coordinate approaches infinity, the corresponding horosphere shrinks to a point.

2.2 The action of $\mathrm{PSL}(2, \mathbf{R})$ **and its discrete subgroups on points in the light cone.** The group of isomorphisms preserving the quadratic form described above is $\mathrm{O}(n, 1)$, called the *Lorentz group* of $(n + 1) \times (n + 1)$ matrices. This group has various components. We are interested in those elements with determinant one which preserve

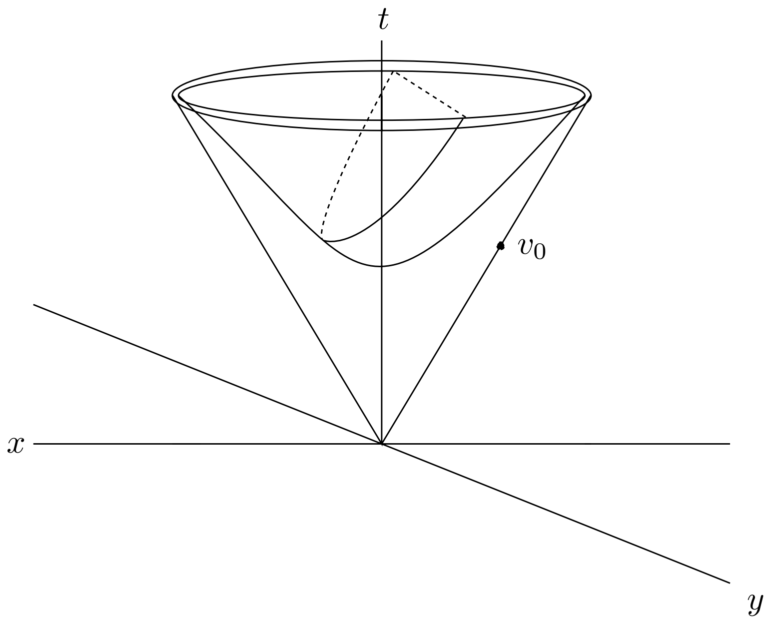

FIGURE 1. The light-cone L^+ and a vector $v_0 \in L^+$ with its corresponding horosphere.

the upper sheet of the hyperboloid, that is, $\mathrm{SO}(n,1) \cap \mathrm{O}^+(n,1)$. This group is isomorphic to the group of isometries of $\mathbf{H}^n$. The isomorphism $\mathrm{SO}(2,1) \to \mathrm{PSL}(2,\mathbf{R})$ is achieved by

$$v = (t,x,y) \longleftrightarrow \begin{bmatrix} t+x & y \\ y & t-x \end{bmatrix} = \Sigma.$$

Note that $\det(\Sigma) = 1$ implies that $t^2 - x^2 - y^2 = 1$, which is precisely the condition that $v \in \mathbf{H}^2$.

A matrix $A \in \mathrm{SL}(2,\mathbf{R})$ acts on Σ via similarity, that is

$$\Sigma \mapsto A^T \Sigma A$$

and factors through $\mathrm{PSL}(2,\mathbf{R})$. Since all elements of $\mathrm{PSL}(2,\mathbf{R})$ have determinant one, this action preserves the quadratic form.

The dynamics of an action of a discrete subgroup of isometries of $\mathbf{H}^2$ on the circle at infinity are quite complicated. However, if we consider the analogue of the circle at infinity in the hyperboloid model, namely the light-cone, it turns out that the action of certain parabolic elements is relatively well-behaved.

Epstein and Penner [**1**] proved several results about the consequences of these actions on the light-cone. The following result is particularly relevant.

Theorem 2.2.1. *Let Γ be a discrete finite co-volume subgroup of $O^+(n,1)$. Then $v \in L^+$ is a parabolic fixed point if and only if, for every $k > 0$, the number of points in the orbit of v with height (t coordinate) less than k is finite.*

2.3 The convex hull construction. If we are given a discrete finite co-volume subgroup of $\mathrm{O}^+(n,1)$ with at least one parabolic fixed point in the light-cone, that is, $\mathbf{H}^n/\Gamma$ has at least one cusp, we can form a convex hull in Minkowski space in the following way. We choose one orbit of parabolic fixed points in L^+ corresponding to each cusp. Since Theorem 2.2.1 ensures us that these orbits are discrete, we can then form the closed convex hull of these points.

The resulting convex hull is an $n+1$-dimensional object.

We will be particularly interested in the boundary of the convex hull. The following lemma of Epstein and Penner [**1**] helps to elucidate its nature.

Lemma 2.3.1. *Let C be the convex hull described above. The boundary of C is the union of $C \cap L^+$ along with a countable collection of codimension 1 faces, where each face is the convex hull of a finite number of points in the orbit of a parabolic fixed point.*

We know that C contains all rays of L^+ originating at each point in the orbit of parabolic fixed points.

For each cusp, we choose a height on the ray in the light-cone corresponding to the cusp. Thus, if p equals the number of cusps of $\mathbf{H}^n/\Gamma$, we see a $(p-1)$ parameter family of tesselations, where each cell has a totally geodesic face. In particular, if $\mathbf{H}^n/\Gamma$ has 1 cusp, there exists a canonical tesselation. Jeff Weeks used this approach in several papers, see [**7, 11**], for example.

Since Γ is torsion-free, the interior of any face injects into $\mathbf{H}^n/\Gamma$, that is, the tesselation of $\mathbf{H}^n$ descends to a natural $(p-1)$ parameter family of decompositions of $\mathbf{H}^n/\Gamma$. We are mainly concerned with once-punctured tori; thus, this construction gives a completely canonical decomposition.

3. Applications of the convex hull construction. We begin this section fleshing out proofs of two propositions which were sketched by Thurston in his notes [**10**]. These propositions deal with various local and global convexity properties of polyhedra. We then go on to discuss a family of once-punctured torus groups whose convex hulls have several surprising and interesting properties.

3.1 Determining convex hulls. We proceed by stating some relevant definitions regarding convexity of polyhedra. We will assume that all polyhedra are connected.

Definition. A polyhedron P is (*globally*) *convex* if every path in P is homotopic (rel endpoints) to a geodesic lying in P.

P is *locally convex* if each point $x \in P$ has a neighborhood isometric to a convex subset of $\mathbf{R}^3$. If $x \in \partial P$, then x will be on the boundary of this set.

P is *boundary convex* if for any two points lying in ∂P, the geodesic connecting them lies entirely in P.

P is *locally boundary convex* if for each point p in the boundary of P, there exists a convex neighborhood of p such that for any two points x and y in this neighborhood and on the boundary of P, the geodesic connecting them also lives in P.

Proposition 3.1.1. *Boundary convexity of a polyhedron implies global convexity.*

Proof. Suppose not, and let P be a boundary convex polyhedron, $x, y \in P$, and let γ be a path connecting x and y lying in P. There exists a geodesic $\gamma^\star \in \mathbf{R}^3$ connecting x and y. Homotope γ to $\gamma^\star$ rel endpoints. If $\gamma^\star$ does not lie entirely within P, it must cross through the boundary of P (possibly in several places). Choose two points on the boundary of P through which $\gamma^\star$ crosses, x_0 and y_0. The boundary convexity of P guarantees that the geodesic connecting x_0 and y_0 lies entirely in P. However, geodesics in $\mathbf{R}^3$ are straight lines, thus the geodesic connecting x_0 and y_0 is a segment of $\gamma^\star$, implying that $\gamma^\star$ never left P. Thus, P is convex. □

Theorem 3.1.2. *Local boundary convexity of a polyhedron implies boundary convexity.*

Proof. The proof is by path straightening. Choose two points a and b on the boundary of the polyhedron P and a path τ lying in the interior of P connecting them. Such a τ exists due to the connectedness of P. Choose $\varepsilon > 0$ small enough so that for each point p in the boundary of the polyhedron, there is an ε neighborhood of p such that for all points x, y in this neighborhood lying on the boundary of the polyhedron, the geodesic connecting x and y lies within the polyhedron. Divide our path τ into subintervals of length at most $\varepsilon/2$, and straighten each subinterval via homotopy to a geodesic. Since the interior of a polyhedron is locally convex, our only worry arises if a neighborhood containing one of the subintervals of τ also contains a segment of the boundary of the polyhedron. If this is the case, then the local boundary convexity condition insures us that the homotopy must not leave the polyhedron. If it did leave, the geodesic connecting the point at which the homotopy exits the polyhedron to the point at which it reenters actually lies in the polyhedron. That is, we can remove the piece of the homotopy lying outside of the polyhedron.

The result of this homotopy process is a piecewise linear path. Resubdivide this new path into $\varepsilon/2$ subintervals and repeat the straightening process described above. Any time there are angles involved which are not close to π, this process significantly shortens our path. The sequence of resulting path lengths is monotonically decreasing and positive, thus converges to a limit. If the path corresponding to this limit were not a geodesic, then it would contain at least one nonlinear section. To show this cannot happen, take an $\varepsilon/2$ neighborhood of the nonlinear section and perform the straightening operation, decreasing the total path length. Thus, our original two points in the boundary of the polyhedron are indeed connected by a geodesic lying entirely in the polyhedron. □

In studying commensurability of finite co-volume subgroups of PSL $(2, \mathbf{R})$ and pseudomodular surfaces [**4**], Long and Reid introduced a convenient normalization for once-punctured torus groups, $\Delta(u^2, 2\tau)$. The parameters involved correspond to placement of the cusp (u^2) and translation length (in the upper-half space model) of the parabolic element stabilizing infinity (2τ). Convenient generators for these groups are given [**4**] by

$$g_1 = \begin{pmatrix} (-1+\tau)/\sqrt{-1+\tau-u^2} & u^2/\sqrt{-1+\tau-u^2} \\ 1/\sqrt{-1+\tau-u^2} & 1/\sqrt{-1+\tau-u^2} \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} u/\sqrt{-1+\tau-u^2} & u/\sqrt{-1+\tau-u^2} \\ 1/u\sqrt{-1+\tau-u^2} & (\tau-u^2)/(u\sqrt{-1+\tau-u^2}) \end{pmatrix}.$$

To further investigate the properties of these groups, we applied the convex hull construction and inspected the decompositions of the punctured tori which resulted.

To determine the actual convex hulls of these groups, we wrote Mathematica code which generates a list of potential faces of the hull. Using the following facts, we can then show that the computer generated result is indeed the correct triangulation of the domain.

Since any two adjacent faces in the boundary of the convex hull form a fundamental domain for the given punctured torus, cf. subsection 3.3, we choose two well-placed faces to use for the local convexity check. Generally, lower vertices tend to make the calculations easier. To check local convexity of the entire hull, we need only to check the convexity across three edges, the edge between the two faces and the two other edges of either one of the triangular faces, as all other edges are the image of one of these three edges under a covering transformation.

Example. $\Delta(1,6) = \ker\{\mathrm{PSL}\,(2,\mathbf{Z}) \to \mathbf{Z}_6\}$. The hull generated by $\Delta(1,6)$ is particularly nice due to its symmetry. After selecting v_∞

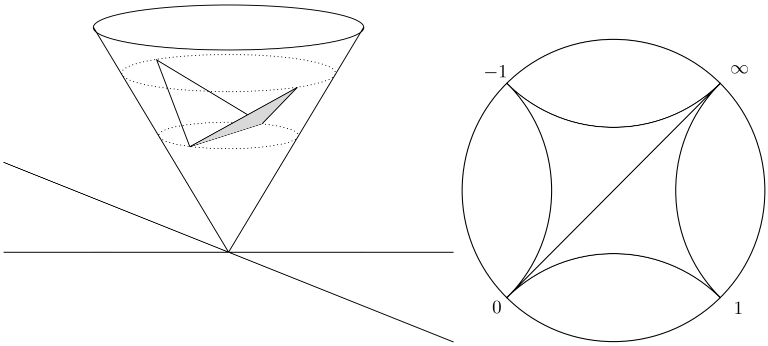

FIGURE 2. The canonical triangulation for the $\Delta(1,6)$ group in the Poincaré model and the faces formed on the boundary of the convex hull.

to be $(1,-1,0)$, one sees that a canonical domain contains the face joining the vertices $v_\infty, (1,1,0)$, and $(2,0,-2)$ with the face joining the vertices $v_\infty, (1,1,0)$ and $(2,0,2)$. This is illustrated in Figure 2. One notes also that the images of the two ideal triangles (in the Poincaré model) making up the fundamental domain for the $\Delta(1,6)$ group form the Farey tesselation of the hyperbolic plane. See, for example, [**8**] for more details of the Farey tesselation.

3.2 A family of once-punctured tori tiled by rectangles. The generic decomposition associated to a convex hull of a punctured torus described in subsection 2.3 is composed of two triangles. In this case, these triangles have a dihedral angle less than π between them, tesselating the boundary of the convex hull by triangles. If we consider $\Delta(u^2, 2\tau)$ groups where $2\tau = 4$, that is, whose commutator is translation by four in the upper-half plane model, we see a surprising result.

Theorem 3.2.1. *The canonical "triangulation" of the once punctured torus corresponding to the group* $\Delta(u^2, 4)$ *for* $0 < u^2 \leq 1/2$ *aris-*

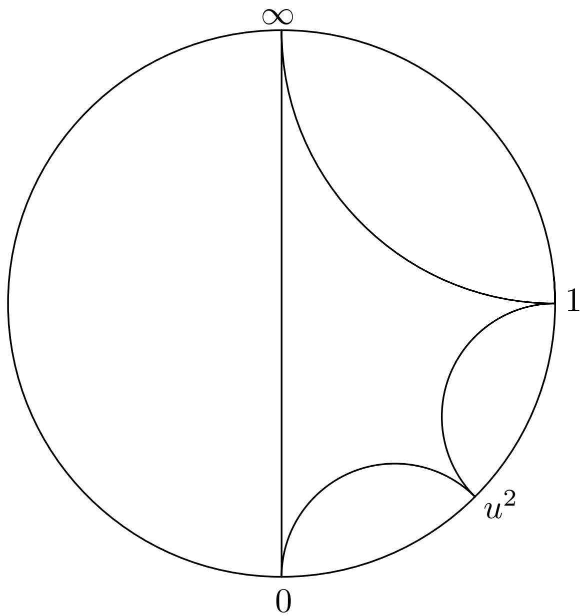

FIGURE 3. $\Delta(u^2, 4)$ domain.

ing from the convex hull in Minkowski space is the rectangle shown in Figure 3*.*

Proof. Given a fundamental domain, it is sufficient to check the convexity across either of the triangles composing this domain, as mentioned above. Further, since the domain in question is flat, we need only check the convexity across two adjacent edges of the domain. These are shown as dashed lines in Figure 4. Appealing to Proposition 3.1.1 and Theorem 3.1.2 above, we see that this local convexity implies the convexity of the entire hull.

The calculation to check convexity is elementary. We first choose a consistent ordering for the domains and calculate the Euclidean normal to the plane containing the domain in Minkowski space. If we choose a clockwise ordering, the resultant normals point outward from the convex hull.

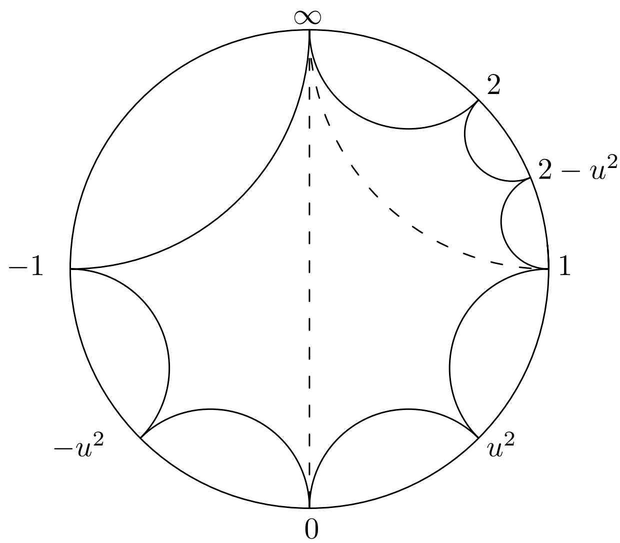

FIGURE 4. Checking for local convexity across adjacent domains.

We now take the Euclidean dot product of this normal with a vector in an adjacent domain.

Continuing in this fashion with all of the domains, we see that the dot products all have the same sign. We can thus conclude that we have determined the convex hull and that these domains are flat. □

It is easy to show [**4**] that the group $\Delta(u^2, 4)$ is equivalent to the group $\Delta(1-u^2, 4)$ after remarking and conjugacy, even though the canonical tesselations derived from the convex hulls of these groups appear to differ. This fact will be used in proving Theorem 4.1.5.

3.3 Properties of flat face groups. In this section, we examine various properties of groups whose convex hulls are tiled by rectangles, in particular the $\Delta(u^2, 4)$ groups. We begin with several definitions as well as a series of lemmata which allow us to prove a theorem describing the nature of the rectangles which comprise the boundary of the hull. We will let T denote a flat face group and C its convex hull.

Remark. Any two adjacent triangles in the boundary of C form a fundamental domain for T.

The following lemma determines when the nongeneric case of tiling by squares arises.

Lemma 3.3.1. *The convex hull of a $\Delta(u^2, 4)$ group is tesselated by squares only when $u^2 = 1/2$.*

Proof. If we choose $v_\infty = \{1, -1, 0\}$, then one calculates that

$$v_1 = g_1 \cdot v_\infty = \left\{ \frac{2}{1-u^2},\ 0,\ \frac{2}{u^2-1} \right\}$$

$$v_0 = (g_1^{-1} \cdot g_2) \cdot v_\infty = \left\{ \frac{1}{u^2},\ \frac{1}{u^2},\ 0 \right\}$$

Using the Minkowski metric, one calculates that the Minkowski length of the side connecting v_∞ and v_0 is $2\sqrt{1/u^2}$, and that the side connecting v_∞ and v_1 is $2\sqrt{1/1-u^2}$. However, these lengths are not well-defined, as they depend on the initial choice of v_∞. Therefore, we must consider the ratio of these side lengths. This ratio is unaffected by the scaling of the horoball, i.e., the height we choose for v_∞ on the light-like ray representing infinity. Thus, the rectangle is a square exactly when the ratio $\sqrt{u^2/(1-u^2)} = 1$. This occurs precisely when $u^2 = 1/2$, proving the lemma. □

Definition. We say that two triangles in the boundary of the convex hull are *coplanar in the boundary of the convex hull* if they live in the affine hull of three points in the boundary of the convex hull. Likewise we say that three vertices in the boundary of the convex hull (these are necessarily light-like vectors) are *coplanar in the boundary of the convex hull* if they live in the affine hull of three points in the boundary of the convex hull. A triangle in a coplanar chain of triangles is called an *endmost triangle* if exactly one of the dihedral angles it forms with the two adjacent triangles in the hull is π radians. Finally, an edge in the hull is called a *flat edge* if the dihedral angle between the faces on either side of the edge is π radians.

Lemma 3.3.2. *Any two coplanar triangles in the boundary of the convex hull are connected by a chain of coplanar triangles, all of which lie in the boundary of C.*

Proof. Let A and B be two coplanar triangles in the boundary of the convex hull. If A and B have an edge in common, then there is nothing to show, so we may assume that they do not. Since C is convex, we know that all of the hull lies to one side of the plane containing A and B. Further, any line connecting these two triangles must also lie in the hull, again due to the convexity of C. The fact that C all lies on one side of this plane implies that the line we drew connecting the triangles must never leave the boundary of C, for if the line left the boundary but stayed in the hull, convexity would force the hull to cross through this plane. Thus, if we take the collection of triangles in the boundary of the hull which meet this line, we have our coplanar chain of triangles. □

Lemma 3.3.3. *The number of coplanar vertices in the boundary of the convex hull is finite.*

Proof. First note that any plane intersecting a cone traces out either an ellipse (nongenerically a circle) if the intersection is compact, or an arc (either a parabola or a hyperbola) if the intersection is noncompact. If we assume by way of contradiction that there existed an infinite number of coplanar vertices in the light-cone, all of which lie on the hull, then the intersection of the plane containing these points with the light-cone would have to be an arc, as depicted in Figure 5, as a compact intersection would give rise to an accumulation point in the light-cone, contradicting Theorem 2.2.1. Further, these points cannot all lie on a single ray as each ray in the light-cone contains at most one point in the orbit of infinity.

But this implies that there is a neighborhood of a ray in the light-cone which never crosses this plane. Since the hull is convex, all vertices must lie to one side of this plane formed by the coplanar vertices. Because light-like rays are dense in the boundary of the hull and a neighborhood of the original ray contains no points in the orbit of infinity, we have a contradiction. □

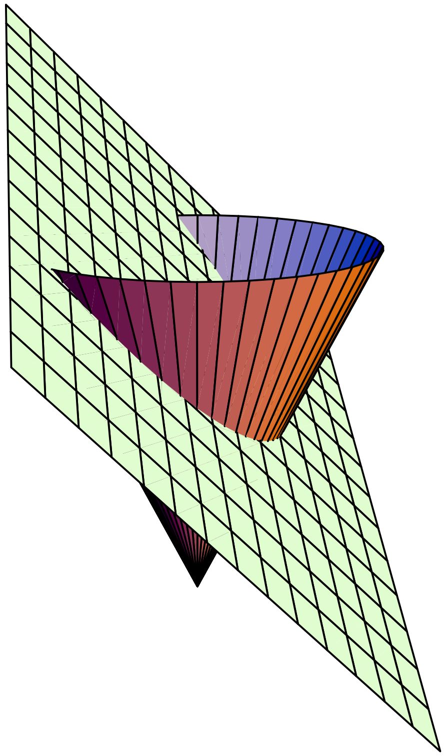

FIGURE 5. Plane intersecting light-cone in an arc.

An immediate consequence of Lemma 3.3.3 is that the chain of coplanar triangles in C discussed in Lemma 3.3.2 must be finite.

Lemma 3.3.4. *There is at least one endmost triangle in any coplanar chain in the boundary of C.*

Proof. Let us assume that we have a coplanar chain in the boundary of C with no endmost triangles, and choose any triangle in the chain. Draw a simple curve which starts in this triangle and exits through a flat edge. Continue passing the curve through triangles in the chain making sure to enter or exit triangles only through flat edges, crossing each edge no more than once. Since there are only a finite number of triangles, and by assumption each triangle has at least two flat edges, the curve must reenter a triangle, forming a closed loop.

One notes, however, that there can be no closed circuits of triangles around a vertex, because each vertex in the light-cone has an infinite number of triangles incident upon it.

This contradiction proves Lemma 3.3.4. □

We are now prepared to prove Theorem 3.3.5.

Theorem 3.3.5. *The affine hull of any three points lying in the boundary of the convex hull contains at most four vertices in the light-cone.*

Proof. To prove the theorem, assume that there are at least five coplanar vertices in the light-cone on the convex hull. Since three vertices form a face in the hull, and any two faces are connected by a coplanar chain, we have at least three coplanar triangles on the boundary of C, any two of which form a fundamental domain.

Of these three triangles in the coplanar chain, the one triangle adjacent to the endmost triangle in the chain must have at least two flat edges. Call the endmost triangle A and its adjacent flat triangle B. Taken together, triangles A and B form a fundamental domain for the punctured torus, as depicted in Figure 6. At least one of triangle B's outer edges is flat. Since one of the outer edges of triangle A is identified with this edge, and neither of these outer edges is flat, we have a contradiction. Thus, there are at most two adjacent coplanar triangles in the boundary of the hull, proving Theorem 3.3.5. □

We can further show that if we have a convex hull whose boundary is tiled by rectangles, then the relevant geodesics which identify opposite edges of the rectangles which form the punctured tori meet at right angles. We first need a theorem from Ratcliffe, see [**6**, page 71], and a definition.

Theorem 3.3.6. *Let v_1, v_2 be linearly independent space-like vectors in Minkowski space. Then the vectors v_1 and v_2 satisfy*

$$|\langle v_1, v_2\rangle| < \|v_1\| \, \|v_2\|$$

if and only if all nonzero vectors in the subspace spanned by v_1 and v_2 are space-like.

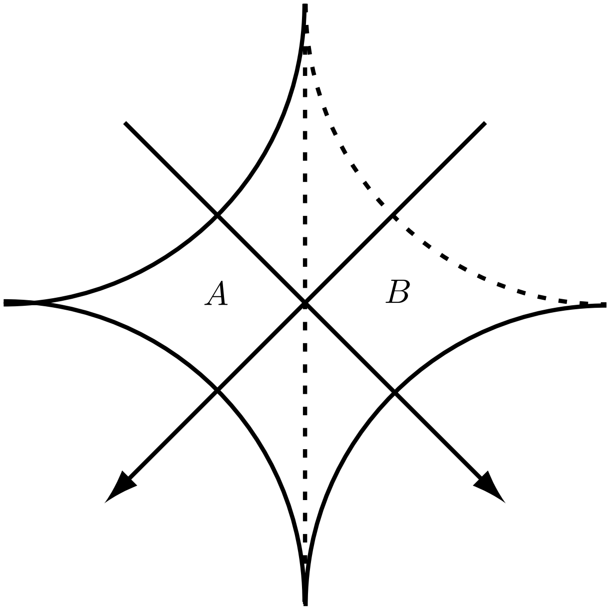

FIGURE 6. Adjacent coplanar triangles– Note dashed lines indicate flat edges.

Definition. We say four coplanar points v_1, v_2, v_3, v_4 in Minkowski space form a *parallelogram* if

$$(v_2 - v_1) + (v_3 - v_1) = v_4 - v_1.$$

We say two vectors in Minkowski space meet at a *right angle* if their Minkowski inner product is zero.

Remark. Let $v_1, \dots, v_4$ form a Minkowski parallelogram with vertices in the light-cone. Then $v_1, \dots, v_4$ form a Minkowski rectangle if any one of the angles of the parallelogram is right. One notes that if any one angle is right, the other three are as well.

Theorem 3.3.7. *Let C be the convex hull of a punctured torus group in Minkowski space. If ∂C is composed of rectangles, then the axes of the geodesics which identify opposite sides of these rectangles meet at right angles.*

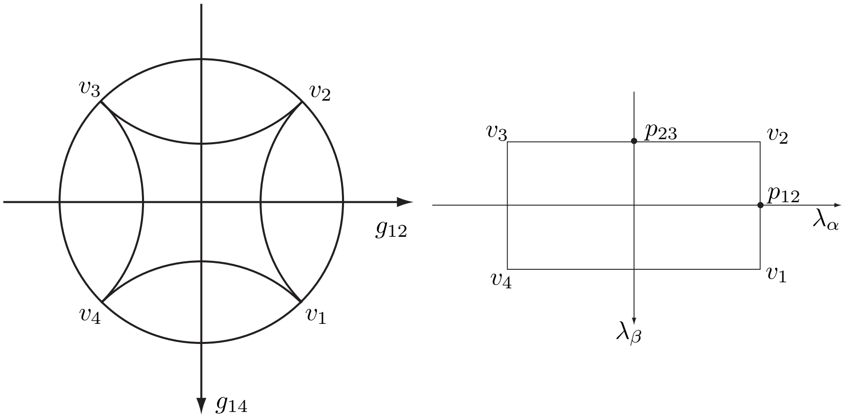

FIGURE 7. Rectangle in the boundary of the hull depicted two ways.

Proof. Choose a rectangle in the boundary of the hull, and label its vertices counterclockwise v_1 through v_4, noting that these vertices lie in the light-cone. In addition, label as g_{12} the geodesic which maps the geodesic running from v_3 to v_4 to the one running from v_1 to v_2. This is pictured in the Poincaré model in the lefthand image of Figure 7. The plane spanned by the light-like rays fixed by g_{12} is a time-like subspace intersecting the given rectangle in a (Euclidean) straight line, which we label λ_α. Note also that the intersection of this plane with the hyperboloid is the axis of g_{12}. We similarly define g_{14} and λ_β. Finally, label as p the point where λ_a and λ_b cross. The image on the righthand side of Figure 7 shows this rectangle in Minkowski space.

Claim 3.3.8. *The Minkowski inner product of two light-like vectors is negative.*

Let v_1 and v_2 be independent light-like vectors. Clearly $v_1 + v_2$ is time-like, so $0 > ||v_1 + v_2||^2$.

Now, observe

$$\begin{aligned} 0 > \|v_1 + v_2\|^2 &= \langle v_1 + v_2,\, v_1 + v_2 \rangle \\ &= \langle v_1,\, v_1 \rangle + \langle v_2,\, v_2 \rangle + 2\langle v_1,\, v_2 \rangle = 2\langle v_1,\, v_2 \rangle, \end{aligned}$$

proving Claim 3.3.8. □

Claim 3.3.9. *If v_1 and v_2 are independent light-like vectors, $v_1 - v_2$ is space-like.*

We need to show that $\|v_1 - v_2\|^2 > 0$. We do this as follows:

$$\begin{aligned}\|v_1 - v_2\|^2 &= \langle v_1 - v_2,\, v_1 - v_2\rangle \\ &= \langle v_1,\, v_1\rangle + \langle v_2,\, v_2\rangle - 2\langle v_1,\, v_2\rangle \\ &= -2\langle v_1,\, v_2\rangle > 0.\end{aligned}$$

So, $v_1 - v_2$ is space-like. A similar calculation shows that $v_1 - v_4$ is also space-like. □

The hyper-elliptic involution τ of the once-punctured torus admits three fixed points. One can choose a lift of τ in which one of the three fixed points in the torus lifts to a point on the edge of the domain connecting v_1 and v_2 which remains fixed in $\mathbf{H}^2$ under $\tilde{\tau}$. This gives rise to a fixed time-like vector. This fixed time-like vector intersects edge $\overline{v_1 v_2}$ of our rectangle at p_{12}. The action of $\tilde{\tau}$ is rotation by π through p_{12}, which interchanges v_1 and v_2, maps g_{12} to $-g_{12}$, and thus maps λ_α to $-\lambda_\alpha$. In short, $\tilde{\tau}$ preserves λ_α up to orientation. Further, $\tilde{\tau}$ is an isometry mapping the portion of $\overline{v_1 v_2}$ between p_{12} and v_1 to the portion between p_{12} and v_2. Thus, λ_α crosses $\overline{v_1 v_2}$ at its midpoint, p_{12}, which exists due to the fact that $v_2 - v_1$ is space-like. By choosing the lift of τ which fixes a point on the edge $\overline{v_1 v_4}$, one sees that λ_β crosses the edge $\overline{v_1 v_4}$ at its midpoint, p_{14}. Finally, since $\overline{v_1 v_2}$ is isometric to $\overline{v_3 v_4}$ via a covering translation, we know that λ_α connects the midpoints of $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$. Likewise λ_β connects the midpoints of $\overline{v_1 v_4}$ and $\overline{v_2 v_3}$. Label the point where λ_α crosses λ_β point p.

We now show that λ_α and λ_β cross at right angles in the plane containing the given rectangle.

Since $v_1 - v_2$ and $v_1 - v_4$ are linearly independent, $\|v_1 - v_2\|$ and $\|v_1 - v_4\|$ are positive (these vectors are space-like) and since v_1, v_2, v_3, v_4 form a rectangle, $\langle v_1 - v_2,\, v_1 - v_4\rangle$ is equal to 0. Thus, $|\langle v_1 - v_2,\, v_1 - v_4\rangle| < \|v_1 - v_2\|\,\|v_1 - v_4\|$. By Theorem 3.3.6 above, the subspace spanned by these vectors is space-like. Thus, the Minkowski inner product restricted to this plane is positive definite.

For convenience, translate this subspace so that p lies at the origin, and relabel the vertices of the rectangle u_1 through u_4, as shown in

Figure 8. Note that these vertices are no longer vectors in the light-cone. In fact, these vectors are all space-like and thus have positive length. Also, $-u_1 = u_3$, and $-u_2 = u_4$.

By assumption,

$$0 = \langle u_1 - u_4, u_2 - u_1 \rangle = \langle u_1 + u_2, u_2 - u_1 \rangle = -||u_1||^2 + ||u_2||^2$$

Thus, $||u_1|| = ||u_2||$, implying that the vector $u_1 + u_2$ bisects $u_2 - u_1$, and that $u_2 + u_3 = u_2 - u_1$ bisects $u_2 + u_1$. Assembling this yields

$$-u_1 + \frac{1}{2}(u_1 + u_2) = \frac{1}{2}(u_2 - u_1) = p_{23}$$

and

$$u_1 + \frac{1}{2}(u_2 - u_1) = \frac{1}{2}(u_1 + u_2) = p_{12}.$$

So,

$$\langle p_{12}, p_{23} \rangle = \left\langle \frac{1}{2}(u_2 - u_1), \frac{1}{2}(u_1 + u_2) \right\rangle = \frac{1}{4}\langle (u_1 + u_2), (u_2 - u_1) \rangle = 0.$$

Thus, λ_α and λ_β meet at right angles in this plane. It remains to be shown that the axes of the geodesics meet at a right angle in hyperbolic space to conclude the proof of Theorem 3.3.7.

When we form the convex hull, we are free to choose the height along the ray representing infinity at which we fix the cusp. Since there is a highest vertex of the rectangle (the vertex with the largest t coordinate), we can choose the cusp to be at a height low enough so that the rectangle does not intersect the hyperboloid.

By continuously varying the height of the cusp, there will be a point at which p intersects the hyperboloid. If this intersection is transverse, the resulting intersection between the plane containing the rectangle and the hyperboloid yields an ellipse. This ellipse must be invariant under the hyper-elliptic involution $\tilde{\tau}$. If we now apply the lift of τ which fixes p (we know this exists, as $\tilde{\tau}$ must preserve the axes of both g_α and g_β, and p is their intersection), we have an ellipse which is invariant under rotation by π through a point which is not its center (p is a point of intersection between the plane and the hyperboloid, so is a point on the

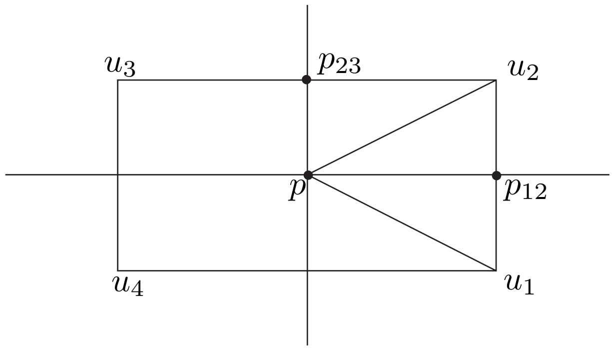

FIGURE 8. Rectangle with p at origin and vertices relabeled.

ellipse), a contradiction. In other words the point at which p intersects this hyperboloid cannot yield a transverse intersection between the plane and the hyperboloid. Thus, when the hull is at a height where p intersects the hyperboloid, the plane containing the rectangle is tangent to the hyperboloid.

The point of tangency for the plane containing the rectangle with the hyperboloid is the point at which the axes of the geodesics meet. We showed above that these axes meet at right angles in this plane. Since this model is conformal, the axes meet at right angles in hyperbolic space. □

4. Further results. In this section we examine some interesting algebraic results which can be derived using the fact that certain punctured-torus groups have convex hulls tiled by rectangles.

4.1 Commensurability results. We follow [**5**] for the definitions of commensurability.

Definition. Fuchsian or Kleinian groups Γ_1 and Γ_2 are *commensurable* if Γ_1 has a subgroup of finite index which is conjugate to a subgroup of finite index in Γ_2. Commensurability can also be defined for hyperbolic manifolds and orbifolds. In this case, we say that two hyperbolic manifolds or orbifolds M_1 and M_2 are *commensurable* if they have a common finite cover.

Many of the results in this section rely on an important theorem of Margulis, details of which can be found in [**5**, Chapter 10.3].

Theorem 4.1.1. (Margulis). *There is a unique minimal element in the commensurability class of a nonarithmetic, finite co-volume group.*

As Long and Reid discuss in [**4**, Theorem 2.2], it follows from [**2, 9**] that a noncocompact Fuchsian group Γ of finite co-area is arithmetic if the traces of γ^2 are rational integers for a generating set $\gamma \in \Gamma$ with γ not of order two. In the case of $\Delta(u^2, 2\tau)$ groups, one need only check $\mathrm{tr}\,(g_1 \cdot g_1)$, $\mathrm{tr}\,(g_2 \cdot g_2)$ and $\mathrm{tr}\,(g_1 \cdot g_2 \cdot g_1 \cdot g_2)$, since they are free groups of rank two.

The minimal element of the commensurability class corresponds to a group containing all elements of the commensurability class up to conjugacy. If we think in terms of quotient groups and we call M the minimal element in the commensurability class of two groups G_1 and G_2, then $\mathbf{H}^2/M$ is covered by both $\mathbf{H}^2/G_1$ and $\mathbf{H}^2/G_2$.

We will now prove the following theorem describing the minimal element of the commensurability class of a nonarithmetic $\Delta(u^2, 4)$ punctured-torus. Recall that a cusp may be defined as a ray stabilized by a parabolic.

Theorem 4.1.2. *Let* $\Gamma = \Delta(u^2, 4)$ *be nonarithmetic. The minimal element in the commensurability class of* Γ *is* $\mathbf{H}^2/\langle\Gamma, \tau\rangle$, *where* τ *is the hyper-elliptic involution of* $\mathbf{H}^2/\Gamma$.

Let Ω be the minimal element of Γ. To prove Theorem 4.1.2, we proceed with several lemmata describing the convex hull of Ω as well as a discussion of the hyper-elliptic involution of a once-punctured torus. We will also need Margulis's theorem 4.1.1 discussed earlier.

Lemma 4.1.3. *Let* Γ *be a finite co-volume group such that* $\mathbf{H}^2/\Gamma$ *has one cusp, and let* $\Gamma < \varpi$ *be a finite extension. After a choice of height,* ϖ *and* Γ *have the same convex hull.*

Proof. We will begin by showing that Γ and ϖ have the same set of lifts of cusps. First observe that ϖ is discrete, as a finite extension of a discrete group is discrete. Now note that since $\mathbf{H}^2/\Gamma$ has one cusp and covers $\mathbf{H}^2/\varpi$, it follows that $\mathbf{H}^2/\varpi$ also has one cusp. Any light-like ray stabilized by a parabolic element of Γ is obviously stabilized by a parabolic element of ϖ, namely the same parabolic. Conversely, if $\omega \in \Omega$ stabilizes a light-like ray, all powers of ω must also stabilize the same ray. Further, since Γ is of finite index in ϖ, ω^n must lie in Γ for some n. Since there is only one cusp, there is only one orbit in the light cone for both groups. That is, both groups have the same set of (lifts of) cusps.

To complete the proof of the lemma, it remains to be shown that the heights on the rays of the orbit of the cusp are the same for both groups, that is, the volumes of the horoballs are the same for both groups. Assume that the lifts of the cusps occur at different heights. Now fix a point v_∞ on a light-like ray, and let $\gamma \in \Gamma$ and $\omega \in \varpi$ be such that $\gamma(v_\infty) \neq \omega(v_\infty)$, that is, $\gamma(v_\infty)$ and $\omega(v_\infty)$ lie on the same light-like ray, but at different heights. Now consider $\gamma^{-1}(\omega(v_\infty))$. It lies on the same ray as v_∞, but at a different height. This contradicts the fact that the stabilizer of v_∞ is parabolic, as parabolic elements stabilize light-like rays pointwise, proving the lemma. □

Since the minimal element Ω is a finite extension of Γ, we know from Lemma 4.13 that Γ and Ω have the same hull. Call this common hull P, and note that we also know that the hull corresponding to Ω has boundary tesselated by rectangles.

We showed in Lemma 3.3.1 that the only instance in which a $\Delta(u^2, 4)$ group has a convex hull tiled by squares, as opposed to nonsquare rectangles, occurs precisely when $u^2 = 1/2$. Since we assumed Γ was nonarithmetic and $\Delta(1/2, 4)$ is an arithmetic group, we know that P is tesselated by nonsquare rectangles. We now continue with an analysis of the hyper-elliptic involution of a once-punctured torus.

Fix $\Gamma = \Delta(u^2, 4)$. After forming the convex hull of Γ, we arrive at a canonical decomposition of $\mathbf{H}^2$ into rectangles which descends into a canonical geodesic rectangular domain for $\mathbf{H}^2/\Gamma$. First note that the hyper-elliptic involution preserves the convex hull. This follows directly from Lemma 4.1.3, as Γ is a subgroup of $\langle \Gamma, \tau \rangle$ of finite index. The hyper-elliptic involution acts on the once-punctured torus with three

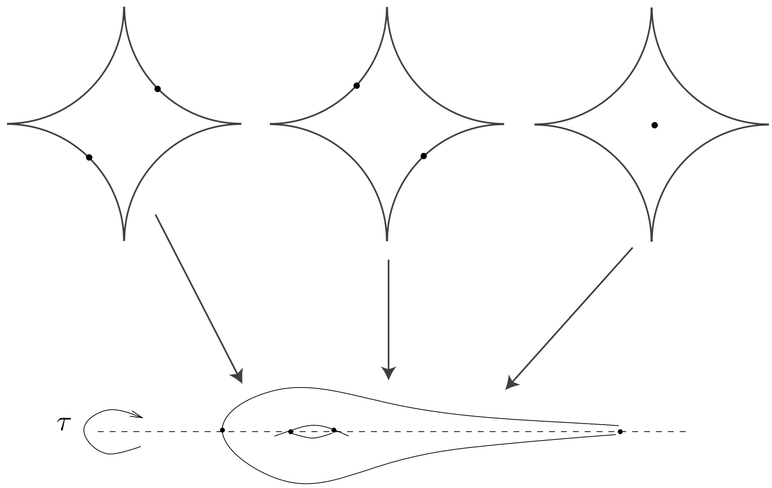

FIGURE 9. Illustration of three lifts of τ.

fixed points. The quotient orbifold is a 2-sphere with three cone points of order two. We will ascertain where on the rectangular domain these points can lie.

First, only one of the points can lie in the interior of the rectangle, as the hyper-elliptic involution rotates the rectangle by π radians. Note also that this interior fixed point must lie in the barycenter of the domain. The other two fixed points must necessarily lie on the geodesic boundary of the ideal rectangle. The rotation through π in a neighborhood of either of the noninterior fixed points of the rectangle lifts to a rotation about the center of one of the edges of the ideal rectangle in the universal cover. Thus, we can lift the hyper-elliptic involution in one of three ways, each corresponding to one of the three categories of fixed points described above. Two of the lifts correspond to rotation of π about an edge of the ideal rectangle, and the third lift corresponds to a rotation of π about the center of a face. These are illustrated in Figure 9.

We know that Ω preserves the convex hull. We now analyze what elliptic elements could preserve P, and prove the following proposition.

Proposition 4.1.4. *With* Γ*,* τ *and* P *as above, the only object* $\mathbf{H}^2/\Gamma$ *can cover is* $\mathbf{H}^2/\langle\Gamma,\tau\rangle$*.*

Proof. Let $\mathbf{H}^2/\varpi$ be covered by $\mathbf{H}^2/\Gamma$. Since ϖ is a finite extension of Γ, Lemma 4.1.3 ensures us that Γ and ϖ generate the same convex hull. Any orientation preserving element of finite order must stabilize a cone point, and any lift of an elliptic element fixes a time-like vector which meets P in exactly one point. The following three cases exhaust the possible choices for hull-preserving elliptics in ϖ. Note that ϖ can't be torsion-free due to Euler characteristic considerations.

Case I. The time-like vector meets P at a vertex. This, however, is impossible as the vertices of P lie in the light-cone, and thus their stabilizers are parabolic.

Case II. The time-like vector meets P in an edge. This implies the elliptic is an order two rotation through π about a point which fixes an edge setwise.

Case III. The time-like vector meets P in a face. As remarked earlier, we know the faces of the hull are non-square rectangles, as square rectangles only occur in an arithmetic group. This implies we must have an order two rotation about a point in the center of a face.

Thus, if $\omega \in \varpi$ is an element which fixes the relevant point, then its action on P agrees with the hyper-elliptic involution in a neighborhood of each of the fixed points. Since we have two linear maps which coincide on an open set, they are the same map, i.e., $\varpi = \langle\Gamma,\tau\rangle$.

We have shown that, in the nonarithmetic case, the only object the $\mathbf{H}^2/\Gamma$ punctured torus can cover is $\mathbf{H}^2/\langle\Gamma,\tau\rangle$, proving the proposition. □

We now appeal to Theorem 4.1.1 for the existence of the unique minimal element in the commensurability class of a nonarithmetic group and complete the proof of Theorem 4.1.2, *The minimal element in the commensurability class of* Γ *is* $\mathbf{H}^2/\langle\Gamma,\tau\rangle$*, where* τ *is the hyper-elliptic involution of* $\mathbf{H}^2/\Gamma$*.*

The orbifold $\mathbf{H}^2/\langle\Gamma,\tau\rangle$ must cover the unique minimal element guaranteed by Theorem 4.1.1. Further, by Proposition 4.1.4 above, it can't

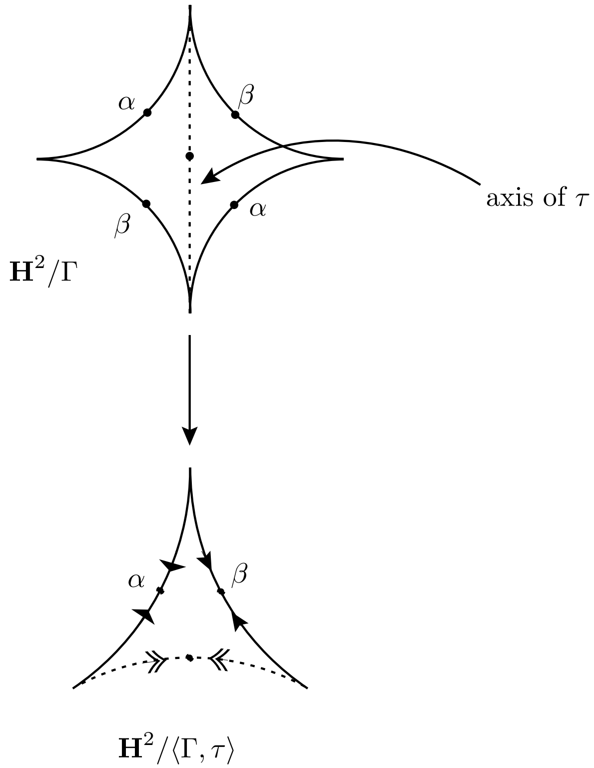

FIGURE 10. Punctured torus covering the orbifold obtained from the quotient of the hyper-elliptic involution.

cover anything else, thus $\mathbf{H}^2/\langle\Gamma,\tau\rangle$ is the minimal element of Γ, and Theorem 4.1.2 is proved. □

In addition to having information about the minimal element of the commensurability class of the flat-face groups, we also can show that there are infinitely many commensurability classes in the $\Delta(u^2, 4)$ groups. To prove this we appeal to the cross-ratio, defined as follows:

Definition. If $x > y > z$ are distinct ordered points of $\partial\mathbf{H}^2$ (represented by the unit circle) the *cross ratio* of ∞, x, y, z is one of $\{x - z/y - z, z - x/x - y\}$.

Theorem 4.1.5. *Let $0 < u^2 < 1/2$. No two nonarithmetic groups in $\Delta(u^2, 4)$ are commensurable.*

Proof. Let G_1 and G_2 be commensurable groups in $\Delta(u^2, 4)$. Fix a vector for infinity in the Minkowski model of $\mathbf{H}^2$ and form the convex hulls for G_1 and G_2. By Lemma 3.2.1 above, this gives rise to two canonical ideal hyperbolic rectangles representing the fundamental domains of $\mathbf{H}^2/G_i$, $i = 1, 2$.

Now consider the group formed via taking the quotient by the hyper-elliptic involution. If τ_i is the relevant hyper-elliptic involution, this new group is $\Omega_i = \langle \Gamma_i, \tau_i \rangle \cong \mathbf{Z}_2 \star \mathbf{Z}_2 \star \mathbf{Z}_2$, and $\mathbf{H}^2/\Omega_i$ is a two-sphere with three cone points of order 2. Since Γ_1 is commensurable with Γ_2, Theorem 4.1.2 implies that Ω_1 and Ω_2 are in fact conjugate.

Claim 4.1.6. *The punctured torus groups Γ_1 and Γ_2 are conjugate.*

Let $\gamma \in \mathrm{PSL}\,(2, \mathbf{R})$ be an element that conjugates Ω_1 to Ω_2. There are seven double covers of Ω_1 (and thus of Ω_2). Of these seven, only one is torsion free. Since Γ_1 is a torsion free index two subgroup of Ω_1, γ must conjugate it to a torsion free index two subgroup of Ω_2, and since there is only one such subgroup, namely Γ_2, we know that Γ_1 and Γ_2 are conjugate, proving the claim.

Claim 4.1.7. *Let P_1 and P_2 be the convex hulls corresponding to Γ_1 and Γ_2 respectively. If γ conjugates Γ_1 to Γ_2 as described above, then γ conjugates P_1 to P_2, that is, $\gamma \cdot P_1 = P_2$.*

First note that $\gamma\Gamma_1\gamma^{-1} = \Gamma_2$ and that $\Gamma_2 \cdot P_2 = P_2$. Now, $\Gamma_2 \cdot (\gamma \cdot P_1) = \gamma\Gamma_1\gamma^{-1} \cdot (\gamma \cdot P_1) = \gamma \cdot P_1$. So $\gamma \cdot P_1$ remains invariant under Γ_2. Thus, $\gamma \cdot P_1 = P_2$, proving the claim.

To complete the proof of Theorem 4.1.5, we use the fact that conjugate hulls imply conjugate faces. Because we know that the hulls in question have rectangular faces, we can examine the rectangles in the Poincaré disk model and appeal to the cross ratio.

Since γ conjugates P_1 to P_2, we know that γ carries faces of P_1 to faces of P_2. In the Poincaré disk model of $\mathbf{H}^2$, γ maps the rectangular domain which descends from P_1 to that of P_2. Rectangles in $\mathbf{H}^2$ are conjugate if and only if they have the same cross ratio. We may assume that the domain for Γ_1 is the canonical domain, R_1, described in Lemma 3.2.1, namely, the ideal rectangle with vertices $\infty, 1, u^2$ and 0, i.e., $\Delta(u^2, 4)$. This rectangle has cross ratio in the set $\{1/u^2, 1/1-u^2\}$. If R_2 is the domain given to us from the convex hull for Γ_2, it must also have cross-ratio in this set. However, this implies that R_2 is the ideal rectangle composed of vertices $\infty, 1, 1-u^2$ and 0, i.e. $\Delta(1-u^2, 4)$. As noted in subsection 3.2, this implies that Γ_1 and Γ_2 are equivalent groups after renaming, proving that G_1 and G_2 are commensurable. □

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