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REGULARITY OF POINTS IN THE SPECTRUM OF A C^*-ALGEBRA

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ABSTRACT. The relationship between different notions of regularity for the points in the spectrum of a C^*-algebra is investigated. Under certain conditions on the points of the spectrum, the Fell regularity implies Glimm regularity and vice versa. A localized version of the Fell-Dixmier theorem on continuous trace of a C^*-algebra is described.

1. Introduction. Let A be a C^*-algebra, and let $\widehat{A}$ be the spectrum of A, the space of all (equivalence classes of) irreducible representations of A. In [**4**, 4.5.3(iii)] and [**5**, Remark to Theorem 6] two notions of regularity of points in the spectrum $\widehat{A}$ are described. A point $\pi \in \widehat{A}$ is said to be Fell-regular (or a Fell-point) if there exists an $a \in A^+$ (the set of positive elements of A) and a neighborhood V of π such that $\sigma(a)$ is a rank-one projection for all $\sigma \in V$. On the other hand, a point $\pi \in \widehat{A}$ is said to be Glimm-regular if, whenever (e, U) is a pair such that $e \in A$ and U is a neighborhood of π and (i) $\sigma(e)$ is a projection for all $\sigma \in U$, (ii) $\pi(e)$ is a rank-one projection, then there exists a neighborhood U_0 of π with $U_0 \subseteq U$ such that $\sigma(e)$ is rank-one for all $\sigma \in U_0$. A point π is said to be a separated point of $\widehat{A}$ if for each $\sigma \in \widehat{A} \setminus \overline{\{\pi\}}$ there exist disjoint open sets U_1 and U_2 such that $\pi \in U_1$ and $\sigma \in U_2$.

It is known [**5, 6**] that the notions agree if A is liminal with $\widehat{A}$ Hausdorff. We investigate the relation between these notions for more general C^*-algebras. Of course, if $\pi \in \widehat{A}$ is Fell-regular, then, since $\pi(A)$ contains nonzero elements of the algebra of compact operators $K(H_\pi)$, we have $\pi(A) \supseteq K(H_\pi)$ [**4**, 4.1.10]. On the other hand, if $\pi(A) \cap K(H_\pi) = \{0\}$, then, although π cannot be Fell-regular, it is automatically Glimm-regular (by vacuous satisfaction). However, we prove that, if $\pi \in \widehat{A}$ is a separated point, then π is Fell-regular if and only if $\pi(A) \supseteq K(H_\pi)$ and π is Glimm-regular. We give examples to show that if π is not a separated point then (even if $\pi(A)$ contains the

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compacts) neither kind of regularity implies the other. We also give an example to show that π can be both Fell regular and Glimm regular without being a separated point.

From [**4**, 4.5.3, 4.5.4] we know that if A is a C^*-algebra with continuous trace, then A is liminal, $\widehat{A}$ is Hausdorff and every point of $\widehat{A}$ is Fell-regular; conversely, A is a C^*-algebra with continuous trace if $\widehat{A}$ is Hausdorff and every point of $\widehat{A}$ is Fell-regular. These latter conditions are known as the "Fell-Dixmier conditions for continuous trace of the C^*-algebra A." In the final section we will prove a localized version of the Fell-Dixmier theorem.

The symbols $B(H)$ and $K(H)$ denote, respectively, the C^*-algebras of bounded linear and compact linear operators acting on a Hilbert space H with adjoint as involution and operator norm. A C^*-algebra A is said to be liminal if $\pi(A) = K(H_\pi)$ for every $\pi \in \widehat{A}$, where H_π is the Hilbert space for π. For the following definitions we refer the reader to [**1**]. If φ and ψ are pure states of a C^*-algebra A and p, q are their respective support projections in A^{**}, then the transition probability between φ and ψ is denoted by $\langle\varphi, \psi\rangle$ and is defined by $\langle\varphi, \psi\rangle = \varphi(q) = \psi(p)$. If φ and ψ are unitarily equivalent, there will be an irreducible representation $\pi : A \to B(H)$ and unit vectors $\xi, \eta \in H$ such that for every $a \in A$ we have $\varphi(a) = \langle\pi(a)\xi, \xi\rangle$, and $\psi(a) = \langle\pi(a)\eta, \eta\rangle$. Hence, the transition probability between φ and ψ is given by $\langle\varphi, \psi\rangle = |\langle\xi, \eta\rangle|^2$. If φ and ψ are inequivalent (that is, their respective GNS irreducible representations are not unitarily equivalent), then $\langle\varphi, \psi\rangle = 0$. In Proposition 2.1 we shall use the subset $R(A)$ of $P(A) \times P(A)$ which is defined by $R(A) = \{(\varphi, \psi) : \varphi$ and ψ are unitarily equivalent pure states of $A\}$.

2. The Fell and Glimm regular points. We give a short proof of the following result using a continuity property [**2**] for transition probabilities for pure states of the C^*-algebra A.

Proposition 2.1. *Let A be C^*-algebra with spectrum $\widehat{A}$. Let $\pi \in \widehat{A}$. Suppose π is Fell-regular. Then π a separated point of $\widehat{A}$ implies π is Glimm regular.*

Proof. Suppose π is a separated point of $\widehat{A}$. Suppose π is not Glimm-regular. Then, there exist an $e \in A$, a neighborhood U of π in $\widehat{A}$, and a net (π_α) in U convergent to π such that (i) $\pi(e)$ is a rank-one projection, (ii) $\sigma(e)$ is a projection for all $\sigma \in U$, (iii) rank $(\pi_\alpha(e)) \geq 2$. By (iii) choose, for each α, orthogonal unit vectors $\xi_\alpha, \eta_\alpha \in \pi_\alpha(e)H_{\pi_\alpha}$, and define $\varphi_\alpha = \langle \pi_\alpha(\cdot)\xi_\alpha, \xi_\alpha \rangle$, $\psi_\alpha = \langle \pi_\alpha(\cdot)\eta_\alpha, \eta_\alpha \rangle$, Clearly, $\varphi_\alpha, \psi_\alpha \in P(A)$, the set of pure states of A with relative w^*-topology. Choose a unit vector $\xi \in \pi(e)H_\pi$, and define $\varphi = \langle \pi(\cdot)\xi, \xi \rangle \in P(A)$. We will show that $\varphi_\alpha, \psi_\alpha \to \varphi$. Let $a \in A$. Since $\pi(e)$ is a one-dimensional projection, $\pi(eae) = \varphi(a)\pi(e)$. Since π is a separated point of $\widehat{A}$, the map $\sigma \to \|\sigma(a)\|$ is continuous at π for all $a \in A$ [**4**, 3.9.4(a)]. Therefore, $|\pi_\alpha(eae - \varphi(a)e)| \leq \|\pi_\alpha(eae - \varphi(a)e)\| = 0$. Now, since $|\varphi_\alpha(a) - \varphi(a)| = |\varphi_\alpha(eae - \varphi(a)e)| \leq \|\pi_\alpha(eae - \varphi(a)e)\| \to 0$, we get $\varphi_\alpha \to \varphi$. Similarly, $\psi_\alpha \to \varphi$. Since π is Fell-regular by [**2**, Theorem 4.2 ((ii) $\to$ (i))] the transition probability map $\langle\ ,\ \rangle$: $R(A) \to [0,1]$, given by $(f,g) \mapsto \langle f, g \rangle$ is continuous at (φ, φ). Therefore, $(\varphi_\alpha, \psi_\alpha) \to (\varphi, \varphi)$ implies $\langle \varphi_\alpha, \psi_\alpha \rangle \to \langle \varphi, \varphi \rangle$. But $\langle \varphi_\alpha, \psi_\alpha \rangle = |\langle \xi_\alpha, \eta_\alpha \rangle|^2 = 0$ (since $\xi_\alpha \perp \eta_\alpha$), whereas $\langle \varphi, \varphi \rangle = |\langle \xi, \xi \rangle|^2 = 1$, a contradiction to the continuity of the map $\langle\ ,\ \rangle$: $R(A) \to [0,1]$ at (φ, φ). Thus, π is Glimm-regular. □

An alternative, but more lengthy, proof of Proposition 2.1 can be given by developing the methods of [**5**] in a more general setting.

Proposition 2.2. *Let A be a C^*-algebra with spectrum $\widehat{A}$. Suppose π is a separated point and a Glimm-regular point of $\widehat{A}$ such that $\pi(A) \supseteq K(H_\pi)$. Then π is Fell-regular.*

Proof. Let E be a rank-one projection in $K(H_\pi)$. Then, since $\pi(A) \supseteq K(H_\pi)$, there exists an $a \in A$ such that $\pi(a) = E$. Let $b = a^*a \geq 0$. Then $\pi(b) = E$ and since $\|\pi(b)\| \leq \|b\|$ so, in particular, $\|b\| \geq 1$. Therefore, $\mathrm{Sp}\,(b) \cap [1, \infty) \neq \varnothing$, where $\mathrm{Sp}\,(b)$ is the spectrum of b. Define $g : [0, \infty) \to R$ by

$$g(t) = \begin{cases} t & \text{if } t \in [0,1] \\ 1 & \text{if } t \in (1, \infty); \end{cases}$$

then g is continuous on $\mathrm{Sp}\,(b)$. Define $c = g(b) \geq 0$. Then $\pi(c) = E$ and $\|c\| = 1$. Since π is a separated point of $\widehat{A}$, the mapping

$\sigma \to \|\sigma(x)\|$ is continuous at π for each $x \in A$ [**4**, 3.9.4 (a)]. Also, $\|\pi(c^2 - c)\| = 0$. Hence, there exists a neighborhood U of π in $\widehat{A}$ such that $\|\sigma(c^2 - c)\| < 3/16$ for all $\sigma \in U$. It follows from this that for $\sigma \in U$, $\mathrm{Sp}\,(\sigma(c)) \subseteq [0, 1/4] \cup [3/4, 1]$. Define $f : [0, 1] \to [0, 1]$ by

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, 1/4] \\ 2t - 1/2 & \text{if } t \in [1/4, 3/4] \\ 1 & \text{if } t \in [3/4, 1]. \end{cases}$$

Then f is continuous. Define $e = f(c)$. If $\sigma \in U$, then by definition of f, $\mathrm{Sp}\,(\sigma(e)) \subseteq \{0, 1\}$. Since $\sigma(e) \geq 0$, it follows that $\sigma(e)$ is a projection for all $\sigma \in U$, and $\pi(e) = \pi(f(c)) = E$. But, by assumption, π is Glimm-regular; therefore, there exists a neighborhood V of π with $V \subseteq U$ such that $\sigma(e)$ is one-dimensional for all $\sigma \in V$. So π is Fell-regular. □

By combining Proposition 2.1 and Proposition 2.2 we obtain the following Theorem.

Theorem 2.1. *Let A be a C^*-algebra with spectrum $\widehat{A}$. Let $\pi \in \widehat{A}$ be a separated point of $\widehat{A}$. Then the following are equivalent*: (1) *π is Fell-regular*; (2) *$\pi(A) \supseteq K(H_\pi)$ and π is Glimm-regular.*

It follows immediately from Theorem 2.1 that if A is liminal and $\widehat{A}$ is Hausdorff, then the notions of Fell regularity and Glimm regularity coincide for elements of $\widehat{A}$ (see [**5**, page 60] and [**6**, page 74]).

Remark 2.1. If $\pi \in \widehat{A}$ and $\pi(A) \not\supseteq K(H_\pi)$, then $\pi(A) \cap K(H_\pi) = \{0\}$ [**4**, 4.1.10] and so π is Glimm-regular (by vacuous satisfaction) but not Fell-regular. However, even if we assume $\pi(A) \supseteq K(H_\pi)$, then neither kind of regularity implies the other in the absence of the hypothesis that π is a separated point. In Kaplansky's example [**4**, 4.7.19], the one-dimensional representation λ (and also μ) is a nonseparated Fell-point that is not Glimm-regular (consider the pair (e, U) where $e = 1$ and $U = \widehat{A}$). On the other hand, in the example in [**3**, page 443], the C^*-algebra A has a one-dimensional representation λ with the property that if $g \in A$ and $\lambda(g) = 1$ then there is no neighborhood V of λ in $\widehat{A}$ such that $\sigma(g)$ is a projection for all $\sigma \in V$. Thus, λ is Glimm-regular (vacuously) and not Fell-regular (in fact the upper multiplicity $M_U(\lambda)$ is 2). By Theorem 2.1, λ cannot be a separated point of $\widehat{A}$. Indeed, the construction in [**3**, page 443] shows that for each $t \in (0,1)$ there is a one-dimensional representation λ_t which cannot be separated from λ by disjoint open sets.

We now give an example to show that π can be both Fell-regular and Glimm-regular without being a separated point.

Example 2.1. Let $X = I \times I$ where $I = [0,1]$. Define $A = \{f \in C(X, M_2)$: $f(0,0) = \operatorname{diag}(\lambda(f), 0), f(0,1) = \operatorname{diag}(\lambda(f), \mu(f))$, $f(0,t) = \operatorname{diag}(\lambda(f), \mu_t(f))$ for all $t \in (0,1)\}$, where $\lambda(f), \mu(f), \mu_t(f) \in$ **C**.

With pointwise operations and sup-norm A is a C^*-subalgebra of $C(X, M_2)$. Let $(x,y) \in X$. Define $\pi_{(x,y)} : A \to M_2(C)$ by $\pi_{(x,y)}(f) = f(x,y)$. One can check that

$$\widehat{A} = \big\{\pi_{(x,y)} : 0 < x \le 1, 0 \le y \le 1\big\} \cup \{\mu_t : 0 < t < 1\} \cup \{\lambda, \mu\}.$$

We show that λ is Fell-regular. Define $h : X \to M_2(\mathbf{C})$ by

$$h(x,y) = \operatorname{diag}(1,0) \quad \text{for all } (x,y) \in X.$$

Let $J = \overline{AhA}$. Then, clearly, J is a norm-closed two sided ideal of A. Let

$$V = \widehat{A} \setminus (\{\mu_t : t \in (0,1)\} \cup \{\mu\}) = \{\sigma \in \widehat{A} : \sigma(J) \neq \{0\}\}.$$

Then V is an open neighborhood of λ in $\widehat{A}$. Since $\sigma(h) = \operatorname{diag}(1,0)$ for all $\sigma \in V \setminus \{\lambda\}$ and $\lambda(h) = 1$, therefore λ is a Fell-point of $\widehat{A}$.

We show that λ is Glimm-regular. For each $\varepsilon > 0$,

$$V_\varepsilon = \{\lambda\} \cup \{\pi_{(x,y)} : 0 < x < \varepsilon, 0 \le y \le 1\}$$

is an open neighborhood of λ, corresponding to the closed two-sided ideal of A consisting of all functions f which vanish on $[\varepsilon, 1] \times [0,1]$ and satisfy $\mu(f) = \mu_t(f) = 0$ for all $t \in (0,1)$. Suppose that (e, U) is a pair such that $e \in A$, U is a neighborhood of λ in $\widehat{A}$, $\sigma(e)$ is a projection for all $\sigma \in U$ and $\lambda(e) = 1$. An elementary compactness argument shows that there exists $\varepsilon > 0$ such that $V_\varepsilon \subseteq U$. The function

$$(x, y) \longrightarrow \operatorname{tr}(e(x, y))$$

is a continuous, integer-valued function on the connected set $\{(0,0)\} \cup ((0,\varepsilon) \times [0,1])$ and takes the value 1 at $(0,0)$. It follows that $\pi_{(x,y)}(e)$ has rank one for all $\pi_{(x,y)} \in V_\varepsilon$. Thus, λ is Glimm-regular.

Finally, we show that λ is not a separated point of $\widehat{A}$. We will construct a net in $\widehat{A}$ and show that it converges to both λ and μ. Let V be some open neighborhood of λ. Therefore, there exist a closed two sided ideal K of A such that $\widehat{K} = V$. Since $\lambda(K) \neq \{0\}$ there exists an $f \in K$ such that $\lambda(f) \neq 0$. Therefore, $f(0,1) = \operatorname{diag}(\lambda(f), \mu(f)) \neq \underline{0}$. Now, since f is continuous, therefore $f(x, 1-x) \to f(0,1)$ as $x \to 0^+$; that is, $\pi_{(x,1-x)}(f) \to \operatorname{diag}(\lambda(f), \mu(f))$ as $x \to 0^+$. So there exist a $\delta > 0$ such that $\pi_{(x,1-x)}(f) \neq 0$, for all $x \in (0,\delta)$, that is, $\pi_{(x,1-x)} \in V$, for all $x \in (0,\delta)$ and, therefore, $\pi_{(x,1-x)} \to \mu$ as $x \to 0^+$. Following the above lines with λ replaced by μ we will get $\pi_{(x,1-x)} \to \mu$ as $x \to 0^+$; thus, we get $\pi_{(x,1-x)} \to \lambda, \mu$ as $x \to 0^+$. Since $\ker \not\subseteq \ker \mu$, λ is not a separated point of $\widehat{A}$. This completes our example.

3. Fell-Dixmier conditions for continuous trace of a C^*-algebra. The following result is a local version of the Fell-Dixmier theorem (see the introduction). The proof uses some classical methods from [**4**] and also a more recent lower semi-continuity result from [**2**].

Theorem 3.1. *Let A be a C^*-algebra with spectrum $\widehat{A}$. Let $\pi \in \widehat{A}$. Then the following are equivalent*: (1) *π is a Fell-point, and π is a separated point*; (2) *there exists a two sided ideal J of A such that* (i) $\ker \pi$ *is strictly contained in $\overline{J}$ (norm closure of J)*, (ii) *for each $a \in J^+$, $\operatorname{tr} \pi(a) < \infty$ and the map $\sigma \mapsto \operatorname{tr} \sigma(a)$ is continuous at π.*

Proof. ((1) $\Rightarrow$ (2)). Define $S = \{a \in A^+ : \sigma \mapsto \operatorname{tr} \sigma(a)$ is finite and continuous at $\pi\}$. Clearly, $S + S \subseteq S$ and if $x \in A$ satisfies $xx^* \in S$, then, since $\operatorname{tr} \sigma(xx^*) = \operatorname{tr} \sigma(x^*x)$, for all $\sigma \in \widehat{A}$, $x^*x \in S$. Let $x \in S$ and $y \in A^+$ be such that $y \leq x$. Note that $\operatorname{tr} \pi(y) \leq \operatorname{tr} \pi(x) < \infty$. Let (π_α) be a net in $\widehat{A}$ such that $\pi_\alpha \to \pi$. Then, by [**4**, 3.5.9], $\lim_\alpha \inf \operatorname{tr} \pi_\alpha(y) \geq$

$\operatorname{tr}\pi(y) \geq 0$, and $\lim_\alpha \inf \operatorname{tr}\pi_\alpha(x-y) \geq \operatorname{tr}\pi(x) - \operatorname{tr}\pi(y) \geq 0$. Or

$$-\lim_\alpha \sup(-\operatorname{tr}\pi_\alpha(x) + \operatorname{tr}\pi_\alpha(y)) \geq \operatorname{tr}\pi(x) - \operatorname{tr}\pi(y).$$

Since $\sigma \mapsto \operatorname{tr}\pi(x)$ is continuous at π, we get

$$-(-\operatorname{tr}\pi(x)) - \lim_\alpha \sup(\operatorname{tr}\pi_\alpha(y)) \geq \operatorname{tr}\pi(x) - \operatorname{tr}\pi(y).$$

Since $\operatorname{tr}\pi(x) < \infty$, we can cancel it out on both sides and get

$$-\lim_\alpha \sup(\operatorname{tr}\pi_\alpha(y)) \geq -\operatorname{tr}\pi(y), \quad \text{or} \quad 0 \leq \lim_\alpha \sup(\operatorname{tr}\pi_\alpha(y)) \leq \operatorname{tr}\pi(y).$$

Thus, $\lim_\alpha \operatorname{tr}\pi_\alpha(y) = \operatorname{tr}\pi(y) < \infty$, and hence $y \in S$. Let $J = \lim(S)$. Then by [**4**, 4.5.1 (c) (ii)], J is a two sided ideal of A such that $J^+ = S$.

We show first that $\ker\pi$ is contained in $\overline{J}$. Let $\sigma \in \widehat{A}$ and suppose $\ker\pi$ is not contained in $\ker\sigma$. It is enough to show that $\overline{J}$ is not contained in $\ker\sigma$. Since π is a separated point of $\widehat{A}$, there exist disjoint open sets V_1, V_2 in $\widehat{A}$ such that $\pi \in V_1$, $\sigma \in V_2$. There exists a closed two-sided ideal K of A such that $\widehat{K} = V_2$. Since $\sigma(K) \neq \{0\}$, there exists $k \in K^+$ such that $\sigma(k) \neq 0$, whereas $\theta(k) = 0$ for all $\theta \in V_1$. Now, since, $\operatorname{tr}\theta(k) = 0$ for all $\theta \in V_1$, the map $\theta \mapsto \operatorname{tr}\theta(k)$ is finite and continuous at π. Thus, $k \in S \subseteq \overline{J}$. Since $\sigma(k) \neq 0$, $k \notin \ker\sigma$ and hence $\overline{J}$ is not contained in $\ker\sigma$. Thus, $\ker\pi$ is not contained in $\ker\sigma$ implies that $\overline{J}$ is not contained in $\ker\sigma$. Or, equivalently, $\overline{J} \subseteq \ker\sigma \to \ker\pi \subseteq \ker\sigma$. This shows that $\ker\pi \subseteq \overline{J}$.

Secondly, we show that $\ker\pi \neq \overline{J}$, that is, $\overline{J}$ strictly contains $\ker\pi$. Since π is a Fell-point, there exist an $e \in A^+$ and an open neighborhood V of π in $\widehat{A}$ such that $\sigma(e)$ is a rank-1 projection for all $\sigma \in V$. Now as $\operatorname{tr}\sigma(e) = 1$ for all $\sigma \in V$, so the map $\sigma \mapsto \operatorname{tr}\sigma(e)$ is finite and continuous at π. Therefore, $e \in S \subset J \subseteq \overline{J}$. Since $\pi(e) \neq 0$, $e \notin \ker\pi$ and therefore $\ker\pi \neq \overline{J}$. Thus, $\ker\pi \subset \overline{J}$.

((2) $\Rightarrow$ (1)). Since $\ker\pi \subset \overline{J}$, there exists $a \in J^+$ such that $\pi(a) \neq 0$. Therefore by [**4**, 4.4.2 (ii)], π is a Fell-point. Now suppose that $\pi_0 \in \widehat{A} \setminus \{\pi\}$ and that (π_α) is a net in $\widehat{A}$ such that $\pi_\alpha \to \pi, \pi_0$. Let $x \in J^+$. Then, by [**2**, Theorem 2.4], $\lim_\alpha \inf \operatorname{tr}\pi_\alpha(x) \geq \operatorname{tr}\pi_0(x) + \operatorname{tr}\pi(x)$. But the map $\sigma \mapsto \operatorname{tr}\sigma(x)$ is finite and continuous at π; therefore, $\lim_\alpha \inf \operatorname{tr}\pi_\alpha(x) = \lim_\alpha \operatorname{tr}\pi_\alpha(x) = \operatorname{tr}\pi(x)$. Hence,

$\operatorname{tr}\pi(x) \geq \operatorname{tr}\pi_0(x) + \operatorname{tr}\pi(x)$. Since $\operatorname{tr}\pi(x)$ is finite and $x \geq 0$, we obtain $\pi_0(x) = 0$. Since $x \in J^+$ is arbitrary, we get $\pi_0(J^+) = \{0\}$. By linearity and continuity of π_0, we get, respectively, $\pi_0(J) = \{0\}$ and $\pi_0(\overline{J}) = \{0\}$. Thus, $\overline{J} \subseteq \ker\pi_0$ and hence $\ker\pi \subset \ker\pi_0$. That is, $\pi_0 \in \overline{\{\pi\}}$. This shows that π is a separated point of $\widehat{A}$. □

Remark 3.1. If π satisfies the equivalent conditions of Theorem 3.1 and if $\ker\pi$ is a maximal closed two-sided ideal of A, then, of course, J is dense in A. However, $\ker\pi$ need not be maximal. Indeed, let $A = B(H)$ for an infinite dimensional Hilbert space H, and let π be the identity representation. Since $\{\pi\}$ is open and dense in $\widehat{A}$, π is a Fell-point and a separated point. The construction in the proof of Theorem 3.1 leads to J being the ideal of trace-class operators.

Corollary 3.1. *Let A be a C^*-algebra with spectrum $\widehat{A}$. Let $\pi \in \widehat{A}$, and let $\pi(A) = K(H_\pi)$. Then the following are equivalent*: (1) *π is a Fell point, and π is a separated point,* (2) *there exists a dense two-sided ideal J of A such that, for each $a \in J^+$, $\operatorname{tr}\pi(a) < \infty$ and the map $\sigma \mapsto \operatorname{tr}\sigma(a)$ is continuous at π.*

Proof. ((1) $\Rightarrow$ (2)). Suppose π is a Fell-point and a separated point of $\widehat{A}$. Then, by Theorem 3.1, there exist a two-sided ideal J of A such that $\ker\pi \subset \overline{J}$ and, for each $a \in J^+$, the map $\sigma \mapsto \operatorname{tr}\sigma(a)$ is finite and continuous at π. But $\pi(A) = K(H_\pi)$ implies that $\ker\pi$ is a maximal ideal of A, therefore $\overline{J} = A$.

((2) $\Rightarrow$ (1)). Since $\overline{J} = A$, $\ker\pi \subset \overline{J}$. Therefore, by Theorem 3.1, π is a Fell-point and a separated point of $\widehat{A}$. □

In Example 3.8 of [**2**] A is a C^*-algebra for which every $\pi \in \widehat{A}$ is a Fell point yet both the separated points and the nonseparated points form dense subsets in $\widehat{A}$. In the following corollary, we show how the Fell-Dixmier theorem can be obtained from our local version.

Corollary 3.2. *Let A be a C^*-algebra with spectrum $\widehat{A}$. Then the following are equivalent*: (1) *$\widehat{A}$ is Hausdorff and every point of $\widehat{A}$ is Fell-regular,* (2) *A is a C^*-algebra with continuous trace.*

Proof. ((2) $\Rightarrow$ (1)). By (2), A is liminal. By Corollary 3.1, every point $\pi \in \widehat{A}$ is Fell-regular and a separated point. Since A is liminal, it follows from [**4**, 4.2.5] that $\widehat{A}$ is a T_1 space, and hence $\widehat{A}$ is Hausdorff (since every $\pi \in \widehat{A}$ is a separated point).

((1) $\Rightarrow$ (2)). Let $\pi \in \widehat{A}$. Since π is a Fell-point, $\pi(A) \supseteq K(H_\pi)$. But $\widehat{A}$ is Hausdorff and so $\pi(A) = K(H_\pi)$. By Corollary 3.1, there is a dense two-sided ideal J_π (associated with π) of A such that for each $a \in J_\pi^+$, $\operatorname{tr}\pi(a) < \infty$ and the map $\sigma \mapsto \operatorname{tr}(a)$ is continuous at π. Let J be the Pedersen ideal of A. Then $J \subseteq J_\pi$. Thus, for each $a \in J^+$, $\operatorname{tr}\pi(a) < \infty$ and the map $\sigma \mapsto \operatorname{tr}\sigma(a)$ is continuous at π. Since $\pi \in \widehat{A}$ is arbitrary, the map $\sigma \mapsto \operatorname{tr}\sigma(a)$ is continuous on $\widehat{A}$, and hence A has continuous trace. □

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