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OSCILLATION OF NONLINEAR IMPULSIVE HYPERBOLIC EQUATION WITH SEVERAL DELAYS

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ABSTRACT. In this paper, oscillatory properties of solutions for certain nonlinear impulsive hyperbolic equations with several delays are investigated and a series of new sufficient conditions and a necessary and sufficient condition for oscillation of the solutions are established.

1. Introduction. The theory of delay partial differential equations can be applied to many fields, such as to biology, population growth, engineering, generic repression, control theory and climate model. In the last few years, the fundamental theory of partial differential equations with deviating argument has undergone intensive development. The qualitative theory of this class of equations, however, is still in an initial stage of development. A few papers have been published on oscillation theory of partial differential equations with delay. We may easily visualize situations in nature where abrupt change such as shock and disasters may occur. These phenomena are short-time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is, in the form of impulses. In 1991, the first paper [**8**] on this class of equations was published. But, for instance, only a few papers have been published on oscillation theory of impulsive partial differential equations. Recently, Bainov, Minchev, Fu, Luo and Liu [**2–5, 9, 18–21**] investigated the oscillation of solutions of impulsive partial differential equations with or without deviating argument.

In this paper we'll discuss the oscillatory properties of solutions for a class of nonlinear impulsive hyperbolic equation with several delays (1),

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under the boundary conditions (4) and (5) separately. Up to now, we do not find work for oscillations of this kind of problem. The problem that paper [**20**] discussed is a special case of Theorem 1 here.

(1)
$$\frac{\partial^2 u}{\partial t^2} = a(t)h(u)\Delta u - q(t,x)f(u(t,x)) - \sum_{j=1}^{n} g_j(t,x)f_j(u(t-\sigma_j,x))$$
$$t \neq t_k, \quad (t,x) \in R_+ \times \Omega = G$$

(2) $$u(t_k^+,x) - u(t_k^-,x) = q_k u(t_k,x),$$

(3) $$u_t(t_k^+,x) - u_t(t_k^-,x) = b_k u_t(t_k,x), \quad t = t_k, \quad k = 1,2,\dots .$$

with the boundary conditions

(4) $$\frac{\partial u}{\partial n} = g(t,x,u), \quad (t,x) \in R_+ \times \partial\Omega$$

(5) $$u = 0, \quad (t,x) \in R_+ \times \partial\Omega$$

and the initial condition $u(t,x) = \Phi(t,x)$, $(t,x) \in [-\delta,0] \times \Omega$. Here $\Omega \subset R^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and n is a unit exterior normal vector of $\partial\Omega$, $\delta = \max\{\sigma_j\}$, $\Phi(t,x) \in C^2([-\delta,0] \times \Omega, R)$.

This article is organized as follows: Section 2 studies the oscillatory properties of solutions for problems (1) and (4). Section 3 discusses problems (1) and (5). In Section 4, we obtain for the linear case a necessary and sufficient condition.

Assume that the following conditions are fulfilled:

H_1) $a(t) \in PC(R_+,R_+)$, $\sigma_j = \text{const} > 0$, $q(t,x), g_j(t,x) \in C(R_+ \times \overline{\Omega}, (0,\infty))$, $j = 1,2,\dots,n$; where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$ and left continuous at $t = t_k$, $k = 1,2,\dots$.

H_2) $h'(u), f(u), f_j(u) \in C(R,R)$; $f(u)/u \geq C = \text{const} > 0$, $f_j(u)/u \geq C_j = \text{const} > 0$, for $u \neq 0$; $uh'(u) \geq 0$,$g(t,x,u)$ is continuous and $uh(u)g(t,x,u) < 0$, $q_k > -1, b_k > -1, b_k < q_k$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{t\to\infty} t_k = \infty$.

H_3) $u(t,x)$ and their derivatives $u_t(t,x)$ are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$ and left continuous at $t = t_k$, $u(t_k,x) = u(t_k^-,x)$, $u_t(t_k,x) = u_t(t_k^-,x)$, $k = 1,2,\dots$.

Let us construct the sequence $\{\bar{t}_k\} = \{t_k\} \cup \{t_{k\sigma_j}\}$, where $t_{k\sigma_j} = t_k + \sigma_j$ and $\bar{t}_k < \bar{t}_{k+1}$, $k = 1, 2, \dots$.

Definition 1. By a solution of problem (1), (4) ((1),(5)) with initial condition, we mean that any function $u(t,x)$ for which the following conditions are valid:

1. If $-\delta \le t \le 0$, then $u(t,x) = \Phi(t,x)$.

2. If $0 \le t \le \bar{t}_1 = t_1$, then $u(t,x)$ coincides with the solution of the problem (1), (2), (3) and (4) ((5)) with initial condition.

3. If $\bar{t}_k < t \le \bar{t}_{k+1}, \bar{t}_k \in \{t_k\} \setminus \{t_{k\sigma_j}\}$, then $u(t,x)$ coincides with the solution of the problem (1), (2), (3) and (4) ((5)).

4. If $\bar{t}_k < t \le \bar{t}_{k+1}, \bar{t}_k \in \{t_{k\sigma_j}\}$, then $u(t,x)$ coincides with the solution of the problem (4) ((5)) and the following equations

$$\frac{\partial^2 u}{\partial t^2} = a(t^+)h(u(t^+,x))\Delta u(t^+,x) - q(t,x)f(u(t^+,x)) - \sum_{j=1}^{n} g_j(t,x)f_j(u((t-\sigma_j)^+,x)), \quad (t,x) \in R_+ \times \Omega = G$$

$$u(\bar{t}_k^+,x) = u(\bar{t}_k,x), \quad u_t(\bar{t}_k^+,x) = u_t(\bar{t}_k,x), \quad \text{for } \bar{t}_k \in \{t_{k\sigma_j}\} \setminus \{t_k\},$$

or

$$u(\bar{t}_k^+,x) = (1+q_{k_i})u(\bar{t}_k,x), \quad u_t(\bar{t}_k^+,x) = (1+b_{k_i})u_t(\bar{t}_k,x), \quad \text{for } \bar{t}_k \in \{t_{k\sigma_j}\} \bigcap \{t_k\}.$$

Here the number k_i is determined by the equality $\bar{t}_k = t_{k_i}$.

We introduce the notations: $\Gamma_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \Omega\}$, $\Gamma = \cup_{k=0}^{\infty}\Gamma_k$, $\overline{\Gamma}_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \overline{\Omega}\}$, $\overline{\Gamma} = \cup_{k=0}^{\infty}\overline{\Gamma}_k$, $v(t) = \int_\Omega u(t,x)\,dx$ and $p(t) = \min q(t,x)$, $p_j(t) = \min g_j(t,x)$, $x \in \overline{\Omega}$.

Definition 2. The solution $u \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ of problem (1), (4) ((1),(5)) is called nonoscillatory in the domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

2. Oscillation properties of the problem (1), (4). The following is the main theorem of this paper, and the proof needs the following lemmas.

Lemma 1. *Let $u \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem* (1), (4) *in G. Then the function $v(t)$ satisfies the following impulsive differential inequality*

$$v''(t) + Cp(t)v(t) + \sum_{j=1}^{n} C_j p_j(t) v(t-\sigma_j) \le 0, \quad t \ne t_k \tag{6}$$

$$v(t_k^+) = (1+q_k)v(t_k), \quad k = 1, 2, \dots \tag{7}$$

$$v'(t_k^+) = (1+b_k)v'(t_k) \quad k = 1, 2, \dots . \tag{8}$$

Proof. Let $u(t,x)$ be a positive solution of the problem (1), (4) in G. Without loss of generality, we may assume that $u(t,x) > 0$, $u(t-\sigma_j, x) > 0$, $j = 1, 2, \dots, n$, for any $(t,x) \in [t_0, \infty) \times \Omega$.

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \dots$, integrating (1) with respect to x over Ω yields

$$\begin{aligned}\frac{d^2}{dt^2}\left[\int_\Omega u\,dx\right] &= a(t)\int_\Omega h(u)\Delta u\,dx - \int_\Omega q(t,x) f(u(t,x))\,dx \\ &\quad - \sum_{j=1}^{n}\int_\Omega g_j(t,x) f_j(u(t-\sigma_j,x)).\end{aligned} \tag{9}$$

By Green's formula and the boundary condition, we have

$$\begin{aligned}\int_\Omega h(u)\Delta u\,dx &= \int_{\partial\Omega} h(u)\frac{\partial u}{\partial n}\,ds - \int_\Omega h'(u)|\mathrm{grad}\,u|^2\,dx \\ &\le -\int_\Omega h'(u)|\mathrm{grad}\,u|^2\,dx \le 0.\end{aligned}$$

From condition H_2), we can easily obtain

$$\int_\Omega q(t,x) f(u(t,x))\,dx \ge Cp(t)\int_\Omega u(t,x)\,dx,$$

$$\int_\Omega g_j(t,x) f_j(u(t-\sigma_j,x))\,dx \ge C_j p_j(t)\int_\Omega u(t-\sigma_j,x)\,dx.$$

It follows that

$$v'' + Cp(t)v(t) + \sum_{j=1}^{n} C_j p_j(t) v(t-\sigma_j) \le 0, \quad t \ge t_0, \quad t \ne t_k \tag{10}$$

where $v(t) > 0$.

For $t > t_0$, $t = t_k$, $k = 1, 2, \ldots$, we have

$$\int_\Omega u(t_k^+, x)\, dx - \int_\Omega u(t_k^-, x)\, dx = q_k \int_\Omega u(t_k, x)\, dx,$$
$$\int_\Omega u_t(t_k^+, x)\, dx - \int_\Omega u_t(t_k^-, x)\, dx = b_k \int_\Omega u_t(t_k, x)\, dx.$$

This implies

$$v(t_k^+) = (1 + q_k) v(t_k) \tag{11}$$
$$v'(t_k^+) = (1 + b_k) v'(t_k), \quad k = 1, 2, \ldots . \tag{12}$$

Hence we obtain that $v(t) > 0$ is a positive solution of differential inequality (6)–(8). This ends the proof of the lemma. □

Definition 3. The solution $v(t)$ of differential inequality (6)–(8) is called eventually positive (negative) if there exists a number t^* such that $v(t) > 0$ ($v(t) < 0$) for $t \ge t^*$.

Lemma 2 [**11**, Theorem 1.4.1]. *Assume that*

(i) $m(t) \in PC^1[R^+, R]$ *is left continuous at* t_k *for* $k = 1, 2, \ldots$,

(ii) *for* $k = 1, 2, \ldots, t \ge t_0$,

$$m'(t) \le p(t) m(t) + q(t), \quad t \ne t_k$$
$$m(t_k^+) \le d_k m(t_k) + e_k,$$

where $p(t), q(t) \in C(R^+, R), d_k \ge 0$ *and* e_k *are real constants,* $PC^1[R^+, R] = \{x : R^+ \to R; x(t)$ *is continuous and continuously differentiable everywhere except some* t_k *at which* $x(t_k^+), x(t_k^-)$, $x'(t_k^+)$ *and* $x'(t_k^-)$ *exist and* $x(t_k) = x(t_k^-)$, $x'(t_k) = x'(t_k^-)\}$.

Then

$$\begin{aligned} m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)\,ds\right) \\ + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(r)\,dr\right) q(s)\,ds \\ + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)\,ds\right) e_k. \end{aligned}$$

From Lemma 2 we can obtain Lemma 3. See also [**20**].

Lemma 3. *Let $v(t)$ be an eventually positive (negative) solution of differential inequality (6)–(8). Assume that there exists $T \ge t_0$ such that $v(t) > 0 (v(t) < 0)$ for $t \ge T$. If the following condition holds,*

$$\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1+b_k}{1+q_k}\,ds = +\infty, \tag{13}$$

then $v'(t) \ge 0$ $(v'(t) \le 0)$ for $t \in [T, t_l] \cup (\cup_{k=l}^{+\infty}(t_k, t_{k+1}])$, where $l = \min\{k : t_k \ge T\}$.

Theorem 1. *Let condition (13) and the following condition (14) hold for some j_0,*

$$\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1+q_k}{1+b_k} p_{j_0}(s)\,ds = +\infty. \tag{14}$$

Then every solution of the problem (1), (4) oscillates in G.

Proof. Let $u(t,x)$ be a nonoscillatory solution of (1), (4). Without loss of generality, we can assume that $u(t,x) > 0$, $u(t-\sigma_j, x) > 0$, $j = 1, 2, \ldots, n$, for any $(t,x) \in [t_0, \infty) \times \Omega$. From Lemma 1, we know that $v(t)$ is a positive solution of (6)–(8). Thus, from Lemma 3, we can find that $v'(t) \ge 0$ for $t \ge t_0$.

For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \ldots$, define

$$w(t) = \frac{v'(t)}{v(t)}, \quad t \geq t_0.$$

Then we have $w(t) > 0$, $t \geq t_0$, $v'(t) - w(t)v(t) = 0$. We may assume that $v(t_0) = 1$, thus we have that for $t \geq t_0$

$$v(t) = \exp\bigg(\int_{t_0}^{t} w(s)\,ds\bigg) \tag{15}$$

$$v'(t) = w(t)\exp\bigg(\int_{t_0}^{t} w(s)\,ds\bigg) \tag{16}$$

$$v''(t) = w^2(t)\exp\bigg(\int_{t_0}^{t} w(s)\,ds\bigg) + w'(t)\exp\bigg(\int_{t_0}^{t} w(s)\,ds\bigg). \tag{17}$$

We substitute (15)–(17) into (6) and can obtain the following inequality

$$w^2(t)\exp\bigg(\int_{t_0}^{t} w(s)\,ds\bigg) + w'(t)\exp\bigg(\int_{t_0}^{t} w(s)\,ds\bigg) + C_{j_0}p_{j_0}(t)\exp\bigg(\int_{t_0}^{t-\sigma_{j_0}} w(s)\,ds\bigg) \leq 0.$$

Hence, we have

$$w^2(t) + w'(t) + C_{j_0}p_{j_0}(t)\exp\bigg(-\int_{t-\sigma_{j_0}}^{t} w(s)\,ds\bigg) \leq 0.$$

From this inequality and condition $b_k < q_k$, it is easy to see that the function $w(t)$ is nonincreasing for $t \geq t_1 \geq \delta + t_0$. Thus, $w(t) \leq w(t_1)$ for $t \geq t_1$ which implies that

$$w'(t) + C_{j_0}p_{j_0}(t)\exp(-\delta w(t_1)\,ds) \leq 0, \quad t \geq t_1.$$

From (7), (8) we can obtain

$$w(t_k^+) = \frac{v'(t_k^+)}{v(t_k^+)} = \frac{1 + b_k}{1 + q_k} w(t_k), \quad k = 1, 2, \ldots .$$

It follows that

$$w'(t) \le -C_{j_0} p_{j_0}(t) \exp(-\delta w(t_1)\, ds), \quad t \ne t_k \tag{18}$$

$$w(t_k^+) = \frac{1+b_k}{1+q_k} w(t_k), \quad t = t_k. \tag{19}$$

By using Lemma 2, we obtain

$$\begin{aligned} w(t) &\le w(t_0) \prod_{t_0<t_k<t} \frac{1+b_k}{1+q_k} \\ &\quad + \int_{t_0}^{t} \prod_{s<t_k<t} \frac{1+b_k}{1+q_k} (-C_{j_0} p_{j_0}(s) \exp(-\delta w(t_1)))\, ds \\ &= \prod_{t_0<t_k<t} \frac{1+b_k}{1+q_k} \{ w(t_0) \\ &\quad - \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1+q_k}{1+b_k} C_{j_0} p_{j_0}(s) \exp(-\delta w(t_1))\, ds \}. \end{aligned}$$

Since $w(t) > 0$, the last inequality contradicts (14). The proof of Theorem 1 is completed. □

It should be noted that obviously every solution of problem (1), (4) is oscillatory if there exists a subsequence n_k of n such that $q_{n_k} < -1$, for $k = 1, 2, \dots$. So we only discuss the case of $q_k > -1$.

3. Oscillation properties of the problem (1), (5). Making use of the following lemma of eigenvalue, we can obtain many similar results for problem (1), (5). In this section, we suppose that $h(u) = 1$.

Lemma 4. *Suppose that λ_0 is the smallest eigenvalue of the problem*

$$\begin{aligned} \Delta\varphi(x) + \lambda\varphi(x) = 0, &\quad x \in \Omega \\ \varphi(x) = 0, &\quad x \in \partial\Omega \end{aligned}$$

and $\varphi(x)$ is the corresponding eigenfunction of λ_0. Then $\lambda_0 > 0, \varphi(x) > 0$, $x \in \Omega$.

Lemma 5. *Let $u(t,x) \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem* (1), (5) *in G. Then the function $v(t) = \int_\Omega u(t,x)\varphi(x)\,dx$ satisfies the impulsive differential inequality*

$$v''(t) + \lambda_0 a(t)v(t) + Cp(t)v(t) + \sum_{j=1}^{n} C_j p_j(t) v(t-\sigma_j) \le 0, \quad t \ne t_k \tag{20}$$

$$v(t_k^+) = (1+q_k)v(t_k) \quad k = 1,2,\dots. \tag{21}$$

$$v'(t_k^+) = (1+b_k)v'(t_k), \quad k = 1,2,\dots. \tag{22}$$

Proof. Let $u(t,x)$ be a positive solution of the problem (1), (5) in G. Without loss of generality, we may assume that $u(t,x) > 0$, $u(t-\sigma_j, x) > 0$, $j = 1,2,\dots,n$, for any $(t,x) \in [t_0,\infty) \times \Omega$.

For $t \ge t_0$, $t \ne t_k$, $k = 1,2,\dots$, multiplying equation (1) with $\varphi(x)$, which is the same as that in Lemma 4 and then integrating (1) with respect to x over Ω yields

$$\frac{d^2}{dt^2}\left[\int_\Omega u\varphi(x)\,dx\right] = a(t)\int_\Omega \Delta u \varphi(x)\,dx - \int_\Omega q(t,x)f(u(t,x))\varphi(x)\,dx - \sum_{j=1}^{n}\int_\Omega g_j(t,x)f_j(u(t-\sigma_j,x))\varphi(x)\,dx.$$

By Green's formula and the boundary condition we have

$$\int_\Omega u\Delta\varphi dx - \int_\Omega \varphi\Delta u\,dx = \int_{\partial\Omega}\frac{\partial\varphi}{\partial n}u\,ds - \int_{\partial\Omega}\frac{\partial u}{\partial n}\varphi\,ds = 0.$$

It follows that

$$\int_\Omega \Delta u(t,x)\varphi(x)\,dx = \int_\Omega \Delta\varphi(x)u(t,x)\,dx = -\lambda_0\int_\Omega \varphi(x)u(t,x)\,dx.$$

From the condition H_2), we can easily obtain

$$\int_\Omega q(t,x)f(u(t,x))\varphi(x)\,dx \ge Cp(t)\int_\Omega u(t,x)\varphi(x)\,dx$$

$$\int_\Omega g_j(t,x)f_j(u(t-\sigma_j,x))\varphi(x)\,dx \ge C_j p_j(t)\int_\Omega u(t-\sigma_j,x)\varphi(x)\,dx.$$

Thus, $v(t) > 0$. Hence, we obtain the following differential inequality

$$v'' + \lambda_0 a(t)v(t) + Cp(t)v(t) + \sum_{j=1}^{n} C_j p_j(t)v(t-\sigma_j) \le 0. \tag{23}$$

For $t = t_k$, we have

$$\int_\Omega u(t_k^+, x)\varphi(x)\,dx - \int_\Omega u(t_k^-, x)\varphi(x)\,dx = q_k \int_\Omega u(t_k, x)\varphi(x)\,dx,$$
$$\int_\Omega u_t(t_k^+, x)\varphi(x)\,dx - \int_\Omega u_t(t_k^-, x)\varphi(x)\,dx = b_k \int_\Omega u_t(t_k, x)\varphi(x)\,dx.$$

This implies

$$v(t_k^+) = (1+q_k)v(t_k), \quad k = 1, 2, \dots \tag{24}$$
$$v'(t_k^+) = (1+b_k)v'(t_k), \quad k = 1, 2, \dots . \tag{25}$$

Hence, we obtain that $v(t) > 0$ is a positive solution of impulsive differential inequality (20)–(22). This ends the proof of Lemma 5. □

Theorem 2. *Let condition* (13) *and the following condition hold*

$$\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1+q_k}{1+b_k}(\lambda_0 a(s) + Cp(s))\,ds = +\infty. \tag{26}$$

Then every solution of the problem (1), (5) *oscillates in* G.

Proof. Let $u(t,x)$ be a nonoscillatory solution of (1), (5). Without loss of generality, we can assume that $u(t,x) > 0$, $u(t-\sigma_j, x) > 0$, $j = 1, 2, \dots, n$, for any $(t,x) \in [t_0, \infty) \times \Omega$. From Lemma 5, we know that $v(t)$ is a positive solution of (20)–(22). Thus, from Lemma 3, we can find that $v'(t) \ge 0$ for $t \ge t_0$.

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \dots$, define

$$w(t) = \frac{v'(t)}{v(t)}, \quad t \ge t_0.$$

Then we have $w(t) > 0$, $t \geq t_0$, $v'(t) - w(t)v(t) = 0$. We may assume that $v(t_0) = 1$, thus we have that for $t \geq t_0$

$$v(t) = \exp\left(\int_{t_0}^{t} w(s)\,ds\right) \tag{27}$$

$$v'(t) = w(t)\exp\left(\int_{t_0}^{t} w(s)\,ds\right) \tag{28}$$

$$v''(t) = w^2(t)\exp\left(\int_{t_0}^{t} w(s)\,ds\right) + w'(t)\exp\left(\int_{t_0}^{t} w(s)\,ds\right). \tag{29}$$

We substitute (27) and (29) into (20) and can obtain the following inequality

$$w^2(t) + w'(t) + \lambda_0 a(t) + Cp(t) \leq 0.$$

From (21), (22) we get

$$w(t_k^+) = \frac{v'(t_k^+)}{v(t_k^+)} = \frac{1+b_k}{1+q_k} w(t_k), \quad k = 1, 2, \dots .$$

It follows that

$$w'(t) \leq -\lambda_0 a(t) - Cp(t), \quad t \neq t_k \tag{30}$$

$$w(t_k^+) = \frac{1+b_k}{1+q_k} w(t_k), \quad t = t_k. \tag{31}$$

By using Lemma 2, we obtain

$$\begin{aligned} w(t) \leq w(t_0) &\prod_{t_0<t_k<t} \frac{1+b_k}{1+q_k} \\ &+ \int_{t_0}^{t} \prod_{s<t_k<t} \frac{1+b_k}{1+q_k}(-\lambda_0 a(s) - Cp(s))\,ds \\ = &\prod_{t_0<t_k<t} \frac{1+b_k}{1+q_k}\Big\{w(t_0) \\ &\qquad - \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1+q_k}{1+b_k}(\lambda_0 a(s) + Cp(t))\,ds\Big\}. \end{aligned}$$

Since $w(t) > 0$, the last inequality contradicts (26). The proof of theorem is completed. □

Theorem 3. *Let condition* (13) *and the following condition hold for some* p_{j_0},

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1+q_k}{1+b_k}C_{j_0}p_{j_0}(s)\,ds=+\infty. \tag{32}$$

Then every solution of the problem (1), (5) *oscillates in* G.

The proof is easy, we just omit it. □

4. Necessary and sufficient condition. In this section, we will establish a necessary and sufficient condition for oscillation of impulsive wave equation with several delays. We consider the following linear problem.

$$u_{tt}=a(t)\Delta u-p(t)u(t,x)-\sum_{j=1}^{n}p_j(t)u(t-\sigma_j,x) \tag{33}$$
$$t\neq t_k,\quad (t,x)\in R_+\times\Omega=G$$
$$u(t_k^+)-u(t_k^-)=q_ku(t_k,x),\quad t=t_k,\quad k=1,2,\dots \tag{34}$$
$$u_t(t_k^+)-u_t(t_k^-)=b_ku_t(t_k,x),\quad t=t_k,\quad k=1,2,\dots, \tag{35}$$

with boundary condition (5).

Theorem 4. *Every solution of the problem* (33)–(35), (5) *is oscillatory in domain* G *if and only if every solution of the following impulsive delay differential equation* (36)–(38) *is oscillatory.*

$$\frac{d^2v(t)}{dt^2}+a(t)\lambda_0v(t)+p(t)v(t)+\sum_{j=1}^{n}p_j(t)v(t-\sigma_j)=0, \tag{36}$$

$$v(t_k^+)-v(t_k^-)=q_kv(t_k),\quad k=1,2,\dots. \tag{37}$$
$$v'(t_k^+)-v'(t_k^-)=b_kv'(t_k),\quad k=1,2,\dots. \tag{38}$$

Proof. Sufficiency. Using reduction to absurdity. Let $u(t,x)$ be a nonoscillatory solution of the problem (33)–(35), (5). Without loss of generality, we may assume that there exists a $t_0 \geq T$ such that $u(t,x) > 0$ and $u(t-\sigma_j, x) > 0$, $j = 1, \ldots, n$, for any $(t,x) \in [t_0, +\infty) \times \Omega$.

For $t \geq t_0, t \neq t_k$, $k = 1, 2, \ldots$, multiplying equation (33) with $\varphi(x)$, which is the same as that in Lemma 4, then integrating (33) with respect to x over Ω yields

$$\begin{aligned}\frac{d^2}{dt^2}\int_\Omega u(t,x)\varphi(x)\,dx = a(t)&\int_\Omega \Delta u(t,x)\varphi(x)\,dx\\ &- p(t)\int_\Omega u(t,x)\varphi(x)\,dx\\ &- \sum_{j=1}^n p_j(t)\int_\Omega u(t-\sigma_j,x)\varphi(x)\,dx.\end{aligned}$$

By Green's formula and boundary condition, we have

$$\int_\Omega u\Delta\varphi(x)\,dx - \int_\Omega \varphi(x)\Delta u\,dx = \int_{\partial\Omega} u\frac{\partial\varphi(x)}{\partial n}\,ds - \int_{\partial\Omega}\varphi(x)\frac{\partial u}{\partial n}\,ds = 0.$$

It follows that

$$\int_\Omega \varphi(x)\Delta u\,dx = \int_\Omega u\Delta\varphi(x)\,dx = -\lambda_0\int_\Omega \varphi(x)u(t,x)\,dx.$$

Denote $v(t) = \int_\Omega \varphi(x)u(t,x)\,dx$. Then $v(t) > 0$. It follows that

$$\frac{d^2v(t)}{dt^2} + a(t)\lambda_0 v(t) + p(t)v(t) + \sum_{j=1}^n p_j(t)v(t-\sigma_j) = 0.$$

For $t \geq t_0$, $t = t_k$, $k = 1, 2, \ldots$, analogous to (11), (12) we have

$$v(t_k^+) - v(t_k^-) = q_k v(t_k) \tag{39}$$

$$v'(t_k^+) - v'(t_k^-) = b_k v'(t_k), \quad k = 1, 2, \ldots. \tag{40}$$

Hence, we obtain that $v(t) > 0$ satisfies equation (36)–(38). This means that impulsive delay differential equation (36)–(38) has a nonoscillatory solution. A contradiction. This ends the proof of sufficient condition.

Necessity. We are still using reduction to absurdity. Let $v(t)$ be a nonoscillatory solution of the equation (36)–(38). Without loss of generality, we may assume that there exists a t_1 large enough such that $v(t) > 0$ and $v(t-\sigma_j) > 0$, $j = 1, \dots, n$, for any $t \in [t_1, +\infty)$.

For $t \geq t_1$, $t \neq t_k$, $k = 1, 2, \dots$, setting $u(t,x) = v(t)\varphi(x)$, we have $u(t,x) > 0$ and we can easily obtain

$$\Delta u(t,x) = \Delta[v(t)\varphi(x)] = v(t)\Delta\varphi(x) = -\lambda_0 v(t)\varphi(x).$$

Making use of this result, we use equation (36). We obtain

$$\frac{d^2}{dt^2}[v(t)\varphi(x)] + a(t)\lambda_0 v(t)\varphi(x) + p(t)v(t)\varphi(x) + \sum_{j=1}^{n} p_j(t)v(t-\sigma_j)\varphi(x) = 0.$$

This means that $u(t,x) = v(t)\varphi(x)$ satisfies equation (33).

For $t \geq t_1$, $t = t_k$, $k = 1, 2, \dots$, from the conditions (37) and (38), it is easy to see that function $u(t,x) = v(t)\varphi(x)$ satisfies (34), (35). And, because $\varphi(x) = 0$, $x \in \partial\Omega$. That is, $u(t,x) = v(t)\varphi(x)$ also satisfies boundary condition (5). This indicates that problem (33)–(35), (5)has a nonoscillatory solution. This is a contradiction. This ends the proof of Theorem 4. □

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