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CROSSED PRODUCTS OF LOCALLY C^*-ALGEBRAS

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ABSTRACT. The crossed products of locally C^*-algebras are defined and a Takai duality theorem for inverse limit actions of a locally compact group on a locally C^*-algebra is proved.

1. Introduction. Locally C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology on a locally C^*-algebra is defined by a directed family of C^*-seminorms. In [**9**], Phillips defines the notion of action of a locally compact group G on a locally C^*-algebra A whose topology is determined by a countable family of C^*-seminorms, and also defines the crossed product of A by an inverse limit action

$$\alpha = \lim_{\substack{\leftarrow \\ n}} \alpha^{(n)}$$

as being the inverse limit of crossed products of A_n by $\alpha^{(n)}$. In this paper, by analogy with the case of C^*-algebras, we define the concept of crossed product, respectively reduced crossed product of locally C^*-algebras.

The Takai duality theorem says that if α is a continuous action of an Abelian locally compact group G on a C^*-algebra A, then we can recover the system (G, A, α) up to stable isomorphism from the double dual system in which $G = \widehat{\widehat{G}}$ acts on the crossed product $(A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}$ by the dual action of the dual group. In [**3**], Imai and Takai prove a duality theorem for C^*-crossed products by a locally compact group that generalizes the Takai duality theorem [**12**]. For a given C^*-dynamical system (G, A, α), they construct a "dual" C^*-crossed product of the reduced crossed product $A \times_{\alpha,r} G$ by an isomorphism

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β from $A \times_{\alpha,r} G$ into $L(H)$, the C^*-algebra of all bounded linear operators on some Hilbert space H and show that this is isomorphic to the tensor product $A \otimes \mathcal{K}(L^2(G))$ of A and $\mathcal{K}(L^2(G))$, the C^*-algebra of all compact operators on $L^2(G)$. If G is commutative, the "dual" C^*-crossed product constructed by Imai and Takai is isomorphic to the double crossed product $(A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}$. Katayama [**6**] shows that a nondegenerate coaction β of a locally compact group on a C^*-algebra A induces an action $\widehat{\beta}$ of G on the crossed product $A \times_\beta G$ and proves that the C^*-algebras $(A \times_\beta G) \times_{\widehat{\beta},r} G$ and $A \otimes \mathcal{K}(L^2(G))$ are isomorphic. In [**13**], Vallin shows that there is a bijective correspondence between the set of all actions of a locally compact group G on a C^*-algebra A and the set of all actions of the commutative Kac C^*-algebra $C^*\mathbf{K}_G^a$ associated with G on A. A coaction of G on A is an action of the symmetric Kac C^*-algebra $C^*\mathbf{K}_G^s$ associated with G. If G is commutative, we can identified $C_r^*(G)$ with $C_0(\widehat{G})$ via the Fourier transform, whence it becomes clear that a coaction of G is the same thing as an action of $\widehat{G}$. Thus, we can regard the coactions of a locally compact group G as "actions of the dual group even there isn't any dual group." Also, Vallin shows that an action α (coaction β) of G on A induces a coaction $\widehat{\alpha}$ (action $\widehat{\beta}$) of G on the crossed product $A \times_{\alpha,r} G$ (respectively $A \times_\beta G$) and proves a version of the Takai duality theorem showing that the double crossed product $(A \times_{\alpha,r} G) \times_{\widehat{\alpha}} G$ is isomorphic to $A \otimes \mathcal{K}(L^2(G))$. We propose to prove a version of the Takai duality theorem for crossed products of locally C^*-algebras.

The paper is organized follows. In Section 2 we present some basic definitions and results about locally C^*-algebras and Kac C^*-algebras. In Section 3 we define the notion of crossed product (reduced crossed product) of a locally C^*-algebra A by an inverse limit action α of a locally compact group G and prove some basic properties of these. Section 4 is devoted to actions of a Kac C^*-algebra on a locally C^*-algebra. We show that there is a bijective correspondence between the set of all inverse limit actions of a locally compact group G on a locally C^*-algebra A and the set of all inverse limit actions of the commutative Kac C^*-algebra $C^*\mathbf{K}_G^a$ on A, Proposition 4.4. As a consequence of this result, we obtain: for a compact group G, any action of the Kac C^*-algebra $C^*\mathbf{K}_G^a$ on A is an inverse limit of actions of the Kac C^*-algebras $C^*\mathbf{K}_G^a$ on A_p, $p \in S(A)$. In Section 5, using the same arguments as in [**13**], we show that any inverse limit action α (coaction β) of a locally compact group G on a locally C^*-algebra A induces an inverse limit coaction $\widehat{\alpha}$ (action $\widehat{\beta}$) of G on the crossed product $A \times_{\alpha,r} G$ (respectively $A \times_{\beta} G$), Proposition 5.5. Finally, we prove that if α is an inverse limit action of a locally compact group G on a locally C^*-algebra A, then there is an isomorphism of locally C^*-algebras from $(A \times_{\alpha,r} G) \times_{\widehat{\alpha}} G$ onto $A \otimes \mathcal{K}(L^2(G))$ and the inverse limit actions $\widehat{\widehat{\alpha}}$ and $\alpha \otimes \operatorname{ad} \rho$ are equivalent, Theorem 5.6.

2. Preliminaries. A locally C^*-algebra is a complete complex Hausdorff topological $*$-algebra A whose topology is determined by a family of C^*-seminorms, see [**1, 2, 4, 9, 10**]. If $S(A)$ is the set of all continuous C^*-seminorms on A, then for each $p \in S(A)$, $A_p = A/\ker(p)$ is a C^*-algebra with respect to the norm induced by p, and

$$A = \lim_{\substack{\leftarrow \\ p \in S(A)}} A_p.$$

The canonical maps from A onto A_p, $p \in S(A)$ are denoted by π_p, the image of a under π_p by a_p and the connecting maps of the inverse system $\{A_p\}_{p \in S(A)}$ by π_{pq}, $p, q \in S(A)$ with $p \geq q$.

A morphism of locally C^*-algebras is a continuous $*$-morphism Φ from a locally C^*-algebra A to a locally C^*-algebra B. An isomorphism of locally C^*-algebras is a morphism of locally C^*-algebras which is invertible and its inverse is a morphism of locally C^*-algebras. An S-morphism of locally C^*-algebras is a morphism $\Phi : A \to M(B)$, where $M(B)$ is the multiplier algebra of B, with the property that for any approximate unit $\{e_i\}_i$ of A the net $\{\Phi(e_i)\}_i$ converges to 1 with respect to the strict topology on $M(B)$. If $\Phi : A \to M(B)$ is an S-morphism of locally C^*-algebras, then it extends to a unique morphism $\overline{\Phi} : M(A) \to M(B)$ of locally C^*-algebras, see [**5**].

A Kac C^*-algebra is a quadruple $\mathbf{K} = (B, d, j, \varphi)$, where B is a C^*-algebra, d is a comultiplication on B, j is a coinvolution on B, and φ is a semi-finite, lower semi-continuous, faithful weight on B, see [**13**].

Let A and B be two locally C^*-algebras. The injective tensor product of the locally C^*-algebras A and B is denoted by $A \otimes B$, see [**2**], and the locally C^*-subalgebra of $M(A \otimes B)$ generated by the elements x in $M(A \otimes B)$ such that $x(1 \otimes B) + (1 \otimes B)x \subseteq A \otimes B$ is denoted by $M(A, B)$. If G is a locally compact group, then $M(A, C_0(G))$

may be identified with the locally C^*-algebra $C_b(G, A)$ of all bounded continuous functions from G to A.

Let G be a locally compact group. $C^*\mathbf{K}_G^a = (C_0(G), d_G^a, j_G^a, ds)$ is the commutative Kac C^*-algebra associated with G and $C^*\mathbf{K}_G^s = (C_r^*(G), d_G^s, j_G^s, \varphi_G)$ is the symmetric Kac C^*-algebra associated with G, see [**13**].

An action of a Kac C^*-algebra $\mathbf{K} = (B, d, j, \varphi)$ on a C^*-algebra A is an injective S-morphism α from A to $M(A, B)$ such that $(\overline{\alpha \otimes \mathrm{id}}) \circ \alpha = (\overline{\mathrm{id}_A \otimes \sigma_B \circ d}) \circ \alpha$, see [**13**].

3. Crossed products. Let A be a locally C^*-algebra, and let G be a locally compact group.

Definition 3.1. An action of G on A is a morphism α from G to $\mathrm{Aut}\,(A)$, the set of all isomorphisms of locally C^*-algebras from A to A. The action α is continuous if the function $(t, a) \to \alpha_t(a)$ from $G \times A$ to A is jointly continuous.

Definition 3.2. A locally C^*-dynamical system is a triple (G, A, α), where G is a locally compact group, A is a locally C^*-algebra and α is a continuous action of G on A.

Definition 3.3. We say that $\{(G, A_\delta, \alpha^{(\delta)})\}_{\delta\in\Delta}$ is an inverse system of C^*-dynamical systems if $\{A_\delta\}_{\delta\in\Delta}$ is an inverse system of C^*-algebras and for each t in G, $\{\alpha_t^{(\delta)}\}_{\delta\in\Delta}$ is an inverse system of C^*-isomorphisms.

Let

$$A = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} A_\delta \quad \text{and} \quad \alpha_t = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \alpha_t^{(\delta)}$$

for each $t \in G$. Then the map $\alpha : G \to \mathrm{Aut}\,(A)$ defined by $\alpha(t) = \alpha_t$ is a continuous action of G on A and (G, A, α) is a locally C^*-dynamical system. We say that (G, A, α) is the inverse limit of the inverse system of C^*-dynamical systems $\{(G, A_\delta, \alpha^{(\delta)})\}_{\delta\in\Delta}$.

Definition 3.4. A continuous action α of G on A is an inverse limit action if we can write A as inverse limit

$$\lim_{\overleftarrow{\delta\in\Delta}} A_\delta$$

of C^*-algebras in such a way that there are actions $\alpha^{(\delta)}$ of G on A_δ such that

$$\alpha_t = \lim_{\overleftarrow{\delta\in\Delta}} \alpha_t^{(\delta)}$$

for all t in G [**9**, Definition 5.1].

Remark 3.5. The action α of G on A is an inverse limit action if there is a cofinal subset of G-invariant continuous C^*-seminorms on A (a continuous C^*-seminorm p on A is G-invariant if $p(\alpha_t(a)) = p(a)$ for all a in A and for all t in G).

The following lemma is Lemma 5.2 of [**9**].

Lemma 3.6. *Any continuous action of a compact group G on a locally C^*-algebra A is an inverse limit action.*

Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. By Remark 3.5, we can suppose that $S(A)$ coincides with the set of all G-invariant continuous C^*-seminorms on A.

Let $C_c(G, A)$ be the vector space of all continuous functions from G to A with compact support.

Lemma 3.7. *Let $f \in C_c(G, A)$. Then there is a unique element $\int_G f(s)\,ds$ in A such that for any nondegenerate $*$-representation (φ, H_φ) of A*

$$\left\langle \varphi\left(\int_G f(s)\,ds \right)\xi, \eta \right\rangle = \int_G \langle \varphi(f(s))\xi, \eta\rangle\, ds$$

for all ξ, η in H_φ. Moreover, we have:

(1) $p(\int_G f(s)\,ds) \leq M \sup\{p(f(s));\ s \in \operatorname{supp}(f)\}$ *for some positive number* M *and for all* $p \in S(A)$;

(2) $(\int_G f(s)\,ds)a = \int_G f(s)a\,ds$ *for all* $a \in A$;

(3) $\Phi(\int_G f(s)\,ds) = \int_G \Phi(f(s))\,ds$ *for any morphism of locally* C^**-algebras* $\Phi : A \to B$;

(4) $(\int_G f(s)\,ds)^* = \int_G f(s)^*\,ds$.

Proof. Let $p \in S(A)$. Then $\pi_p \circ f \in C_c(G, A_p)$ and so there is a unique element $\int_G (\pi_p \circ f)(s)\,ds$ in A_p such that for any nondegenerate $*$-representation $(\varphi_p, H_{\varphi_p})$ of A_p

$$\left\langle \varphi_p\left(\int_G (\pi_p \circ f)(s)\,ds\right)\xi, \eta\right\rangle = \int_G \langle \varphi_p((\pi_p \circ f)(s))\xi, \eta\rangle\,ds$$

for all ξ, η in H_{φ_p}; see, for instance, [**11**, Lemma 7].

To show that $(\int_G (\pi_p \circ f)(s)\,ds)_p$ is a coherent net in A, let $p, q \in S(A)$ with $p \geq q$. Then we have

$$\begin{aligned}\pi_{pq}\left(\int_G (\pi_p \circ f)(s)\,ds\right) &= \int_G \pi_{pq}((\pi_p \circ f)(s))\,ds \text{ using Lemma 7 of [11]}\\ &= \int_G (\pi_q \circ f)(s)\,ds.\end{aligned}$$

Therefore, $(\int_G (\pi_p \circ f)(s)\,ds)_p \in A$, and we define $\int_G f(s)\,ds = (\int_G (\pi_p \circ f)(s)\,ds)_p$.

Suppose that there is another element b in A such that for any nondegenerate $*$-representation (φ, H_φ) of A

$$\langle \varphi(b)\xi, \eta\rangle = \int_G \langle \varphi(f(s))\xi, \eta\rangle\,ds$$

for all ξ, η in H_φ. Then for any $p \in S(A)$ and for any nondegenerate $*$-representation $(\varphi_p, H_{\varphi_p})$ of A_p

$$\langle \varphi_p(\pi_p(b))\xi, \eta\rangle = \int_G \langle \varphi_p((\pi_p \circ f)(s))\xi, \eta\rangle\,ds$$

for all ξ, η in H_{φ_p}. From these facts and [**11**, Lemma 7], we conclude that

$$\pi_p(b) = \int_G (\pi_p \circ f)(s)\, ds$$

for all $p \in S(A)$. Therefore, $b = \int_G f(s)\, ds$ and the uniqueness is proved.

Using [**11**, Lemma 7] it is easy to check that $\int_G f(s)\, ds$ satisfies the conditions (1)–(4). □

Let f, h in $C_c(G, A)$. It is easy to check that the map $(s,t) \to f(t)\alpha_t(h(t^{-1}s))$ from $G \times G$ to A is an element in $C_c(G \times G, A)$ and the relation

$$(f \times h)(s) = \int_G f(t)\alpha_t(h(t^{-1}s))\, dt$$

defines an element in $C_c(G, A)$, called the convolution of f and h. Also it is not hard to check that $C_c(G, A)$ becomes a $*$-algebra with convolution as product and involution defined by

$$f^\sharp(t) = \gamma(t)^{-1}\alpha_t(f(t^{-1})^*)$$

where γ is the modular function on G.

For any $p \in S(A)$, define N_p from $C_c(G, A)$ to $[0, \infty)$ by

$$N_p(f) = \int_G p(f(s))\, ds.$$

Straightforward computations show that N_p, $p \in S(A)$, are submultiplicative $*$-seminorms on $C_c(G, A)$.

Let $L^1(G, A, \alpha)$ be the Hausdorff completion of $C_c(G, A)$ with respect to the topology defined by the family of submultiplicative $*$-seminorms $\{N_p\}_{p \in S(A)}$. Then by [**7**, Theorem III 3.1]

$$L^1(G, A, \alpha) = \lim_{\substack{\leftarrow \\ p \in S(A)}} (L^1(G, A, \alpha))_p$$

where $(L^1(G, A, \alpha))_p$ is the completion of the $*$-algebra $C_c(G, A)/\ker(N_p)$ with respect to the norm $\|\cdot\|_p$ induced by N_p.

Lemma 3.8. *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. Then*

$$(L^1(G, A, \alpha))_p = L^1(G, A_p, \alpha^{(p)})$$

for all $p \in S(A)$, up to a topological algebraic $*$-isomorphism.

Proof. Let $p \in S(A)$ and f in $C_c(G, A)$. Then

$$\|f + \ker(N_p)\|_p = \int_G p(f(s))\, ds = \int_G \|\pi_p(f(s))\|_p\, ds = \|\pi_p \circ f\|_1.$$

Therefore we can define a linear map ψ_p from $C_c(G, A)/\ker(N_p)$ to $C_c(G, A_p)$ by

$$\psi_p\,(f + \ker(N_p)) = \pi_p \circ f.$$

It is not hard to check that ψ_p is a $*$-morphism, and since ψ_p is an isometric $*$-morphism from $C_c(G, A)/\ker(N_p)$ to $C_c(G, A_p)$, it can be uniquely extended to an isometric $*$-morphism ψ_p from $(L^1(G, A, \alpha))_p$ to $L^1(G, A_p, \alpha^{(p)})$.

To show that ψ_p is surjective, let $a \in A$ and $f \in C_c(G)$. Define $\widetilde{f}$ from G to A by $\widetilde{f}(s) = f(s)a$. Clearly $\widetilde{f} \in C_c(G, A)$ and

$$\psi_p(\widetilde{f} + \ker(N_p))(s) = f(s)\pi_p(a)$$

for all s in G. This implies that

$$A_p \otimes_{\text{alg}} C_c(G) \subseteq \psi_p((L^1(G, A, \alpha))_p) \subseteq L^1(G, A_p, \alpha^{(p)})$$

whence, since $A_p \otimes_{\text{alg}} C_c(G)$ is dense in $L^1(G, A_p, \alpha^{(p)})$ and since ψ_p is an isometric $*$-morphism, we deduce that ψ_p is surjective and the lemma is proved. $\square$

Corollary 3.9. *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. Then*

$$L^1(G, A, \alpha) = \lim_{\substack{\leftarrow \\ p \in S(A)}} L^1(G, A_p, \alpha^{(p)})$$

up to an algebraic and topological $*$-isomorphism.

Remark 3.10. If $\{e_i\}_{i\in I}$ is an approximate unit for A and $\{f_j\}_{j\in J}$ is an approximate unit for $L^1(G)$, then $\{\widetilde{f}_{(i,j)}\}_{(i,j)\in I\times J}$, where $\widetilde{f}_{(i,j)}(s) = f_j(s)e_i$, $s \in G$, is an approximate unit for $L^1(G, A, \alpha)$, see [**7**, Lemma XIV.1.2]. Then by [**1**, Definition 5.1], we can construct the enveloping algebra of $L^1(G, A, \alpha)$.

Definition 3.11. A covariant representation of (G, A, α) is a triple (φ, u, H), where (φ, H) is a $*$-representation of A and (u, H) is a unitary representation of G such that

$$\varphi(\alpha_t(a)) = u_t\varphi(a)u_t^*$$

for all $t \in G$ and for all $a \in A$.

We say that the covariant representation (φ, u, H) of (G, A, α) is nondegenerate if the $*$-representation (φ, H) of A is nondegenerate.

Remark 3.12. (1) If (φ, u, H) is a covariant representation of (G, A, α) such that $\|\varphi(a)\| \leq p(a)$ for all $a \in A$, then there is a unique covariant representation (φ_p, u, H) of the C^*-dynamical system $(G, A_p, \alpha^{(p)})$ such that $\varphi_p \circ \pi_p = \varphi$.

(2) If (φ_p, u, H) is a covariant representation of the C^*-dynamical system $(G, A_p, \alpha^{(p)})$, then $(\varphi_p \circ \pi_p, u, H)$ is a covariant representation of the locally C^*-dynamical system (G, A, α).

If $R(G, A, \alpha)$ denotes the nondegenerate covariant representations of (G, A, α), then it is easy to check that

$$R(G, A, \alpha) = \bigcup_{p\in S(A)} R_p(G, A, \alpha)$$

where $R_p(G, A, \alpha) = \{(\varphi, u, H) \in R(G, A, \alpha); \|\varphi(a)\| \leq p(a) \text{ for all } a \in A\}$. Also it is easy to check that the map $\varphi_p \mapsto \varphi_p \circ \pi_p$ from $R(G, A_p, \alpha^{(p)})$ to $R_p(G, A, \alpha)$ is bijective.

Proposition 3.13. *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. Then there is a bijection between the covariant nondegenerate representations of (G, A, α) and the nondegenerate $*$-representations of $L^1(G, A, \alpha)$.*

Proof. Let $(\varphi, u, H) \in R(G, A, \alpha)$. Then, there is a $p \in S(A)$ and $(\varphi_p, u, H) \in R(G, A_p, \alpha^{(p)})$ such that $\varphi = \varphi_p \circ \pi_p$. Since $(\varphi_p, u, H) \in R(G, A_p, \alpha^{(p)})$, there is a unique nondegenerate $*$-representation $(\varphi_p \times u, H)$ of $L^1(G, A_p, \alpha^{(p)})$ such that

$$(\varphi_p \times u)(f) = \int_G \varphi_p(f(t))u_t\,dt$$

for all $f \in L^1(G, A_p, \alpha^{(p)})$, see, for instance, [**8**, Proposition 7.6.4].

Let $\varphi \times u = (\varphi_p \times u) \circ \widetilde{\pi}_p$, where $\widetilde{\pi}_p$ is the canonical map from $L^1(G, A, \alpha)$ to $L^1(G, A_p, \alpha^{(p)})$, $\widetilde{\pi}_p(f) = \pi_p \circ f$ for all f in $L^1(G, A, \alpha)$. Then, clearly $(\varphi \times u, H)$ is a nondegenerate $*$-representation of $L^1(G, A, \alpha)$ and moreover,

$$(\varphi \times u)(f) = (\varphi_p \times u)(\pi_p \circ f) = \int_G \varphi_p((\pi_p \circ f)(t))u_t\,dt = \int_G \varphi(f(t))u_t\,dt$$

for all $f \in L^1(G, A, \alpha)$. Thus, we have obtained a map $(\varphi, u, H) \to (\varphi \times u, H)$ from $R(G, A, \alpha)$ to $R(L^1(G, A, \alpha))$. To show that this map is bijective, let (Φ, H) be a nondegenerate $*$-representation of $L^1(G, A, \alpha)$. Then, there is $p \in S(A)$ and a nondegenerate $*$-representation (Φ_p, H) of $L^1(G, A_p, \alpha^{(p)})$ such that $\Phi = \Phi_p \circ \pi_p$. By [**8**, Proposition 7.6.4] there is a unique nondegenerate covariant representation (φ_p, u, H) of $(G, A_p, \alpha^{(p)})$ such that $(\Phi_p, H) = (\varphi_p \times u, H)$. Therefore, there is a nondegenerate covariant representation (φ, u, H) of (G, A, α), where $\varphi = \varphi_p \circ \pi_p$, such that $(\Phi, H) = (\varphi \times u, H)$.

To show that (φ, u, H) is unique, let (ψ, v, K) be another nondegenerate covariant representation of (G, α, A) such that $(\psi \times v, K) = (\Phi, H)$. Then there is a $q \in S(A)$ with $q \geq p$ such that $(\psi, v, K) \in R_q(G, A, \alpha)$ and $(\Phi, K) \in R_q(L^1(G, A, \alpha))$. Therefore $\Phi = \Phi_q \circ \widetilde{\pi}_q$ with $(\Phi_q, H) \in R(L^1(G, A_q, \alpha^{(q)}))$ and $\psi = \psi_q \circ \pi_q$ with $(\psi_q, v, K) \in R(G, A_q, \alpha^{(q)})$ and moreover, $(\Phi_q, H) = (\psi_q \times v, K)$.

On the other hand, $(\varphi_p \circ \pi_{qp}, u, H) \in R(G, A_q, \alpha^{(q)})$ and

$$\begin{aligned}((\varphi_p \circ \pi_{qp}) \times u)(f) &= \int_G (\varphi_p \circ \pi_{qp})(f(t))u_t\,dt \\ &= \int_G \varphi_p(\widetilde{\pi_{qp}}(f)(t))u_t\,dt \\ &= \Phi_p(\widetilde{\pi_{qp}}(f)) = (\Phi_p \circ \widetilde{\pi_{qp}})(f) = \Phi_q(f)\end{aligned}$$

for all $f \in L^1(G, A_q, \alpha^{(q)})$. From these facts and [**8**, Proposition 7.6.4], we conclude that the covariant representations (ψ_q, v, K) and $(\varphi_p \circ \pi_{qp}, u, H)$ of $(G, A_q, \alpha^{(q)})$ coincide, and so the covariant representations (ψ, v, K) and (φ, u, H) of (G, A, α) coincide. □

Definition 3.14. Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. The crossed product of A by the action α, denoted by $A \times_\alpha G$, is the enveloping algebra of the complete locally m-convex $*$-algebra $L^1(G, A, \alpha)$.

Remark 3.15. By Corollary 3.9 and Corollary 5.3 of [**2**], $A \times_\alpha G$ is a locally C^*-algebra and

$$A \times_\alpha G = \lim_{\substack{\longleftarrow \\ p \in S(A)}} A_p \times_{\alpha^{(p)}} G$$

up to an isomorphism of locally C^*-algebras.

Proposition 3.16. *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. Then there is a bijection between nondegenerate covariant representations of (G, A, α) and the nondegenerate representations of $A \times_\alpha G$.*

Proof. Since $A \times_\alpha G$ is the enveloping locally C^*-algebra of the complete locally m-convex $*$-algebra $L^1(G, A, \alpha)$, there is a bijection between the nondegenerate representations of $A \times_\alpha G$ and the nondegenerate representations of $L^1(G, A, \alpha)$, [**2**, pages 37]. From this fact and Proposition 3.13 we conclude that there is a bijection between the nondegenerate representations of $A \times_\alpha G$ and the nondegenerate covariant representations of (G, A, α). □

For each $p \in S(A)$, we denote by $(\varphi_{p,u}, H_{p,u})$ the universal representation of A_p and by $(\varphi_p, H_{p,u})$ the representation of A associated with $(\varphi_{p,u}, H_{p,u})$, that is, $\varphi_p = \varphi_{p,u} \circ \pi_p$.

Lemma 3.17. *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. Then $(\widetilde{\varphi_p}, \lambda, L^2(G, H_{p,u}))$, where*

$$\widetilde{\varphi_p}(a)(\xi)(t) = \varphi_p\left(\alpha_{t^{-1}}(a)\right)(\xi(t))$$

and

$$\lambda_s(\xi)(t) = \xi(s^{-1}t)$$

for all a in A, ξ in $L^2(G, H_{p,u})$ and s, t in G, is a nondegenerate covariant representation of (G, A, α).

Proof. It is a simple verification. □

Let $p \in S(A)$. The map $r_p : L^1(G, A, \alpha) \to [0, \infty)$ defined by

$$r_p(f) = \|(\widetilde{\varphi_p} \times \lambda)(f)\|$$

is a C^*-seminorm on $L^1(G, A, \alpha)$ with the property that $r_p(f) \leq N_p(f)$ for all f in $L^1(G, A, \alpha)$.

Let

$$I = \bigcap_{p \in S(A)} \ker(r_p).$$

Clearly I is a closed two-sided ideal of $L^1(G, A, \alpha)$ and $L^1(G, A, \alpha)/I$ is a pre-locally C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\{\widehat{r}_p\}_{p \in S(A)}$, $\widehat{r}_p(f + I) = \inf\{r_p(f + h); h \in I\}$.

Definition 3.18. The reduced crossed product of A by the action α, denoted by $A \times_{\alpha,r} G$, is the Hausdorff completion of $(L^1(G, A, \alpha), \{r_p\}_{p \in S(A)})$, that is, $A \times_{\alpha,r} G$ is the completion of the pre-locally C^*-algebra $(L^1(G, A, \alpha)/I, \{\widehat{r}_p\}_{p \in S(A)})$.

Lemma 3.19. *Let (G, A, α) be a locally C^*-dynamical system such that α is an inverse limit action. Then*

$$(A \times_{\alpha,r} G)_p = A_p \times_{\alpha^{(p)},r} G$$

for all $p \in S(A)$, up to an isomorphism of C^*-algebras.

Proof. Let $p \in S(A)$. If $f \in L^1(G, A, \alpha)$, then we have

$$\begin{aligned} \|(f + I) + \ker(\widehat{r}_p)\|_{\widehat{r}_p} &= \widehat{r}_p(f + I) = \inf\{\|(\widetilde{\varphi_p} \times \lambda)\,(f + h)\|\,; h \in I\} \\ &= \inf\{\|(\widetilde{\varphi_p} \times \lambda)\,(f)\|\,; h \in I\} \\ &= r_p(f) = \|f + \ker(r_p)\|_{r_p}. \end{aligned}$$

From this relation, we conclude that $(A \times_{\alpha,r} G)_p$ is isomorphic to the completion of $L^1(G, A, \alpha)/\ker(r_p)$ with respect to the C^*-norm induced by r_p.

On the other hand, $A_p \times_{\alpha^{(p)},r} G$ is the completion of $L^1(G, A_p, \alpha^{(p)})/I_p$, where $I_p = \{f \in L^1(G, A_p, \alpha^{(p)})/(\widetilde{\varphi_{p,u}} \times \lambda)(f) = 0\}$, with respect to the norm $\|\cdot\|'$ given by $\|f + I_p\|' = \|(\widetilde{\varphi_{p,u}} \times \lambda)(f)\| \leq \|f\|_1$. But the completion of $L^1(G, A, \alpha)/\ker(r_p)$ with respect to the norm $\|\cdot\|_{r_p}$ is isomorphic to the completion of $L^1(G, A, \alpha^{(p)})/I_p$ with respect to the norm $\|\cdot\|'$, since

$$\begin{aligned} \|f + \ker(r_p)\|_{r_p} &= r_p(f) = \|(\widetilde{\varphi_p} \times \lambda)\,(f)\| \\ &= \|(\widetilde{\varphi_{p,u}} \times \lambda)\,(\pi_p \circ f)\| \\ &= \|\widetilde{\pi}_p(f) + I_p\|' \end{aligned}$$

for all $f \in L^1(G, A, \alpha)$. Therefore, the C^*-algebras $(A \times_{\alpha,r} G)_p$ and $A_p \times_{\alpha^{(p)},r} G$ are isomorphic. □

Corollary 3.20. *If (G, A, α) is a locally C^*-dynamical system such that α is an inverse limit action, then*

$$A \times_{\alpha,r} G = \lim_{\substack{\leftarrow \\ p \in S(A)}} A_p \times_{\alpha^{(p)},r} G$$

up to an isomorphism of locally C^*-algebras.

4. Actions of a Kac C^*-algebra on a locally C^*-algebra. Let $C^*\mathbf{K} = (B, d, j, \varphi)$ be a Kac C^*-algebra, and let A be a locally C^*-algebra.

Definition 4.1. An action of $C^*\mathbf{K}$ on A is an injective S-morphism α from A to $M(A, B)$ such that

$$\left(\overline{\alpha \otimes \mathrm{id}_B}\right) \circ \alpha = \left(\overline{\mathrm{id}_A \otimes (\sigma_B \circ d)}\right) \circ \alpha.$$

An action α of $C^*\mathbf{K}$ on A is an inverse limit action if we can write A as an inverse limit

$$\lim_{\substack{\leftarrow \\ \delta \in \Delta}} A_\delta$$

of C^*-algebras in such a way that there are actions $\alpha^{(\delta)}$ of $C^*\mathbf{K}$ on A_δ, $\delta \in \Delta$ such that

$$\alpha = \lim_{\substack{\leftarrow \\ \delta \in \Delta}} \alpha^{(\delta)}.$$

Two actions α_1 and α_2 of $C^*\mathbf{K}$ on the locally C^*-algebras A_1, respectively A_2, are said to be equivalent if there is an isomorphism of locally C^*-algebras $\Phi : A_1 \to A_2$ such that $\alpha_2 \circ \Phi = \left(\overline{\Phi \otimes \mathrm{id}_B}\right) \circ \alpha_1$.

Proposition 4.2. *Let G be a locally compact group. If α is an action of $C^*\mathbf{K}_G^a$ on A, then the map $\Sigma(\alpha)$ that applies $t \in G$ to a map $\Sigma(\alpha)_t$ from A to A defined by $\Sigma(\alpha)_t(a) = \alpha(a)(t^{-1})$, is a continuous action of G on A.*

Proof. Since α is a continuous $*$-morphism from A to $C_b(G, A)$, $\Sigma(\alpha)_t$ is a continuous $*$-morphism from A to A for each t in G. Using the same arguments as in the proof of Proposition 5.1.5 of [**13**], it is not difficult to see that $\Sigma(\alpha)_t$ is invertible and, moreover, $(\Sigma(\alpha)_t)^{-1} = \Sigma(\alpha)_{t^{-1}}$ for all t in G. Therefore $\Sigma(\alpha)_t \in \mathrm{Aut}\,(A)$ for each t in G.

To show that the map $(t, a) \to \Sigma(\alpha)_t(a)$ from $G \times A$ to A is continuous, let $(t_0, a_0) \in G \times A$, and let $W_{p,\varepsilon} = \{a \in A; p(a - \Sigma(\alpha)_{t_0}(a_0)) < \varepsilon\}$ be a neighborhood of $\Sigma(\alpha)_{t_0}(a_0)$. Since $\alpha(a_0) \in C_b(G, A)$, there is a neighborhood U_0 of t_0 such that

$$p(\alpha(a_0)(t^{-1}) - \alpha(a_0)(t_0^{-1})) < \frac{\varepsilon}{2}$$

for all t in U_0, and since α is a continuous $*$-morphism, there is a neighborhood V_0 of a_0 such that

$$\|\alpha\,(a) - \alpha\,(a_0)\|_p = \sup\{p(\alpha\,(a)\,(t) - \alpha\,(a_0)\,(t));\ t \in G\} < \frac{\varepsilon}{2}$$

for all a in V_0. Then

$$\begin{aligned} p\left(\Sigma\,(\alpha)_t\,(a) - \Sigma\,(\alpha)_{t_0}\,(a_0)\right) &\le p\left(\alpha\,(a)\left(t^{-1}\right) - \alpha\,(a_0)\left(t^{-1}\right)\right) \\ &\quad + p\left(\alpha\,(a_0)\left(t^{-1}\right) - \alpha\,(a_0)\,(t_0^{-1})\right) \\ &\le \|\alpha\,(a) - \alpha\,(a_0)\|_p + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for all $(t, a) \in U_0 \times V_0$ and the proposition is proved. $\square$

Remark 4.3. According to Proposition 4.2, we can define a map Σ from the set of all actions of $C^*\mathbf{K}_G^a$ on A to the set of all continuous actions of G on A by $\alpha \to \Sigma(\alpha)$. Moreover, Σ is injective.

The following proposition is a generalization of [**13**, Proposition 5.1.5] for inverse limit actions of locally compact groups on locally C^*-algebras.

Proposition 4.4. *Let G be a locally compact group. Then the map Σ defined in Proposition* 4.2 *is a bijective correspondence between the set of all inverse limit actions of $C^*\mathbf{K}_G^a$ on A and the set of all continuous inverse limit actions of G on A.*

Proof. Let α be an inverse limit action of $C^*\mathbf{K}_G^a$ on A. Then A may be written as an inverse limit

$$\lim_{\substack{\leftarrow \\ \delta\in\Delta}} A_\delta$$

of C^*-algebras, and there are actions $\alpha^{(\delta)}$ of $C^*\mathbf{K}_G^a$ on A_δ, $\delta \in \Delta$ such that

$$\alpha = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \alpha^{(\delta)}.$$

According to [**13**, Proposition 5.1.5], for each $\delta \in \Delta$ there is a continuous action $\Sigma(\alpha^{(\delta)})$ of G on A_δ such that $\Sigma(\alpha^{(\delta)})_t(a_\delta) = \alpha^{(\delta)}(a_\delta)(t^{-1})$ for all a_δ in A_δ and for all t in G. Since $\{\alpha^{(\delta)}\}_{\delta\in\Delta}$ is an inverse system of morphisms of C^*-algebras, it is not difficult to check that $\{\Sigma(\alpha^{(\delta)})_t\}_{\delta\in\Delta}$ is an inverse system of C^*-isomorphisms for each t in G. Also it is easy to check that

$$\Sigma(\alpha)_t = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \Sigma(\alpha^{(\delta)})_t$$

for each t in G.

To show that Σ is surjective, let β be a continuous inverse limit action of G on A. Then A may be written as an inverse limit

$$A = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} A_\delta$$

of C^*-algebras and there are continuous actions $\beta^{(\delta)}$ of G on A_δ, $\delta \in \Delta$, such that

$$\beta_t = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \beta_t^{(\delta)}$$

for each t in G. By [**13**, Proposition 5.1.5], for each $\delta \in \Delta$ there is an action $\alpha^{(\delta)}$ of $C^*\mathbf{K}_G^a$ on A_δ such that $\Sigma(\alpha^{(\delta)}) = \beta^{(\delta)}$. It is not difficult to verify that $\{\alpha^{(\delta)}\}_{\delta\in\Delta}$ is an inverse system of injective S-morphisms of C^*-algebras. Let

$$\alpha = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \alpha^{(\delta)}.$$

Then α is an injective S-morphism of locally C^*-algebras and

$$\begin{aligned}\left(\overline{\alpha \otimes \mathrm{id}_{C_0(G)}}\right) \circ \alpha &= \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \left(\overline{\alpha^{(\delta)} \otimes \mathrm{id}_{C_0(G)}}\right) \circ \alpha^{(\delta)} \\ &= \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \left(\overline{\mathrm{id}_{A_p} \otimes \sigma_{C_0(G)} \circ d_G^a}\right) \circ \alpha^{(\delta)} \\ &= \left(\overline{\mathrm{id}_A \otimes \sigma_{C_0(G)} \circ d_G^a}\right) \circ \alpha.\end{aligned}$$

Therefore α is an inverse limit action of $C^*\mathbf{K}_G^a$ on A and $\Sigma(\alpha) = \beta$. Thus, we showed that Σ is bijective. □

Corollary 4.5. *If G is compact, then any action of $C^*\mathbf{K}_G^a$ on A is an inverse limit action.*

Proof. Let α be an action of $C^*\mathbf{K}_G^a$ on A. By Proposition 4.2, $\Sigma(\alpha)$ is a continuous action of G on A which is a limit inverse action, since the group G is compact, Lemma 3.6. From this fact and Proposition 4.4, we conclude that α is an inverse limit action. □

5. The Takai duality theorem. Let G be a locally compact group, and let A be a locally C^*-algebra.

Lemma 5.1. *Let α be an inverse limit action of G on A. Then the reduced crossed product of A by the action α is isomorphic to the locally C^*-subalgebra of $M(A \otimes \mathcal{L}(L^2(G)))$ generated by $\{\alpha(a)(1_{M(A)} \otimes \lambda(f)); a \in A, f \in C_c(G)\}$, where λ is the left regular representation of $L^1(G)$.*

Proof. Let $p \in S(A)$. By [**13**, Remark 5.2.1.1], the map Φ_p from the C^*-subalgebra of $M(A_p \otimes \mathcal{L}(L^2(G)))$ generated by $\{\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)); a_p \in A_p, f \in C_c(G)\}$ to $A_p \times_{\alpha^{(p)},r} G$, that applies $\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f))$ to $\widetilde{f} + I_p$, where $\widetilde{f}(t) = f(t)a_p$, $t \in G$, see the proof of Lemma 3.19, is an isomorphism of C^*-algebras.

If π'_{pq}, p, $q \in S(A)$, $p \geq q$ are the connecting maps of the inverse system $\{M(A_p \otimes \mathcal{L}(L^2(G)))\}_{p \in S(A)}$ and $\widehat{\pi}_{pq}$, $p, q \in S(A)$, $p \geq q$ are the connecting maps of the inverse system $\{A_p \times_{\alpha^{(p)},r} G\}_{p \in S(A)}$, then we have

$$\begin{aligned}(\Phi_q \circ \pi'_{pq})(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)))&\\ &= \Phi_q(\alpha^{(q)}(\pi_{pq}(a_p))(1_{M(A_q)} \otimes \lambda(f)))\\ &= \pi_{pq}(a_p) \otimes f + I_q = \widetilde{\pi_{pq}}(a_p \otimes f) + I_q\\ &= \widehat{\pi}_{pq}(a_p \otimes f + I_q)\\ &= (\widehat{\pi}_{pq} \circ \Phi_p)(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)))\end{aligned}$$

for all a_p in A_p, for all f in $C_c(G)$ and for all $p, q \in S(A)$ with $p \geq q$. Therefore, $\{\Phi_p\}_{p \in S(A)}$ is an inverse system of isomorphisms of C^*-algebras and the lemma is proved. □

Definition 5.2. A coaction of G on A is an action β of $C^*\mathbf{K}_G^s$ on A. We say that a coaction β of G on A is an inverse limit coaction if it is an inverse limit action of $C^*\mathbf{K}_G^s$ on A.

The reduced crossed product of A by the coaction β, denoted by $A \times_\beta G$, is the locally C^*-subalgebra of $M(A \otimes \mathcal{L}(L^2(G)))$ generated by $\{\beta(a)(1_{M(A)} \otimes f); a \in A, f \in C_c(G)\}$.

Remark 5.3. Let

$$\beta = \lim_{\substack{\leftarrow\\ \delta \in \Delta}} \beta^{(\delta)}$$

be an inverse limit coaction of G on A such that the connecting maps of the inverse system $\{A_\delta\}_{\delta \in \Delta}$ are all surjective. Then, by [**10**, Theorem 3.14],

$$M(A \otimes \mathcal{L}\left(L^2\left(G\right)\right)) = \lim_{\substack{\leftarrow\\ \delta \in \Delta}} M(A_\delta \otimes \mathcal{L}\left(L^2\left(G\right)\right))$$

up to an isomorphism of locally C^*-algebras, and by [**7**, Lemma III 3.2],

$$A \times_\beta G = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} A_\delta \times_{\beta(\delta)} G$$

up to an isomorphism of locally C^*-algebras.

Remark 5.4. Let G be a commutative locally compact group. Exactly as in the proof of Proposition 5.1.6 of [**13**], we show that if β is an inverse limit coaction of G on A, then $\beta' = (\mathrm{id}_A \otimes \mathrm{ad}\,\mathcal{F}) \circ \beta$, where $\mathcal{F}$ is the Fourier-Plancherel isomorphism from $L^2(G)$ onto $L^2(\widehat{G})$, is an inverse limit action of $\widehat{G}$ on A and conversely, if α is an inverse limit action of $\widehat{G}$ on A then $\alpha' = (\mathrm{id}_A \otimes \mathrm{ad}\,\mathcal{F}^*) \circ \alpha$ is an inverse limit coaction of G on A. Therefore, an inverse limit coaction of G can be identified with an inverse limit action of $\widehat{G}$ and $\mathrm{id}_A \otimes \mathrm{ad}\,\mathcal{F}$ is an isomorphism between $A \times_\beta G$ and $A \times_{\beta',r} \widehat{G}$.

The following proposition is a generalization of [**13**, Theorem 5.2.6] for inverse limit actions of a locally compact group on a locally C^*-algebra.

Proposition 5.5. *Let A be a locally C^*-algebra, and let G be a locally compact group.*

(1) *If α is an inverse limit action of G on A, then there is an inverse limit coaction $\widehat{\alpha}$ of G on $A \times_{\alpha,r} G$, called the dual coaction associated to α, such that*

$$(*)\qquad \widehat{\alpha}(\alpha(a)(1_{M(A)} \otimes \lambda(f))) = (\alpha(a) \otimes 1_G)(1_{M(A)} \otimes d_G^s(\lambda(f)))$$

for all a in A and for all f in $C_c(G)$.

(2) *If*

$$\beta = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \beta^{(\delta)}$$

is an inverse limit coaction of G on A such that the connecting maps of the inverse system $\{A_\delta\}_{\delta\in\Delta}$ are all surjective, then there is an inverse limit action $\widehat{\beta}$ of G on $A \times_\beta G$, called the dual action associated to β, such that

$$(**)\ \ \widehat{\beta}(\beta(a)(1_{M(A)}\otimes f)) = (\beta(a)\otimes 1_G)(1_{M(A)}\otimes(\overline{\mathrm{id}_{C_0(G)}\otimes j_G^a})d_G^a(f))$$

for all a in A and for all f in $C_c(G)$.

Proof. (1) Since α is an inverse limit action,

$$\alpha = \lim_{\substack{\leftarrow \\ p\in S(A)}} \alpha^{(p)},$$

where $\alpha^{(p)}$ is a continuous action of G on A_p. By [**13**, Theorem 5.2.6 (i)], for each $p \in S(A)$ there is a dual coaction $\widehat{\alpha}^{(p)}$ of G on $A_p \times_{\alpha^{(p)},r} G$ such that

$$\widehat{\alpha}^{(p)}(\alpha^{(p)}(a_p)(1_{M(A_p)}\otimes\lambda(f))) = (\alpha^{(p)}(a_p)\otimes 1_G)(1_{M(A_p)}\otimes d_G^s(\lambda(f)))$$

for all a_p in A_p and for all f in $C_c(G)$. It is not difficult to check that $\{\widehat{\alpha}^{(p)}\}_{p\in S(A)}$ is an inverse system of injective S-morphisms and

$$\widehat{\alpha} = \lim_{\substack{\leftarrow \\ p\in S(A)}} \widehat{\alpha}^{(p)}$$

is a coaction of G on $A \times_{\alpha,r} G$ which verifies the condition $(*)$.

(2) By Theorem 5.2.6 (ii) of [**13**], for each $\delta \in \Delta$ there is a continuous action $\widehat{\beta}^{(\delta)}$ of G on $A_\delta \times_{\beta^{(\delta)}} G$ such that

$$\widehat{\beta}^{(\delta)}(\beta^{(\delta)}(a_\delta)(1_{M(A_\delta)}\otimes f)) \\ = (\beta^{(\delta)}(a_\delta)\otimes 1_G)(1_{M(A_\delta)}\otimes(\overline{\mathrm{id}_{C_0(G)}\otimes j_G^a})d_G^a(f))$$

for all a_δ in A_δ and for all f in $C_c(G)$. Using this relation and Remark 5.3 it is not difficult to check that $\{\widehat{\beta}^{(\delta)}\}_{\delta\in\Delta}$ is an inverse system of injective S-morphisms. Let

$$\widehat{\beta} = \lim_{\substack{\leftarrow \\ \delta\in\Delta}} \widehat{\beta}^{(\delta)}.$$

Then $\widehat{\beta}$ is a continuous action of G on $A \times_\beta G$ and moreover, it verifies the condition $(**)$. □

The following theorem is a version of the Takai duality theorem for inverse limit actions of a locally compact group on a locally C^*-algebra.

Theorem 5.6. *Let G be a locally compact group, let A be a locally C^*-algebra, and let α be an inverse limit action of G on A. Then there is an isomorphism Π from $A \otimes \mathcal{K}(L^2(G))$ onto $(A \times_{\alpha,r} G) \times_{\widehat{\alpha}} G$ such that*

$$\widehat{\widehat{\alpha}} \circ \Pi = (\overline{\Pi \otimes \mathrm{id}_{C_0(G)}}) \circ (\alpha \otimes \mathrm{ad}\,\rho)$$

where ρ is the right regular representation of $L^1(G)$.

Proof. By [**10**, Proposition 3.2],

$$A \otimes \mathcal{K}(L^2(G)) = \lim_{\substack{\leftarrow \\ p \in S(A)}} A_p \otimes \mathcal{K}(L^2(G))$$

up to an isomorphism of locally C^*-algebras.

Since α is an inverse limit action, according to the proof of Proposition 5.5 (1),

$$\widehat{\alpha} = \lim_{\substack{\leftarrow \\ p \in S(A)}} \widehat{\alpha}^{(p)}$$

where $\widehat{\alpha}^{(p)}$ is the dual coaction associated to $\alpha^{(p)}$ for each $p \in S(A)$. Then, since the connecting maps of the inverse system $\{A_p \times_{\alpha^{(p)},r} G\}_{p \in S(A)}$ are all surjective, by Proposition 5.5 (2),

$$\widehat{\widehat{\alpha}} = \lim_{\substack{\leftarrow \\ p \in S(A)}} \widehat{\widehat{\alpha}}^{(p)}$$

and by Remark 5.3,

$$(A \times_{\alpha,r} G) \times_{\widehat{\alpha}} G = \lim_{\substack{\leftarrow \\ p \in S(A)}} \big(A_p \times_{\alpha^{(p)},r} G\big) \times_{\widehat{\alpha}^{(p)}} G$$

up to an isomorphism of locally C^*-algebras.

Let $p \in S(A)$. According to [**13**, Theorem 5.2], there is an isomorphism $\Pi^{(p)}$ from $A_p \otimes \mathcal{K}(L^2(G))$ onto $(A_p \times_{\alpha^{(p)},r} G) \times_{\widehat{\alpha}^{(p)}} G$ such that

$$\widehat{\widehat{\alpha}}^{(p)} \circ \Pi^{(p)} = (\overline{\Pi^{(p)} \otimes \mathrm{id}_{C_0(G)}}) \circ (\alpha^{(p)} \otimes \mathrm{ad}\,\rho).$$

Moreover,

$$\Pi^{(p)}(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)h)) = \widehat{\alpha}^{(p)}(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)))(1_{M(A_p)} \otimes 1_G \otimes h)$$

and

$$\Pi^{(p)}((1_{M(A_p)} \otimes \lambda(f)h)\alpha^{(p)}(a_p)) = \widehat{\alpha}^{(p)}((1_{M(A_p)} \otimes \lambda(f))\alpha^{(p)}(a_p))(1_{M(A_p)} \otimes 1_G \otimes h)$$

for all f and h in $C_c(G)$ and for all a_p in A_p. Using these relations and the fact that $A_p \otimes \mathcal{K}(L^2(G))$ is the C^*-subalgebra of $M(A_p \otimes \mathcal{K}(L^2(G)))$ generated by $\{\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)h), (1_{M(A_p)} \otimes \lambda(f)h)\alpha^{(p)}(a_p); f, h \in C_c(G), a_p \in A_p\}$, see [**13**, Lemma 5.2.10], it is not difficult to check that $\{\Pi^{(p)}\}_{p\in S(A)}$ is an inverse system of C^*-isomorphisms.

Let

$$\Pi = \lim_{\substack{\leftarrow \\ p\in S(A)}} \Pi^{(p)}.$$

Then, clearly Π is an isomorphism of locally C^*-algebras from $A \otimes \mathcal{K}(L^2(G))$ onto $(A \times_{\alpha,r} G) \times_{\widehat{\alpha}} G$ which satisfies the condition

$$\widehat{\widehat{\alpha}} \circ \Pi = (\overline{\Pi \otimes \mathrm{id}_{C(G)}}) \circ (\alpha \otimes \mathrm{ad}\,\rho)$$

and the theorem is proved. □

Since any action of a compact group on a locally C^*-algebra is an inverse limit action, we have:

Corollary 5.7. *Let G be a compact group, let A be a locally C^*-algebra, and let α be a continuous action of G on A. Then there is an isomorphism Π from $A \otimes \mathcal{K}(L^2(G))$ onto $(A \times_{\alpha,r} G) \times_{\widehat{\alpha}} G$ such that*

$$\widehat{\widehat{\alpha}} \circ \Pi = (\overline{\Pi \otimes \mathrm{id}_{C_0(G)}}) \circ (\alpha \otimes \mathrm{ad}\,\rho)$$

where ρ is the right regular representation of $L^1(G)$.

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