ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 5, 2007

ON AN ADDITIVE REPRESENTATION ASSOCIATED WITH THE L_1-NORM OF AN EXPONENTIAL SUM

M.Z. GARAEV

ABSTRACT. Let N be a large positive integer parameter, $f(n)$ be an integer valued strictly increasing function of the natural argument n. It is well known that a nontrivial upper bound estimate for the number of solutions of the diophantine equation

$$f(x)+f(y)=f(u)+f(v), \quad 1 \le x,y,u,v \le N$$

has an important application in obtaining a lower bound for the L_1-norm of an exponential sum. In this paper by a short argument we obtain a result which implies a well-known estimate of Konyagin.

1. Introduction. Let N be a large positive integer parameter, $f(n)$ a strictly increasing integer-valued function of the integer argument $n, 1 \le n \le N$. A famous Littlewood conjecture states that

$$\int_0^1 \left| \sum_{n=1}^{N} \exp(2\pi i \alpha f(n)) \right| d\alpha \gg \log N. \tag{1}$$

This conjecture was independently proved in 1981 by Konyagin [**5**] and McGehee, et al. [**7**]. The relation [**8**, page 67]

$$\int_0^1 \left| \sum_{n=1}^{N} \exp(2\pi i \alpha n) \right| d\alpha = \frac{4}{\pi^2} \log N + O(1)$$

shows that the order $\log N$ in (1) is sharp. However, for a wide class of sequences $f(n)$, the estimate (1) can be improved. Bochkarev [**1**] has improved (1) for sequences of the type $f(n)=[e^{Ax^\beta}]$, where $0<\beta<1$.

2000 AMS *Mathematics subject classification.* Primary 11D45, 11L03.
Keywords and phrases. Exponential sums, L_1-norm, diophantine equation.
Received by the editors on November 22, 2004, and in revised form on May 3, 2005.

The approach in [**1**] is based on harmonic analysis. Karatsuba [**4**] has found a connection between the Littlewood problem and the problem of obtaining a nontrivial upper bound estimate for the number of solutions of the corresponding diophantine equation. Let $J := J(N)$ denote the number of solutions of the equation

$$f(x) + f(y) = f(u) + f(v), \quad 1 \le x, y, u, v \le N.$$

One of the results of [**4**] reads as follows.

Lemma 1. *For arbitrary coefficients $\gamma(x)$, $|\gamma(x)| = 1$, the estimate*

$$I := I(N) = \int_0^1 \left| \sum_{x=1}^{N} \gamma(x) e^{2\pi i \alpha f(x)} \right| d\alpha \ge \left(\frac{N^3}{J} \right)^{1/2}$$

holds.

This statement allows to approach the problem of Littlewood in different ways. For example, in the above mentioned work [**4**] it is shown how the result of [**1**] can be improved using tools of analytic number theory. Konyagin [**6**] has used some methods from combinatorial number theory to get a substantial improvement even for a more general sequence. He established that if

$$(2) \qquad 0 < f(2) - f(1) < f(3) - f(2) < \cdots < f(N) - f(N-1)$$

then $J \ll N^{5/2}$. In particular, $I \gg N^{1/4}$.

In our works [**2, 3**], among other results, new proofs of Konyagin's estimate have been obtained. The aim of the present paper is to obtain by a short argument new information which also includes Konyagin's estimate. Our argument can also be useful on other related questions.

2. The result and its corollary. For a given element h of the set

$$H = \{f(x) + f(y) \mid 1 \le x, y \le N\},$$

we denote by $m(h)$ the number of solutions of the equation

$$(3) \qquad f(x) + f(y) = h, \quad 1 \le x \le y \le N.$$

Let $h_1, h_2, \dots, h_\omega$ be all the elements of the set H numerated such that $m(h_1) \geq m(h_2) \geq \dots \geq m(h_\omega)$. It is clear that

$$J \leq 4\sum_{i=1}^{\omega} m^2(h_i) \tag{4}$$

and

$$\sum_{i=1}^{\omega} m(h_i) = \frac{N(N+1)}{2}. \tag{5}$$

For a fixed L, $1 \leq L < N$, we denote by $J_L := J_L(N)$ the number of solutions of the equation

$$f(x) + f(y) = f(x+t) + f(z) \tag{6}$$

in positive integers x, y, z, t subject to the condition

$$1 \leq x \leq y \leq N, \quad 1 \leq t \leq L, \quad x+t \leq z \leq N. \tag{7}$$

Theorem 2. *For any integers L, r with $1 \leq L \leq N-1$ and $1 \leq r \leq \omega$, the inequality*

$$m(h_r) \leq \frac{N-1}{L} + \frac{J_L}{r} + 1$$

holds.

Proof. For simplicity, denote $m_i = m(h_i)$. If $m_r = 1$, then the statement of Theorem 2 is trivial.

Let $m_r \geq 2$. Following (3) we write out all the representations of the numbers $h_1, \dots, h_r$ as

$$\begin{cases} h_1 = f(x_{11}) + f(y_{11}) = f(x_{12}) + f(y_{12}) = \cdots = f(x_{1m_1}) + f(y_{1m_1}), \\ \cdots \\ h_r = f(x_{r1}) + f(y_{r1}) = f(x_{r2}) + f(y_{r2}) = \cdots = f(x_{rm_r}) + f(y_{rm_r}), \end{cases}$$

where

$$x_{11} < x_{12} < \cdots < x_{1m_1},$$
$$\cdots$$
$$x_{r1} < x_{r2} < \cdots < x_{rm_r}$$

and $x_{i,j} \le y_{i,j}$. Suppose that among the $(m_1 - 1) + (m_2 - 1) + \cdots + (m_r - 1)$ collections of positive integers

$$x_{12} - x_{11},\ x_{13} - x_{12}, \dots ,\ x_{1m_1} - x_{1,m_1-1},$$
$$\cdots$$
$$x_{r2} - x_{r1},\ x_{r3} - x_{r2}, \dots ,\ x_{rm_r} - x_{r,m_r-1}$$

there are V numbers greater than L and v numbers less than or equal to L. Each number l of our collection is associated with the corresponding solution of the equation

$$f(x) + f(y) = f(x + l) + f(z), \quad 1 \le x \le y \le N,\ x + l \le z \le N.$$

Therefore,

$$v \le J_L.$$

On the other hand the total sum of the numbers of our collection is $\le Nr - r$. Hence,

$$VL \le Nr - r.$$

Besides,

$$(m_1 - 1) + (m_2 - 1) + \ldots + (m_r - 1) = V + v.$$

Therefore,

$$r(m_r - 1) \le \frac{(N-1)r}{L} + J_L$$

and the result follows. □

From Theorem 2 we can easily deduce Konyagin's estimate.

Corollary 3. *If the function $f(n)$ satisfies the condition* (2), *then the estimate*

$$J \ll N^{5/2}$$

holds.

Indeed, from (2), (6) and (7) we see that the inequality $1 \le y - z < t \le L$ holds, since otherwise

$$f(y) - f(z) \ge f(z+t) - f(z) > f(x+t) - f(x),$$

contradicting (6). Therefore, the number of all the possible triples (y, z, t) is less than L^2N. For fixed y, z and t there is at most one value for x satisfying (6). Therefore, $J_L \le L^2N$. From Theorem 2, by taking $L = [r^{1/3}]$, we obtain

$$m(h_r) \ll Nr^{-1/3}.$$

Combining this with (4) and (5) we conclude

$$\begin{aligned} J &\ll \sum_{i \le N^{3/2}} m^2(h_i) + \sum_{i > N^{3/2}} m^2(h_i) \\ &\ll N^{5/2} + N^{1/2} \sum_{i \le \omega} m(h_i) \ll N^{5/2}. \end{aligned}$$

Acknowledgments. The author is thankful to the referee for helpful remarks.

REFERENCES

1. S.V. Bochkarev, *A method for estimating the L_1-norm of an exponential sum*, Proc. Steklov Inst. Math. **218** (1997), 74–78.

2. M.Z. Garaev, *On lower bounds for the L_1-norm of exponential sums*, Mat. Zametki **68** (2000), 842–850 (in Russian); Math. Notes **68** (2000), 713–720 (in English).

3. ———, *Upper bounds for the number of solutions of a diophantine equation*, Trans. Amer. Math. Soc. **357** (2005), 2527–2534.

4. A.A. Karatsuba, *An estimate of the L_1-norm of an exponential sum*, Mat. Zametki **64** (1998), 465–468 (in Russian); Math. Notes **64** (1998), 401–404 (in English).

5. S.V. Konyagin, *On the problem of Littlewood*, Izv. Acad. Nauk SSSR **45** (1981), 243–265.

6. ———, *An estimate of the L_1-norm of an exponential sum*, in *The theory of approximations of functions and operators*, Abstracts of Papers of the International Conference Dedicated to Stechkin's 80th Anniversary (in Russian), Ekaterinburg, 2000, pages 88–89.

7. O.C. McGehee, L. Pigno and B. Smith, *Hardy's inequality and the L_1 norm of exponential sums*, Annals Math. **113** (1981), 613–618.

8. A. Zygmund, *Trigonometric series*, Vol. 1., Cambridge University Press, Cambridge, 1959.

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, Apartado Postal 61-3 (Xangari), C.P. 58089, Morelia, Michoacán, México
Email address: garaev@matmor.unam.mx