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A NON-COMMUTATIVE n-NOMIAL FORMULA

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ABSTRACT. We state and prove a general n-nomial expansion formula for $(u_1+\cdots+u_n)^m$ in terms of degree m monomials in $u_1,\ldots,u_n$, under the assumption $u_j u_i = \rho_{ij} u_i u_j$ for all $i<j$.

0. Introduction. The following binomial expansion holds for variables u_i, $i=1,2$, satisfying $u_2u_1 = \rho_{12}u_1u_2$, $\rho_{12}\in \mathbf{C}$,

$$(u_1+u_2)^m = \sum_{k=0}^{m} \binom{m}{k}_{\rho_{12}} u_1^k u_2^{m-k}, \quad m \in \mathbf{N},$$

where $\binom{m}{k}_q = (q)_m/((q)_k(q)_{m-k})$, $(q)_k = (1-q^k)\cdots(1-q)$, $(q)_0=1$, for $q\in\mathbf{C}$ and $m,k\in\mathbf{N}$, $m\ge k\ge 1$. This result is usually attributed to Schützenberger [**6**]. However, it was published three years earlier by H.S.A. Potter. (See [**4**] for an excellent discussion.)

The Potter/Schützenberger q-multinomial coefficient is the generating function for sequences of n_1 1's, n_2 2's, $\ldots$, n_k k's by inversions. This combinatorial interpretation is sometimes attributed to Netto but certainly appears in [**5**]. Equivalent results about partitions were discovered by Cayley and Sylvester.

The main result of this note is a generalization of the Potter/Schützenberger binomial formula to the case of n variables $u_1,\ldots,u_n$, under the assumption $u_ju_i = \rho_{ij}u_iu_j$ for all $i<j$, $\rho_{ij}\in\mathbf{C}$. (See Theorems 1.10 and 2.7.) The coefficient $\binom{m}{k}_{\rho_{12}}$ is replaced by

$$\prod_{k<j} \rho_{kj}^{a_k a_j - T_{k,r_k}} \prod_{k>j} \rho_{jk}^{T_{k,r_k}} \prod_{\substack{k<\ell \\ k,\ell\neq j}} \rho_{k\ell}^{(a_k-r_k)r_\ell} C(r_1,\ldots,r_n;j).$$

For $n=3$ there is a concrete and simple description of $C(r_1,r_2,r_3;j)$, see Lemma 1.8. On the contrary for $n\ge 4$ n-nomial coefficients are quite complicated, cf. Definition 2.3 and Lemma 2.5.

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In Section 1 we will derive the noncommutative n-nomial formula for $n = 3$, while we detail the general case in Section 2.

Throughout this note we will assume $u_j u_i = \rho_{i,j} u_i u_j$ for all $i < j$, with $\rho_{i\,j} \in \mathbf{C}$, and denote by $\alpha(a_1, \dots, a_n)$ the coefficient of $u_1^{a_1}, \dots, u_n^{a_n}$ in the expansion of $(u_1 + \cdots + u_n)^{a_1 + \cdots + a_n}$. The following lemma will be repeatedly used in the sequel.

Lemma 0.1. *We have*

$$\alpha(a_1, \dots, a_n) = \sum_{i:a_i>0} \bigg(\prod_{w<i} \rho_{wi}^{a_w} \alpha(a - e_i) \bigg),$$

where $a = (a_1, \dots, a_n) \in \mathbf{N}^n$, and $(e_i)_{i=1,\dots,n}$ is the canonical basis of $\mathbf{Z}^n$.

Proof. Note that $(u_1 + \cdots + u_n)^{a_1 + \cdots + a_n} = (u_1 + \cdots + u_n)(u_1 + \cdots + u_n)^{a_1 + \cdots + a_n - 1}$. Now, by using induction, $\alpha(a_1, \dots, a_n) u_1^{a_1} \cdots u_n^{a_n} = \sum_{i:a_i>0} u_i \alpha(a - e_i) u_1^{a_1} \dots u_i^{a_i - 1} \dots u_n^{a_n}$. Since $u_j u_i = \rho_{ij} u_i u_j$ for all $i, j = 1, \dots, n$, the conclusion follows easily. Note also that $\alpha(k e_j) = 1$ for all $j = 1, \dots, n$, for all $k \in \mathbf{N}$. $\square$

1. The three-dimensional case. We will start by detailing two lemmas that will be useful in expressing $\alpha(\ell, k, s)$ in terms of lower order coefficients of type $\alpha(0, k - n, s - m)$, $0 \le n < k$, $0 \le m < s$.

Lemma 1.1. *We have, for all $k, s \ge 1$*

$$\alpha(\ell, k, s) = \sum_{j=1}^{\ell} \rho_{12}^{j} \alpha(j, k-1, s) + \sum_{j=1}^{\ell} \rho_{13}^{j} \rho_{23}^{k} \alpha(j, k, s-1) + \alpha(0, k, s).$$

Proof. By induction on $\ell + k + s$. For

$\ell = 0 : \alpha(0, k, s) = \alpha(0, k, s)$

$\ell = 1 : \alpha(1, k, s) = \alpha(0, k, s) + \rho_{12} \alpha(1, k-1, s) + \rho_{13} \rho_{23}^{k} \alpha(1, k, s-1)$

Hence Lemma 1.1 is true for $\ell \le 1$. Now suppose the formula is true for $\alpha(\ell, k, s)$. We will compute $\alpha(\ell + 1, k, s)$ by using Lemma 0.1.

$$\alpha(\ell+1, k, s) = \alpha(\ell, k, s) + \rho_{12}^{(\ell+1)} \alpha(\ell+1, k-1, s) + \rho_{13}^{\ell+1} \rho_{23}^{k} \alpha(\ell+1, k, s-1).$$

By using the induction hypothesis, we obtain

$$\begin{aligned}
&= \sum_{j=1}^{\ell} \rho_{12}^{j} \alpha(j, k-1, s) + \sum_{j=1}^{\ell} \rho_{13}^{j} \rho_{23}^{k} \alpha(j, k, s-1) + \alpha(0, k, s) \\
&\quad + \rho_{12}^{(\ell+1)} \alpha(\ell+1, k-1, s) + \rho_{13}^{\ell+1} \rho_{23}^{k} \alpha(\ell+1, k, s-1) \\
&= \sum_{j=1}^{\ell+1} \rho_{12}^{j} \alpha(j, k-1, s) + \sum_{j=1}^{\ell+1} \rho_{13}^{j} \rho_{23}^{k} \alpha(j, k, s-1) + \alpha(0, k, s). \qquad \square
\end{aligned}$$

By applying Lemma 1.1 twice, we obtain

Lemma 1.2. *For $k, s \geq 2$, we have*

$$\begin{aligned}
\alpha(\ell, k, s) &= \sum_{j=1}^{\ell} \sum_{t=1}^{j} \rho_{12}^{j+t} \alpha(t, k-2, s) + \sum_{j=1}^{\ell} \sum_{t=1}^{j} \rho_{13}^{j+t} \rho_{23}^{2k} \alpha(t, k, s-2) \\
&\quad + \sum_{j=1}^{\ell} \sum_{t=1}^{j} \rho_{12}^{j} \rho_{13}^{t} \rho_{23}^{k-1} (1 + \rho_{13}^{j-t} \rho_{23} \rho_{12}^{-(j-t)}) \alpha(t, k-1, s-1) \\
&\quad + \sum_{j=1}^{\ell} [\rho_{12}^{j} \alpha(0, k-1, s) + \rho_{13}^{j} \rho_{23}^{k} \alpha(0, k, s-1)] + \alpha(0, k, s)
\end{aligned}$$

Remark 1.3. Lemma 1.1 and Lemma 1.2 (and further applications of them) allow us to compute the coefficients of $\alpha(0, k-n, s-m)$, $0 \leq n < s$, $0 \leq m < s$, in the expansion of $\alpha(\ell, k, s)$, cf. Theorem 1.10. For example, we have the following terms with associated coefficients $(k, s \geq 3)$

Term: $\alpha(0, k-1, s)$, Coefficient: $\sum_{j=1}^{\ell} \rho_{12}^{j}$;

Term: $\alpha(0, k-2, s)$, Coefficient: $\sum_{j=1}^{\ell} \sum_{t=1}^{j} \rho_{12}^{j+t}$;

Term: $\alpha(0, k-1, s-1)$,

Coefficient: $\sum_{j=1}^{\ell} \sum_{t=1}^{j} \rho_{12}^{j} \rho_{13}^{t} \rho_{23}^{k-1} (1 + \rho_{13}^{j-t} \rho_{23} \rho_{12}^{-(j-t)})$.

In analogy with Lemma 1.1 and 1.2, one can compute the coefficients of $\alpha(\ell-n, 0, s-m)$, $0 \leq n < \ell$, $0 \leq m < s$, in the expansion of $\alpha(\ell, k, s)$. We will omit the proofs for brevity's sake.

Lemma 1.4. *We have*

$$\begin{aligned}\alpha(\ell, k, s) = \sum_{j=1}^{k} \alpha(\ell-1, j, s)\rho_{12}^{\ell(k-j)} + \sum_{j=1}^{k} \rho_{13}^{\ell}\rho_{23}^{j}\rho_{12}^{\ell(k-j)}\alpha(\ell, j, s-1) \\ + \rho_{12}^{\ell k}\alpha(\ell, 0, s), \quad \forall \ell, s \geq 1.\end{aligned}$$

Lemma 1.5. *For $s, t \geq 2$, we have*

$$\begin{aligned}\alpha(\ell, k, s) = {} & \sum_{j=1}^{k}\sum_{t=1}^{j} \rho_{12}^{\ell(k-t)-(j-t)}\alpha(\ell-2, t, s) \\ & + \sum_{j=1}^{k}\sum_{t=1}^{j} \rho_{12}^{\ell(k-t)-(j-t)}\rho_{13}^{\ell-1}\rho_{23}^{t}(1+\rho_{12}^{j-t}\rho_{13}\rho_{23}^{j-t}) \\ & \qquad\qquad \times \alpha(\ell-1, t, s-1) \\ & + \sum_{j=1}^{k}\sum_{t=1}^{j} \rho_{12}^{\ell(k-t)}\rho_{13}^{2\ell}\rho_{23}^{j+t}\alpha(\ell, t, s-2) \\ & + \sum_{j=1}^{k} \rho_{12}^{\ell k-j}[\alpha(\ell-1, 0, s) + \rho_{12}^{j}\rho_{13}^{\ell}\rho_{23}^{j}\alpha(\ell, 0, s-1)] \\ & + \rho_{12}^{\ell k}\alpha(\ell, 0, s).\end{aligned}$$

Finally, the following two lemmas will give expressions for the coefficients of $\alpha(\ell-n, k-m, 0)$ in the expansion of $\alpha(\ell, k, s)$, $0 \leq n < \ell$, $0 \leq m < k$.

Lemma 1.6. *For $\ell, k \geq 1$, we have*

$$\alpha(\ell,k,s) = \sum_{j=1}^{s} \rho_{13}^{\ell(s-j)} \rho_{23}^{k(s-j)} \alpha(\ell-1,k,j) + \sum_{j=1}^{s} \rho_{12}^{\ell} \rho_{13}^{\ell(s-j)} \rho_{23}^{k(s-j)} \alpha(\ell,k-1,j) + \rho_{13}^{\ell s} \rho_{23}^{ks} \alpha(\ell,k,0).$$

Lemma 1.7. *For $\ell, k \geq 2$, we have*

$$\begin{aligned} \alpha(\ell,k,s) = \sum_{j=1}^{s} \sum_{t=1}^{j} [&\rho_{13}^{\ell(s-t)-(j-t)} \rho_{23}^{k(s-t)} \alpha(\ell-2,k,t) \\ &+ \rho_{12}^{2\ell} \rho_{13}^{\ell(s-t)} \rho_{23}^{k(s-t)-(j-t)} \alpha(\ell,k-2,t)] \\ &+ \sum_{j=1}^{s} \sum_{t=1}^{j} \rho_{12}^{\ell-1} \rho_{13}^{\ell(s-t)-(j-t)} \rho_{23}^{k(s-t)} (1 + \rho_{13}^{j-t} \rho_{12} \rho_{23}^{-(j-t)}) \\ &\times \alpha(\ell-1,k-1,t) \\ &+ \sum_{j=1}^{s} [\rho_{13}^{\ell s-j} \rho_{23}^{ks} \alpha(\ell-1,k,0) + \rho_{12}^{\ell} \rho_{13}^{\ell s} \rho_{23}^{ks-j} \alpha(\ell,k-1,0)] \\ &+ \rho_{13}^{\ell s} \rho_{23}^{ks} \alpha(\ell,k,0). \end{aligned}$$

The following lemma will be used in the proof of the noncommutative 3-nomial formula.

Lemma 1.8. *For $N = 1,2,3$, let $N_1 = \min\{\{1,2,3\} - \{N\}\}$, $N_2 = \max\{\{1,2,3\} - \{N\}\}$ and $\rho = \rho_{N_1 N} \rho_{N N_2}$. For $n, m \geq 0$, and $z_1, \ldots, z_n; w_1, \ldots, w_m \in \mathbf{N}$, define*

$$C^{z_1,\ldots,z_n;w_1,\ldots,w_m}(n,m;N) = 1 + \sum_{r=1,\ldots,\min(n,m)} + \sum_{\substack{\alpha_1 < \cdots < \alpha_r \in \{1,\ldots,n\} \\ \beta_1 < \cdots < \beta_r \in \{1,\ldots,m\}}} \rho^{(z_{\alpha_1} + \cdots + z_{\alpha_r}) - (w_{\beta_1} + \cdots + w_{\beta_r})} \rho_{N_1 N_2}^{nr + \sum_{j=1}^{r} (\beta_j - \alpha_j)}.$$

(We will suppress the superscript $z_1, \ldots, z_n; w_1, \ldots, w_m$ whenever it is clear from the context which variables we are using.) Then, for $i_1, \ldots, i_n, j_1, \ldots, j_m \in \mathbf{N}$, we have

$$\begin{aligned} C^{i_1,\ldots,i_n;j_1,\ldots,j_m}(n,m;N) &= C^{i_2,\ldots,i_n;j_1,\ldots,j_m}(n-1,m;N) \\ &\quad + \rho^n_{N_1N_2}\rho^{(i_1-j_1)}C^{i_2,\ldots,i_n,j_1;j_2,\ldots,j_m}(n,m-1;N). \end{aligned}$$

Proof. The terms in $C(n,m;N)$ not containing i_1 add up exactly to $C(n-1,m;N)$. Now consider a term S in $C(n,m;N)$ of the form

$$\begin{aligned} S &= \rho^{(i_1+i_{\alpha_2}+\cdots+i_{\alpha_r})-(j_{\beta_1}+\cdots+j_{\beta_r})}\rho_{N_1N_2}^{nr+\sum_{j=2}^r(\beta_j-\alpha_j)+(\beta_1-1)} \\ &= \rho^n_{N_1N_2}\rho^{(i_1-j_1)}\rho^{(i_{\alpha_2}+\cdots+i_{\alpha_r}+j_1)-(j_{\beta_1}+\cdots+j_{\beta_r})} \\ &\quad\times \rho_{N_1N_2}^{n(r-1)+\sum_{j=2}^r(\beta_j-\alpha_j)+(\beta_1-1)}. \end{aligned}$$

The term $\widetilde{S}$ in $C(n,m-1;N)$ that corresponds to S is given by

$$\begin{aligned} \widetilde{S} &= \rho^{(i_{\alpha_2}+\cdots+i_{\alpha_r}+j_1)-(j_{\beta_1}+\cdots+j_{\beta_r})}\rho_{N_1N_2}^{nr+\sum_{j=2}^r(\beta_j-1-(\alpha_j-1))+(\beta_1-1-n)} \\ &= \rho^{(i_{\alpha_2}+\cdots+i_{\alpha_r}+j_1)-(j_{\beta_1}+\cdots+j_{\beta_r})}\rho_{N_1N_2}^{n(r-1)+\sum_{j=2}^r(\beta_j-\alpha_j)+(\beta_1-1)}. \end{aligned}$$

Hence $S = \rho^n_{N_1N_2}\rho^{(i_1-j_1)}\tilde{S}$, which proves the lemma. □

Example 1.9. If $N=3$, we have $\rho = \rho_{13}\rho_{32}$, and

$$\begin{aligned} C^{i_1i_2;j_1j_2}(2,2;3) &= 1 + \rho^{i_2-j_1}\rho_{12} + \rho^{i_1-j_1}\rho_{12}^2 + \rho^{i_2-j_2}\rho_{12}^2 \\ &\quad + \rho^{i_1-j_2}\rho_{12}^3 + \rho^{(i_1+i_2)-(j_1+j_2)}\rho_{12}^4, \\ C^{i_2;j_1j_2}(1,2;3) &= 1 + \rho^{i_2-j_1}\rho_{12} + \rho^{i_2-j_2}\rho_{12}^2, \\ C^{i_2j_1;j_2}(2,1;3) &= 1 + \rho^{j_1-j_2}\rho_{12} + \rho^{i_2-j_2}\rho_{12}^2. \end{aligned}$$

Hence $C(2,2;3) = C(1,2;3) + \rho_{12}^2\rho^{i_1-j_1}C(2,1;3)$. We can now formulate our noncommutative trinomial theorem. In the formula below, C is computed with respect to the standard variables $i_1, \ldots, i_n; j_1, \ldots, j_m$.

Theorem 1.10. *For $\ell, k, s > 0$, we have*

$$\begin{aligned}\alpha(\ell,k,s) = &\sum_{\substack{0\le n\le k-1\\ 0\le m\le s-1}} \sum_{T(\ell;n,m)} \rho_{12}^{I_n}\rho_{13}^{J_m}\rho_{23}^{m(k-n)} \\ &\qquad \times C(n,m;1)\binom{(k-n)+(s-m)}{k-n}_{2,3} \\ &+ \sum_{\substack{0\le n\le \ell-1\\ 0\le m\le s-1}} \sum_{T(k;n.m)} \rho_{12}^{\ell k-I_n}\rho_{13}^{m(\ell-n)}\rho_{23}^{J_m} \\ &\qquad \times C(n,m;2)\binom{(\ell-n)+(s-m)}{\ell-n}_{1,3} \\ &+ \sum_{\substack{0\le n\le \ell-1\\ 0\le m\le k-1}} \sum_{T(s;n.m)} \rho_{12}^{m(\ell-n)}\rho_{13}^{\ell s-I_n}\rho_{23}^{ks-J_m} \\ &\qquad \times C(n,m;3)\binom{(\ell-n)+(k-m)}{\ell-n}_{1,2},\end{aligned}$$

where we defined $I_0 = J_0 = 0$, $I_n = i_1 + \cdots + i_n$, for $n > 0$, $J_m = j_1 + \cdots + j_m$, for $m > 0$, and

$$\sum_{T(k;n,m)} = \sum_{i_1=1}^{k}\sum_{i_2=1}^{i_1}\cdots\sum_{i_n=1}^{i_{n-1}}\sum_{j_1=1}^{i_n}\sum_{j_2=1}^{j_1}\cdots\sum_{j_m=1}^{j_{m-1}}, \quad \text{for} \quad n>0,\ m\ge 0,$$

together with

$$\sum_{T(k;n,m)} = \sum_{j_1=1}^{k}\sum_{j_2=1}^{j_1}\cdots\sum_{j_m=1}^{j_{m-1}}, \quad \text{for} \quad n=0.$$

We also set, for brevity's sake, $\binom{r}{s}_{i,j} = \binom{r}{s}_{\rho_{ij}}$, $i,j = 1,2,3$, $i<j$.

Remark 1.11. We have, cf. [**2**],

$$\begin{aligned}\alpha(0,k-n,s-m) &= \binom{(k-n)+(s-m)}{k-n}_{2,3}\\ \alpha(\ell-n,0,s-m) &= \binom{(\ell-n)+(s-m)}{\ell-n}_{1,3}\\ \alpha(\ell-n,k-m,0) &= \binom{(\ell-n)+(k-m)}{\ell-n}_{1,2}\end{aligned}$$

Proof. We will prove the theorem by induction on $\ell+k+s=M\geq 3$. If $M=3$, we have, by Lemma 0.1,

$$\alpha(1,1,1)=\alpha(0,1,1)+\rho_{12}\alpha(1,0,1)+\rho_{13}\rho_{23}\alpha(1,1,0).$$

Hence the theorem is true for $M=3$. For the induction step, assuming the theorem is true for $\alpha(\ell,k,s)$, we will show it for $\alpha(\ell+1,k,s)$, and similar computations show it for $\alpha(\ell,k+1,s)$ and $\alpha(\ell,k,s+1)$ as well.

First compute $\alpha(\ell+1,k,s)$ by using Lemma 0.1,

$$\alpha(\ell+1,k,s)=\alpha(\ell,k,s)+\rho_{12}^{\ell+1}\alpha(\ell+1,k-1,s)+\rho_{13}^{\ell+1}\rho_{23}^{k}\alpha(\ell+1,k,s-1).$$

Using the induction hypothesis express $\alpha(\ell+1,k,s)$ in terms of lower weight α's. Then a straightforward computation by using Lemma 1.8 ends the proof of Theorem 1.10. $\square$

We can also express the coefficients of (1) in a slightly different form, by using the method illustrated above.

Proposition 1.12. *If $k,\ell>0$, $k+\ell=m$, we have (notation as in Theorem 1.10)*

$$\binom{m}{k}_{1,2}=\sum_{t=0}^{\ell-1}\sum_{j_1=1}^{k}\sum_{j_2=1}^{j_1}\cdots\sum_{j_t=1}^{j_{t-1}}\rho_{12}^{J_t}+\sum_{t=0}^{k-1}\sum_{i_1=1}^{\ell}\sum_{i_2=1}^{i_1}\cdots\sum_{i_t=1}^{i_{t-1}}\rho_{12}^{\ell k-I_t}.$$

Proof. Define

$$\beta(k,\ell)=\sum_{t=0}^{\ell-1}\sum_{j_1=1}^{k}\sum_{j_2=1}^{j_1}\cdots\sum_{j_t=1}^{j_{t-1}}\rho_{12}^{J_t}+\sum_{t=0}^{k-1}\sum_{i_1=1}^{\ell}\sum_{i_2=1}^{i_1}\cdots\sum_{i_t=1}^{i_{t-1}}\rho_{12}^{\ell k-I_t}.$$

A straightforward computation shows that

$$\beta(k+1,\ell)=\beta(k,\ell)+\rho_{12}^{k+1}\beta(k+1,\ell-1).$$

Moreover,

$$\binom{m+1}{k+1}_{1,2}=\binom{m}{k}_{1,2}+\rho_{12}^{k+1}\binom{m}{k+1}_{1,2}.$$

Now we can use induction on $m = k + \ell$ to show that Proposition 1.12 holds. $\square$

2. The n-dimensional case ($n \geq 3$). We will start by deriving analogues of Lemmas 1.1, 1.4 and 1.6.

Lemma 2.1. *For any $j \in \{1, \dots, n\}$, and for all $a = (a_1, \dots, a_n) \in \mathbf{N}^n$, we have*

$$\begin{aligned}\alpha(a) = &\prod_{k<j} \rho_{kj}^{a_k a_j} \alpha(a - a_j e_j) \\ &+ \sum_{t=1}^{a_j} \Bigg(\sum_{\substack{i: a_i > 0 \\ i \neq j}} \prod_{k<j} \rho_{kj}^{a_k(a_j - t)} \prod_{\substack{k<i \\ k \neq j}} \rho_{ki}^{a_k} \begin{Bmatrix} 1 & if \quad j > i \\ \rho_{ji}^t & if \quad j < i \end{Bmatrix} \\ &\qquad\qquad\qquad\qquad \times \alpha(a + (t - a_j)e_j - e_i) \Bigg),\end{aligned}$$

where e_i, $i = 1, \dots, n$, is the canonical basis for $\mathbf{R}^n$.

Proof. The proof is a straightforward generalization of the proofs of Lemmas 1.1, 1.4 and 1.6. $\square$

We now need a higher-dimensional version of Lemma 1.8. For simplicity's sake, we will only detail the case $j = 1$.

Definition 2.2 ($j = 1$). A path $\mathcal{P}$ from $a - a_1 e_1 = (0, a_2, \dots, a_n)$ to $a - a_1 e_1 - \sum_{k=2}^n r_k = (0, a_2 - r_2, \dots, a_n - r_n)$, $0 \leq r_t \leq a_t - 1$, is any ordered string $\mathcal{P} = \{i_k\}_{k=1,\dots,r_2+\cdots+r_n}$, $i_k \in \{1, \dots, n\}$, such that $i_k = t$ for exactly r_t distinct values $k(t,1) < \cdots < k(t, r_t)$ of k, $t = 2, \dots, n$. We will say that the path $\mathcal{S}$

$$\begin{aligned}\mathcal{S} = \{i_k = 2, \text{ for } k = 1, \dots, r_2; \dots; i_k = n, \text{ for } k = r_2 + \cdots + r_{n-1}, \dots, \\ r_2 + \cdots + r_{n-1} + r_n\}\end{aligned}$$

is the standard path from $a - a_1 e_1$ to $a - a_1 e_1 - \sum_{k=2}^n r_k e_k$. We will also call $t_{s\,w}$, $s = 2, \dots, n$, $w = 1, \dots, r_s$, the standard variables associated to the standard path $\mathcal{S}$. Set

$$t'_1 = t_{2\,1}, \dots, t'_{r_2} = t_{2\,r_2}, t'_{r_2+1} = t_{3\,1}, \dots, t'_{r_2+\cdots+r_{n-1}} = t_{n-1\,r_{n-1}}, t'_{r_2+\cdots+r_{n-1}+1} = t_{n\,1}, \dots, t'_{r_2+\cdots+r_n} = t_{n\,r_n}.$$

Now define $E(\mathcal{P})$ by the following formula:

$$E(\mathcal{P}) = \left[\rho_{23}^{\sum_{w=1}^{r_3} a^{\mathcal{P}}_{2,k(3,w)}} \left(\rho_{24}^{\sum_{w=1}^{r_4} a^{\mathcal{P}}_{2,k(4,w)}} \rho_{34}^{\sum_{w=1}^{r_4} a^{\mathcal{P}}_{3,k(4,w)}}\right) \cdots \left(\rho_{2n}^{\sum_{w=1}^{r_n} a^{\mathcal{P}}_{2,k(n,w)}} \cdots \rho_{n-1n}^{\sum_{w=1}^{r_n} a^{\mathcal{P}}_{n-1,k(n,w)}}\right)\right] \left(\rho_{12}^{\sum_{w=1}^{r_2} t'_{k(2,w)}} \cdots \rho_{1n}^{\sum_{w=1}^{r_n} t'_{k(n,w)}}\right),$$

where we set $a^{\mathcal{P}}_{s,k(\ell,w)}$ equal to $a_s - q_w$, where q_w is the number of indices less than $k(\ell, w)$ such that $i_k = s$, $w = 1, \dots, r_\ell$.

Now, for $s, \ell = 2, \dots, n$, the minimum value $(= (a_s - r_s)r_\ell)$ of

$$\sum_{w=1}^{r_\ell} a^{\mathcal{P}}_{s,k(\ell,w)}$$

is attained on the paths $\mathcal{Q}$ with $i_k = \ell$ for $k = (r_2 + \cdots + r_n) - r_\ell + 1, \dots, (r_2 + \cdots + r_\ell + \cdots + r_n)$. Hence, for any path $\mathcal{P}$ from $a - a_1 e_1$ to $a - a_1 e_1 - \sum_{k=2}^n r_k e_k$,

$$E(\mathcal{P}) = \prod_{1<s<\ell} \rho_{s\ell}^{(a_s - r_s) r_\ell} \prod_{1<s<\ell} \rho_{s\ell}^{\sum_{w=1}^{r_\ell} (a^{\mathcal{P}}_{s,k(\ell,w)} - (a_s - r_s))} \prod_{\ell=2}^{n} \rho_{1\ell}^{t_{\ell\,1} + \cdots + t_{\ell\,r_\ell}} \prod_{\ell=2}^{n} \rho_{1\ell}^{\sum_{w=1}^{r_\ell} (t'_{k(\ell,w)} - t_{\ell\,w})}.$$

Definition 2.3 ($j = 1$). Define $C^{t_{2\,1},\dots,t_{2\,r_2}; t_{3\,1},\dots,t_{3\,r_3};\dots;t_{n\,1},\dots,t_{n\,r_n}} \times (r_1, r_2, \dots, r_n; 1)$ with respect to the standard variables $t_{2\,1}, \dots, t_{2\,r_2}$; $t_{3\,1}, \dots, t_{3\,r_3}; \dots; t_{n\,1}, \dots, t_{n\,r_n}$ by

$$C^{t_{2\,1},\dots,t_{2\,r_2}; t_{3\,1},\dots,t_{3\,r_3};\dots;t_{n\,1},\dots,t_{n\,r_n}}(r_1, r_2, \dots, r_n; 1) = \sum_{\substack{\text{all paths } \mathcal{P} \\ \text{from } a - a_1 e_1 \text{ to} \\ a - a_1 e_1 - \sum_{k=2}^n r_k e_k}} \prod_{1<s<\ell} \rho_{s\ell}^{\sum_{w=1}^{\ell} (a^{\mathcal{P}}_{s,k(\ell,w)} - (a_s - r_s))} \prod_{\ell=2}^{n} \rho_{1\ell}^{\sum_{w=1}^{r_\ell} (t'_{k(\ell,w)} - t_{\ell\,w})}.$$

(We will drop the superscript when no confusion can arise.)

Example 2.4.

1.

$$\begin{aligned}C^{t_{2\,1};t_{3\,1};t_{4\,1}}(0,1,1,1;1) = 1 &+ \rho_{34}\rho_{13}^{t_{4\,1}-t_{3\,1}}\rho_{14}^{t_{3\,1}-t_{4\,1}} + \rho_{23}\rho_{12}^{t_{3\,1}-t_{2\,1}}\rho_{13}^{t_{2\,1}-t_{3\,1}}\\ &+ \rho_{23}\rho_{24}\rho_{34}\rho_{12}^{t_{4\,1}-t_{2\,1}}\rho_{14}^{t_{2\,1}-t_{4\,1}}\\ &+ \rho_{23}\rho_{24}\rho_{12}^{t_{4\,1}-t_{2\,1}}\rho_{13}^{t_{2\,1}-t_{3\,1}}\rho_{14}^{t_{3\,1}-t_{4\,1}}\\ &+ \rho_{24}\rho_{34}\rho_{12}^{t_{3\,1}-t_{2\,1}}\rho_{13}^{t_{4\,1}\rho_{14}^{t_{2\,1}-t_{4\,1}}-t_{3\,1}}\end{aligned}$$

2.

$$\begin{aligned}C^{t_{2\,1};t_{3\,1},t_{3\,2};t_{4\,1}}(0,1,2,1;1) = 1 &+ \rho_{23}\rho_{[2,3]}^{t_{2\,1}t_{3\,1}} + \rho_{23}^2\rho_{[2,3]}^{t_{2\,1}t_{3\,2}}\\ &+ \rho_{34}\rho_{[3,4]}^{t_{3\,2}t_{4\,1}} + \rho_{34}^2\rho_{[3,4]}^{t_{3\,1}t_{4\,1}}\\ &+ \rho_{23}\rho_{34}\rho_{[2,3,4,3]}^{t_{2\,1}t_{3\,1}t_{4\,1}t_{3\,2}}\\ &+ \rho_{23}\rho_{24}\rho_{34}\rho_{[2,3,4,3]}^{t_{2\,1}t_{3\,2}t_{4\,1}t_{3\,1}}\\ &+ \rho_{23}^2\rho_{2\,4}\rho_{[2,4,3]}^{t_{2\,1}t_{4\,1}t_{3\,2}}\\ &+ \rho_{23}^2\rho_{24}\rho_{34}\rho_{[2,4,3]}^{t_{2\,1}t_{4\,1}t_{3\,1}} + \rho_{24}\rho_{34}^2\rho_{[2,3,4]}^{t_{2\,1}t_{3\,1}t_{4\,1}}\\ &+ \rho_{24}\rho_{34}^2\rho_{23}\rho_{[2,3,4]}^{t_{2\,1}t_{3\,2}t_{4\,1}} + \rho_{24}\rho_{34}^2\rho_{23}^2\rho_{[2,4]}^{t_{2\,1}t_{4\,1}}\end{aligned}$$

where $\rho_{[i,j]}^{t_{i\,k}t_{j\,\ell}} = \rho_{i1}^{t_{i\,k}-t_{j\,\ell}}\rho_{j1}^{t_{j\,\ell}-t_{i\,k}}$ and $\rho_{[i,j,r]}^{t_{i\,k}t_{j\,\ell}t_{r\,s}} = \rho_{i1}^{t_{i\,k}-t_{j\,\ell}}\rho_{j1}^{t_{j\,\ell}-t_{r\,s}} \times \rho_{r1}^{t_{r\,s}-t_{i\,k}}$, cf. Lemma 2.5.

The following lemma, used in the proof of Theorem 2.7, describes an addition law for the $C(r_1,\ldots,r_n;1)$'s.

Lemma 2.5. *We have*

$$\begin{aligned}&C^{t_{2\,1},\ldots,t_{2\,r_2};t_{3\,1},\ldots,t_{3\,r_3};\ldots;t_{n\,1},\ldots,t_{n\,r_n}}(r_1,r_2,\ldots,r_n;1)\\ &= C^{t_{2\,2},\ldots,t_{2\,r_2};t_{3\,1},\ldots,t_{3\,r_3};\ldots;t_{n\,1},\ldots,t_{n\,r_n}}(r_1,r_2-1,r_3,\ldots,r_n;1)\\ &\quad+ \rho_{23}^{r_2}\rho_{[2,3]}^{t_{2\,1}t_{3\,1}}C^{t_{2\,2},\ldots,t_{2\,r_2},t_{3\,1};\ldots;t_{n\,1},\ldots,t_{n\,r_n}}(r_1,r_2,r_3-1,\ldots,r_n;1)+\cdots\\ &\quad+\rho_{2s}^{r_2}\rho_{3s}^{r_3}\cdots\rho_{s-1\,s}^{r_{s-1}}\rho_{[2,3,\ldots,s]}^{t_{2\,1}t_{3\,1}\ldots t_{s\,1}}C^{t_{2\,2},\ldots,t_{2\,r_2},t_{3\,1};\ldots;t_{s\,2},\ldots,t_{s\,r_s};\ldots;t_{n\,1},\ldots,t_{n\,r_n}}\\ &\quad\times(r_1,r_2,\ldots,r_s-1,\ldots,r_n;1) + \rho_{2n}^{r_2}\rho_{3n}^{r_3}\cdots\rho_{n-1\,n}^{r_{n-1}}\rho_{[2,3,\ldots,n]}^{t_{2\,1}t_{3\,1}\ldots t_{n\,1}}\\ &\quad\times C^{t_{2\,2},\ldots,t_{2\,r_2},t_{3\,1};\ldots;t_{n\,2},\ldots,t_{n\,r_n}}(r_1,r_2,\ldots,r_n-1;1),\end{aligned}$$

where

$$\rho_{[2,3,\dots,s]}^{t_{2\,1}t_{3\,1},\cdots,t_{s\,1}} = \rho_{12}^{t_{3\,1}-t_{2\,1}}\rho_{13}^{t_{4\,1}-t_{3\,1}}\cdots\rho_{1s}^{t_{2\,1}-t_{s\,1}}$$

Proof. Straightforward computation. □

As can be seen from Example 2.4, formulas quickly become very complicated, even in dimension 4. However, it is in principle possible to compute $C(r_1,\dots,r_n;1)$, for any $(r_1,\dots,r_n)$, by using the addition formula in Lemma 2.5.

Definition 2.6. Define $T(k;(r_1,\dots,r_n);j)$, $r_1,\dots,r_{j-1},r_{j+1},\dots,r_n \in \mathbf{N}$ (there is no r_j), $k,n\in\mathbf{N}$, $j\in\{1,\dots,n\}$, by

$$T(k;(r_1,\dots,r_n);j) = \sum_{t_{1\,1}=1}^{k}\sum_{t_{1\,2}=1}^{t_{1\,1}}\cdots\sum_{t_{1\,r_1}=1}^{t_{1\,r_1-1}}\sum_{t_{2\,1}=1}^{t_{1\,r_1}}\cdots\sum_{t_{j-1\,r_{j-1}}=1}^{t_{j-1\,r_{j-1}-1}}\sum_{t_{j+1\,1}=1}^{t_{j-1\,r_{j-1}}}\cdots\sum_{t_{n\,r_n}=1}^{t_{n\,r_n-1}},\ j\neq 1,n,$$

$$T(k;(r_1,\dots,r_n);j) = \sum_{t_{2\,1}=1}^{k}\sum_{t_{2\,2}=1}^{t_{2\,1}}\cdots\sum_{t_{2\,r_2}=1}^{t_{2,r_2-1}}\sum_{t_{3\,1}=1}^{t_{2\,r_2}}\cdots\sum_{t_{n\,r_n}=1}^{t_{n\,r_n-1}}, \quad j=1,$$

$$T(k;(r_1,\dots,r_n);j) = \sum_{t_{1\,1}=1}^{k}\sum_{t_{1\,2}=1}^{t_{1\,1}}\cdots\sum_{t_{1\,r_1}=1}^{t_{1\,r_1-1}}\sum_{t_{2\,1}=1}^{t_{1\,r_1}}\cdots\sum_{t_{n-1\,r_{n-1}}=1}^{t_{n-1\,r_{n-1}-1}}, \quad j=n.$$

Also define, for $s=1,\dots,n$, $s\neq j$, $T_{s,0}=0$ and $T_{s,r_s}=\sum_{j_1=1}^{r_s} t_{s\,j_1}$, $r_s>0$.

Theorem 2.7 (Non-commutative n-nomial formula). *We have, for* $a = (a_1, \dots, a_n) \in \mathbf{N}^n$, $a_j > 0$, *for all* $j = 1, \dots, n$

$$\alpha(a) = \sum_{j=1}^{n} \sum_{\substack{0 \le r_t \le a_t - 1 \\ t=1,\dots,n, t \ne j}} \sum_{T(a_j;(r_1,\dots,r_n);j)} \prod_{k<j} \rho_{kj}^{a_k a_j - T_{k,r_k}}$$
$$\prod_{k>j} \rho_{jk}^{T_{k,r_k}} \prod_{\substack{k<\ell \\ k,\ell \ne j}} \rho_{k\ell}^{(a_k - r_k) r_\ell} C(r_1, \dots, r_n; j) \alpha\left(\sum_{\substack{t=1 \\ t \ne j}}^{n} (a_t - r_t) e_t \right).$$

Proof. We will proceed by induction on $M = \sum_{t=1}^n a_t$. If $M = n$, $a_t = 1$, for all t, and by Lemma 0.1

$$\alpha(1, 1, \dots, 1) = \sum_{j=1}^{n} \prod_{k<j} \rho_{kj} \alpha((1, \dots .1) - e_j).$$

Hence, the result is true for $M = n$.

For the induction step, it is enough to show that our formula holds for $\alpha(a + e_1)$, provided it holds for $\alpha(a)$. (Analogously it can be shown that it also holds for $\alpha(a + e_t)$, $t = 2, \dots, n$.)

The proof is analogous to the proof of Theorem 1.10. It involves the induction hypothesis, Lemma 0.1, a change of variables and Lemma 2.5. □

In the case $\rho_{ij} = \rho_j$, for all $i < j$, the noncommutative n-nomial formula simplifies to an n variable generalization of Potter/Schützenberger's theorem.

Example 2.8. For any $m \in \mathbf{N}$, we have

$$(u_1 + \dots + u_n)^m = \sum_{k_1 + \dots + k_n = m} C(k_1, \dots, k_n)\ u_1^{k_1} \dots u_n^{k_n}$$

where

$$C(k_1,\dots,k_n) = \binom{k_1}{k_1}_{\rho_1}\binom{k_1+k_2}{k_2}_{\rho_2}\binom{k_1+k_2+k_3}{k_3}_{\rho_3}\cdots \binom{k_1+\cdots+k_n}{k_n}_{\rho_n}, \quad \rho_j \in \mathbf{C}.$$

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