CURVES IN BANACH SPACES–DIFFERENTIABILITY VIA HOMEOMORPHISMS

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ABSTRACT. We prove several results on curves $f : [0, 1] \to X$, where X is an arbitrary real Banach space. They generalize theorems which were proved by Zahorski, Tolstov, Choquet and Bari in the case $X = \mathbf{R}^n$. First we give a complete characterization of those f that admit an equivalent parametrization which has a continuous derivative (respectively with continuous derivative which is non-zero everywhere or almost everywhere). Further we establish theorems characterizing curves allowing boundedly or finitely differentiable parametrizations (with almost everywhere nonzero derivative). As a tool, we prove versions of the aforementioned theorems for metric analogues of derivatives. Finally, we discuss the case of curves allowing almost everywhere differentiable parametrizations. We also answer several questions posed by Bruckner.

1. Introduction. We prove several results on curves $f : [0, 1] \to X$, where X is an arbitrary real Banach space. Our results give a complete characterization of several situations when there exists an equivalent parametrization of a curve possessing various differentiability properties. They generalize theorems which were known (to our knowledge) for the case $X = \mathbf{R}^n$ only. For some proofs we need, besides the known methods used in the case $X = \mathbf{R}^n$ and results on metric differentiability of Lipschitz (and pointwise-Lipschitz) mappings (from [**10, 14**]), also some new ideas.

Our result on C^1-parametrizations (Theorem 3.1) generalizes a theorem of Tolstov [**17**] for curves with values in the Euclidean space $\mathbf{R}^n$. Note that in [**17**], only curves, which are non-constant on any interval, are considered, and that the result for real functions (possibly constant on an interval) was proved independently by Bruckner and

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where f is strictly monotone, Tolstov considers intervals where f is tangentially smooth. (A related notion of one-sidedly smooth curves was studied in [**1**] for $X = \mathbf{R}^n$ and in [**9**] in the general case.) In our proof we use a result from [**9**] on this notion, see Lemma 2.5, and the (simplified) method from [**13**]. Following [**17**], we also characterize those curves, which allow a parametrization with a continuous derivative that is nonzero almost everywhere, respectively everywhere, see Theorem 3.3, respectively Theorem 3.4.

A simple characterization for $X = \mathbf{R}^n$, namely boudedness of variation, of those f that allow boundedly differentiable parametrizations was proved by Zahorski [**19**] and Choquet [**8**] (and independently in [**6**], cf. [**5**, page 87]). This result clearly holds (with the "same" proof) if (and only if) X has the Radon-Nikodým property. We give in Theorem 4.3 a more complicated characterization which holds for an arbitrary X. The proof is an application of a result on metric differentiability of Kirchheim [**14**]. We also consider (following [**8, 17, 19**]) the case when the bounded derivative can be taken almost everywhere nonzero.

Zahorski [**19**] and Choquet [**8**] (see also Tolstov [**17**]) also proved a (more difficult) result characterizing curves (with $X = \mathbf{R}^n$) that allow a differentiable parametrization as those curves having the VBG_* property. In Section 5 we also generalize this result. Theorem 5.6, which characterizes the situation when a vector-valued function allows a differentiable parametrization with an almost everywhere nonzero derivative, is the deepest result of this paper. In the proof, we combine ideas from [**8, 17**] together with a new idea. As a consequence, we prove the generalization of Zahorski's theorem on differentiable parametrization of a curve having a tangent at all points except a countable set.

The results mentioned above are proved via analogues of these theorems for the notions arising in the theory of metric differentiability, see Definition 2.1. These analogues are formulated for mappings $f : [0, 1] \to X$, where X is a real Banach space. However, every metric space (M, ρ) embeds into a suitable $\ell_\infty(\Gamma)$, (see, e.g., [**7**, Lemma 1.1]), and thus all the results involving the notion of $\mathrm{md}\,(f, \cdot)$ (and metric differentiability) are true for mappings $f : [a, b] \to (M, \rho)$ and are direct consequences of our theorems.

bility, which might be interesting in their own right, see e.g., Lemma 2.4 and Theorem 2.11.

We also observe that in several situations the obtained homeomorphic changes of variables can be taken Lipschitz and differentiable, and thus answer three question posed by Bruckner in [**5**].

Finally, we show that each continuous $f : [0,1] \to X$ allows a parametrization, which is differentiable at all points of a prescribed first category set. This improves a result of Bari [**2**] who proved the existence of an almost everywhere differentiable parametrization in the case $X = \mathbf{R}$, cf. [**4**], where also discontinuous f are considered in the scalar case.

2. Auxiliary results. By X we shall denote a real Banach space. By λ we shall denote the Lebesgue measure on $\mathbf{R}$, and by $\mathcal{H}^1$ we denote the one-dimensional Hausdorff measure. If $x, r \in \mathbf{R}$, $r > 0$, then $B(x,r) := \{y \in \mathbf{R} : |x-y| < r\}$. Let $f : [a,b] \to X$. By $\mathrm{O}sc\,(f, M)$ we mean the oscillation of f on a set M. The symbol $\bigvee_c^d f$ stands for the variation of f on $[c,d] \subset [a,b]$. We say that f *has bounded variation on* $[a,b]$ provided $\bigvee_a^b f < \infty$. We will denote $v_f(x) := \bigvee_a^x f$ for $x \in [a,b]$. For $E \subset [a,b]$, we define $V(f,E)$ to be the supremum of the set of the sums

$$\sum_{i=1}^{n} \|f(d_i) - f(c_i)\|$$

over sequences $c_1 < d_1 \le c_2 < \cdots \le c_n < d_n$ with $c_i, d_i \in E$ for all $i = 1, \ldots, n$. (We put $V(f,E) = 0$ provided $\mathrm{card}\,(E) \le 1$.)

We say that $f : [a,b] \to X$ is VBG_* provided $[a,b] = \cup_{n \in \mathbf{N}} E_n$, where E_n, $n \in \mathbf{N}$, are closed sets such that for all $n \in \mathbf{N}$, $V(f, E_n) < \infty$, and $\sum_{i \in \mathcal{I}_n} \mathrm{O}sc\,(f, I_n^i) < \infty$, where I_n^i $(i \in \mathcal{I}_n \subset \mathbf{N})$ are all (closed) intervals contiguous to E_n. (Note that these properties are trivially satisfied provided $\mathrm{card}\, E_n \le 1$.) For a formally different but equivalent definition, see Remark 2.15.

We say that $f : [a,b] \to X$ is *pointwise Lipschitz at* $x \in [a,b]$, provided

$$\limsup_{\substack{y \to x, \\ y \in [a,b]}} \frac{\|f(x) - f(y)\|}{|x-y|} < \infty \tag{2.1}$$

Lipschitz. If

$$\|f(x) - f(y)\| \le C \cdot |x - y| \quad \text{for all} \quad x, y \in [a, b],$$

for some $C > 0$, then we say that f is *Lipschitz* (*or* C*-Lipschitz*) *on* $[a, b]$.

If $x \in [a, b]$, then we define the derivative $f'(x)$ as

$$f'(x) := \lim_{\substack{t \to 0 \\ x+t \in [a,b]}} \frac{f(x+t) - f(x)}{t},$$

provided the limit exists.

An important tool for us will be Kirchheim's theory of "metric differentiation" developed in [**14**]. For our purposes (we deal with the one-dimensional domain only) it is sufficient to define (instead of $MD(f, x)(v)$ in [**14**]) the simpler notion $\operatorname{md}(f, x)$.

Definition 2.1. Let I be an interval, X a Banach space and $f : I \to X$ be given. Then we put

$$\operatorname{md}(f, x) = \lim_{\substack{t \to 0 \\ t+x \in I}} \frac{\|f(x+t) - f(x)\|}{|t|} \quad \text{for} \quad x \in I,$$

whenever the finite limit exists. We say that f *is metrically differentiable at* x provided $\operatorname{md}(f, x)$ exists and

$$\begin{aligned} \|f(z) - f(y)\| - \operatorname{md}(f, x) \cdot |z - y| = o(|z - x| + |y - x|), \\ (y, z) \to (x, x), \quad y, z \in I. \end{aligned} \tag{2.2}$$

Let us remark that $\operatorname{md}(f, x)$ exists if and only if $\operatorname{MD}(f, x)(1) = \operatorname{MD}(f, x)(-1)$; in that case all three numbers coincide. The above notion of metric differentiability clearly coincides with that of [**14**]. It is easy to see that if $f'(x)$ exists, then $\operatorname{md}(f, x) = \|f'(x)\|$, and f is metrically differentiable at x.

Let $f : [0, 1] \to X$ and $A \subset [0, 1]$. By $N(f|_A, y)$ we will denote the number of elements (possibly ∞) of the set $f^{-1}(\{y\}) \cap A$ and set $\langle f \rangle := f([0, 1])$. We will use the following version of Sard's theorem:

Then

$$\mathcal{H}^1(f(\{x \in [0,1]: \mathrm{md}\,(f,x) = 0\})) = 0.$$

Proof. Extend f to $\mathbf{R}$ by taking $f(x) = 0$ for $x \in \mathbf{R} \setminus [0,1]$. Let $\tilde{A} = \{x \in \mathbf{R} : \mathrm{md}\,(f,x) = 0\}$. For $m,n \in \mathbf{N}$, let

$$A_{mn} = \{x \in \mathbf{R} : \|f(x+h) - f(x)\| \le m^{-1}\,|h| \quad \text{for} \quad |h| < n^{-1}\}.$$

Each A_{mn} is closed by, e.g., [**3**, Lemma 1]. Note that $\tilde{A} = \cap_m \cup_n A_{mn}$, and so $\tilde{A}$ (and thus also $A := \tilde{A} \cap [0,1]$) is Borel. By [**10**, Theorem 2.12] we obtain that

$$0 = \int_A \mathrm{md}\,(f,x)\,dx = \int_{f(A)} N(f|_A, y)\,d\mathcal{H}^1(y) \ge \mathcal{H}^1(f(A)). \qquad \square$$

Theorem 2.5 from [**10**] has the following:

Corollary 2.3. *Let X be a Banach space and $f : [a,b] \to X$ pointwise Lipschitz. Then f is metrically differentiable at almost all $x \in [a,b]$.*

The following lemma is simple:

Lemma 2.4. *Let $f : [c,d] \to X$, $x \in [c,d]$. Then the following hold.*

(i) *If $\mathrm{md}\,(f,x) = 0$, then f is metrically differentiable at x.*

(ii) *If $h : [a,b] \to [c,d]$ is differentiable at $w \in [a,b]$, $h(w) = x$, and f is metrically differentiable at x, then $f \circ h$ is metrically differentiable at w, and $\mathrm{md}\,(f \circ h, w) = \mathrm{md}\,(f,x) \cdot |h'(w)|$.*

Proof. Without any loss of generality, we can assume that $w = 0$, $x = 0$, $f(0) = 0$. Concerning (i), consider an arbitrary $\varepsilon > 0$. There exists $\delta > 0$ such that for $y,z \in B(0,\delta) \cap [c,d]$ we have

$$\|f(y) - f(z)\| \le \|f(y)\| + \|f(z)\| \le \varepsilon(|y| + |z|),$$

and the conclusion follows.

0. Thus using (i), we can assume that $h'(0) \neq 0$. Then we get

$$\begin{aligned}
&\big|\|f(h(y)) - f(h(z))\| - \mathrm{md}\,(f,0)\cdot|h'(0)|\cdot|y-z|\big| \\
&\le \big|\|f(h(y)) - f(h(z))\| - \mathrm{md}\,(f,0)\cdot|h(y)-h(z)|\big| \\
&\quad + \big|\mathrm{md}\,(f,0)\cdot|h(y)-h(z)| - \mathrm{md}\,(f,0)\cdot|h'(0)|\cdot|y-z|\big| \\
&\le o(|h(y)|+|h(z)|) + \mathrm{md}\,(f,0)\cdot|h(y)-h(z)-h'(0)(y-z)| \\
&\le o(|h(y)|+|h(z)|) + \mathrm{md}\,(f,0)\cdot\big(|h(y)-h'(0)\cdot y| + |h(z)-h'(0)\cdot z|\big) \\
&= o(|y|+|z|),
\end{aligned}$$

when $(y,z)\to(0,0)$, $y,z\in[a,b]$. □

Let now X be a Banach space, $I\subset \mathbf{R}$ an arbitrary interval, and $f: I\to X$ given. The *unit tangent vector* of f at $x\in I$ is defined as the limit

$$\tau(f,x) = \lim_{\substack{t\to 0\\ x+t\in I}} \mathrm{sgn}\,(t)\cdot\frac{f(x+t)-f(x)}{\|f(x+t)-f(x)\|}.$$

A continuous function $f: I\to X$ is said to be *tangentially smooth* if $\tau(f,\cdot)$ exists and is continuous on I. We will frequently use the obvious fact that $\tau(f,x) = (f'(x)/\|f'(x)\|)$ whenever $f'(x)\neq 0$. It is easy to see that if $f'(x)\neq 0$ for all $x\in I$ and f' is continuous on I, then f is tangentially smooth on I. Note that if h is a homeomorphism of an interval J onto I and $x\in J$, then

(2.3) $\tau(f\circ h, x)$ exists if and only if $\tau(f,h(x))$ exists.

[**9**, Theorem 3.5] implies that tangentially-smooth curves are one-sidedly smooth in the sense of [**9**]. Note that if f is tangentially smooth in an interval I, then f has finite variation on any compact subinterval of I (this follows from [**9**, Corollary 3.4]). Thus, the following result follows from [**9**, Theorem 3.5 and Proposition 3.6]. (In case when $X=\mathbf{R}^n$ we can use [**1**, Theorem 3.3.3] or [**17**, Lemma 1] instead.)

Note that if I is an interval and $f: I\to X$ is tangentially smooth, then f is not constant on any subinterval of I.

Lemma 2.5. *Let X be a Banach space, $f:(c,d)\to X$ tangentially smooth and $k:(c,d)\to(c^*,d^*)$ a homeomorphism such that* $\vee_x^y f =$

nonzero derivative on (c^*, d^*).

Proof. Choose any $[a,b] \subset (c^*, d^*)$. It is clearly sufficient to prove that the function $g(x) := f \circ k^{-1}(x)$, $x \in [a,b]$, is C^1. We see that g is not constant on any interval and by [**9**, Theorem 3.5] we obtain that g is one-sidedly smooth in the sense of [**9**]. Put $\ell = \bigvee_a^b g$. Since $g^*(y) := g(y+a)$, $y \in [0,\ell]$, is the arc-length parametrization of $f|_{k^{-1}([a,b])}$, by [**9**, Proposition 3.6] we clearly get that $(g^*)'$ is continuous and nonzero on $[0,\ell]$ (as $(g^*)'(x) = \tau(g^*, x)$ for all $x \in [0,\ell]$). Thus $g' = (f \circ k^{-1})'$ is continuous and nonzero on $[a,b]$. □

Several times we will need the following easy consequence of Sard's theorem.

Lemma 2.6. *Let X be a Banach space and suppose that for $f : [0,1] \to X$ there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has derivative everywhere in $[0,1]$. Then*

$$\mathcal{H}^1(f(\{x \in [0,1] : \tau(f,x) \quad \textit{does not exist}\})) = 0.$$

Proof. We define

$$A := \{x \in [0,1] : \tau(f \circ h, x) \text{ does not exist}\},$$

and

$$D := \{x \in [0,1] : \tau(f,x) \text{ does not exist}\}.$$

Since

$$A \subset \{x \in [0,1] : (f \circ h)'(x) = 0\} =: B,$$

by Lemma 2.2, we have $\mathcal{H}^1((f \circ h)(B)) = 0$. By (2.3), $f(D) = (f \circ h)(A)$, and thus $\mathcal{H}^1(f(D)) = 0$. □

We shall need the following lemma.

closed and $f : [0,1] \to X$ *continuous. If* $\mathcal{H}^1(f(B)) = 0$, *then*

$$\bigvee_0^1 f = \sum_{i \in \mathcal{I}} \bigvee_{c_i}^{d_i} f,$$

where $I_i = (c_i, d_i)$, $(i \in \mathcal{I} \subset \mathbf{N})$ *are all (pairwise different) components of* $[0,1] \setminus B$.

Proof. Using the vector version of Banach indicatrix theorem ([**12**, Theorem 2.10.13]) and the obvious equality $N(f,y) = \sum_{i \in I} N(f|_{I_i}, y)$ for $y \in \langle f \rangle \setminus f(B)$, we obtain

$$\begin{aligned} \bigvee_0^1 f &= \int_{\langle f \rangle} N(f,y)\, d\mathcal{H}^1(y) = \int_{\langle f \rangle \setminus f(B)} N(f,y)\, d\mathcal{H}^1(y) \\ &= \sum_{i \in \mathcal{I}} \int_{\langle f \rangle \setminus f(B)} N(f|_{I_i}, y)\, d\mathcal{H}^1(y) = \sum_{i \in \mathcal{I}} \bigvee_{c_i}^{d_i} f. \qquad \square \end{aligned}$$

Lemma 2.8. *Let* X *be a Banach space and* $g : [a,b] \to X$. *Suppose that*

$$\mathcal{H}^1(g(\{x \in [a,b] \colon \tau(g,x) \text{ does not exist}\})) = 0,$$

and $\mathrm{md}\,(g,x)$ *exists for almost all* $x \in [a,b]$. *Then* $g'(x)$ *exists for almost all* $x \in [a,b]$.

Proof. Suppose that $\mathrm{md}\,(g,x)$ exists for all $x \in [a,b] \setminus N$ with $\lambda(N) = 0$. Denote

$$\begin{aligned} A &= \{x \in [a,b] \colon \tau(g,x) \text{ does not exist}\}, \\ M &= \{x \in A \setminus N \colon \mathrm{md}\,(g,x) > 0\}. \end{aligned}$$

We shall prove that $\lambda(M) = 0$. To see this, for $j \in \mathbf{N}$ write

$$M_j := \{x \in M \colon \|g(x) - g(z)\| \geq (1/j)|x - z| \text{ for } z \in B(x, 1/j)\}$$

and write $M_j = \cup_k M_{j,k}$, where $\mathrm{diam}\,(M_{j,k}) < 1/j$ for each $k \in \mathbf{N}$. Then $M = \cup_{j,k} M_{j,k}$ and $g(M_{j,k}) \subset g(A)$ for all $j,k \in \mathbf{N}$. It is easy to as $\mathcal{H}^1(g(A)) = 0$ by our assumptions. Thus $\lambda(M) = 0$.

Now, we prove that $g'(x)$ exists for all $x \in [a,b] \setminus (M \cup N)$. First, consider $x \in [a,b]$ with $\operatorname{md}(g,x) = 0$. Then, obviously,

$$g'(x) = \lim_{t \to 0} \frac{g(x+t) - g(x)}{t} = 0.$$

Second, consider $x \in [a,b] \setminus A$ with $md(g,x) > 0$. Then

$$\begin{aligned} g'(x) &= \lim_{t \to 0} \operatorname{sgn}(t) \cdot \frac{g(x+t) - g(x)}{\|g(x+t) - g(x)\|} \cdot \frac{\|g(x+t) - g(x)\|}{|t|} \\ &= \tau(g,x) \cdot md(g,x). \end{aligned} \tag{2.4}$$

Thus $g'(x)$ exists for all $x \in [a,b] \setminus (M \cup N)$. □

Lemma 2.9. *Let X be a Banach space. Suppose that $f : [c,d] \to X$ is continuous, has bounded variation, and is not constant on any interval. Let $\varphi(x) = v_f^{-1}(x)$ for $x \in [0, v_f(d)]$, and $g = f \circ \varphi$. Then g is 1-Lipschitz and, for almost all $x \in [0, v_f(d)]$, g is metrically differentiable at x with* $\operatorname{md}(g,x) = 1$.

Proof. It is easy to see that φ is an increasing homeomorphism of $[0, v_f(d)]$ onto $[c,d]$ such that $g = f \circ \varphi$ is 1-Lipschitz (see [**12**, Section 2.5.16]). By [**14**, Theorem 2] we obtain that g is metrically differentiable for almost all $x \in [0, v_f(d)]$. Further,

$$\int\limits_{[0,v_f(d)]} \operatorname{md}(g,x)\,dx = \int\limits_{g([0,v_f(d)])} N(g|_{[0,v_f(d)]}, x)\,d\mathcal{H}^1(x) = \bigvee_0^{v_f(d)} g = v_f(d),$$

where the first equality follows from [**14**, Theorem 7] and the second equality from [**12**, Theorem 2.10.13]. The third equality is obvious since φ is a homeomorphism. Thus $\operatorname{md}(g,x) = 1$ for almost all $x \in [0, v_f(d)]$ (as $\operatorname{md}(g,x) \le 1$ where it exists) and the claim follows. □

Lemma 2.10. *Let $f : [0,1] \to X$ be continuous and VBG_*. Then there exists a Lipschitz homeomorphism θ of $[0,1]$ onto itself such that $f \circ \theta$ is pointwise Lipschitz.*

from the definition of VBG_* for f. Define a continuous f_n on $[0,1]$ such that $f_n(x) = f(x)$ for $x \in E_n$ and f_n is linear on each component of $[0,1] \setminus E_n$, and f_n is constant on the (at most) two components of $[0,1] \setminus E_n$ which contain 0 or 1 as an endpoint.

Let $v_n(x) = \vee_0^x f_n$, let $\varphi_n(x) = \sum_i \mathrm{Osc}\,(f, I_n^i)$, where I_n^i are all (pairwise different) components of $[0,x] \setminus E_n$, and $\psi_n(x) = \sum_i \mathrm{Osc}\,(f, J_n^i)$, where J_n^i are all (pairwise different) components of $[x,1] \setminus E_n$. It follows that $v_n, \varphi_n, -\psi_n$ are continuous and nondecreasing on $[0,1]$. Let

$$v(x) = x + \sum_{n=1}^{\infty} (v_n(x) + \varphi_n(x) - \psi_n(x)) \cdot \varepsilon_n, \tag{2.5}$$

where $\varepsilon_n > 0$ ($n \in \mathbf{N}$) are taken such that $v(x)$ is finite for $x = 0$ and $x = 1$. Then clearly v is continuous and strictly increasing on $[0,1]$. Let $[a,b] := v([0,1])$ and consider arbitrary points $p, q \in [a,b]$, $p \neq q$. Denote $s = v^{-1}(p)$ and $t = v^{-1}(q)$, and choose $n \in \mathbf{N}$ such that $s \in E_n$. We easily obtain

$$\|f(t) - f(s)\| \le |v_n(t) - v_n(s) + \varphi_n(t) - \varphi_n(s) - (\psi_n(t) - \psi_n(s))|. \tag{2.6}$$

Indeed, suppose first $t > s$. If $t \in E_n$, then $\|f(t) - f(s)\| = \|f_n(t) - f_n(s)\| \le |v_n(t) - v_n(s)|$. If $t \in I_n^i =: (a_i, b_i)$, then $\|f(t) - f(a_i)\| \le |\varphi_n(t) - \varphi_n(a_i)|$, and therefore $\|f(t) - f(s)\| \le \|f(a_i) - f(s)\| + \|f(t) - f(a_i)\| \le |v_n(a_i) - v_n(s)| + |\varphi_n(t) - \varphi_n(a_i)|$, which easily implies (2.6). In case $t < s$ we proceed symmetrically using $-\psi_n$ instead of φ_n.

Since all the terms in the definition of v are nondecreasing, it follows that

$$\begin{aligned} |v(s) - v(t)| \ge {} & |v_n(s) - v_n(t) + \varphi_n(s) - \varphi_n(t) - (\psi_n(s) - \psi_n(t))| \\ & \cdot \varepsilon_n + |s - t|, \end{aligned} \tag{2.7}$$

and we obtain

$$\big\|f(v^{-1}(q)) - f(v^{-1}(p))\big\| \le \frac{1}{\varepsilon_n} |q - p|. \tag{2.8}$$

This shows that $f \circ v^{-1}$ is pointwise Lipschitz. Since (2.7) implies that v^{-1} is Lipschitz, $\theta(x) = v^{-1}\big(x(b-a) + a\big)$ is clearly a homeomorphism with the desired properties. □

independent interest.

Theorem 2.11. *Let X be a Banach space and suppose that $f : [0,1] \to X$ is continuous and VBG_*. Then f is metrically differentiable at almost all $x \in [0,1]$.*

Proof. Lemma 2.10 applied to f yields a Lipschitz homeomorphism θ of $[0,1]$ onto itself such that $f \circ \theta$ is pointwise Lipschitz. By Corollary 2.3, $f \circ \theta$ is metrically differentiable at all points of $[0,1] \setminus N$, where $\lambda(N) = 0$. Since θ is Lipschitz, we have $\lambda(\theta(N)) = 0$, and since θ^{-1} is monotone, we have $\lambda(M) = 0$ for $M := \{x \in [0,1] : (\theta^{-1})'(x) \text{ does not exist}\}$. Thus, $\lambda(\theta(N) \cup M) = 0$ and, by Lemma 2.4, $f = (f \circ \theta) \circ \theta^{-1}$ is metrically differentiable at all $x \in [0,1] \setminus (\theta(N) \cup M)$. □

The following lemma is an easy consequence of a lemma of Zahorski [**19**], which was independently proved by Choquet [**8**], and in a slightly weaker form already by Tolstov [**18**]. Note that Zahorski proved his lemma in May 1940 (by [**19**, page 7]), after submission of [**18**]. However, Tolstov in [**18**] does not consider absolute continuity of h^{-1} (where h is as in the proof of the following lemma).

Lemma 2.12. *Let X be a Banach space, and let $g : [a,b] \to X$ be pointwise Lipschitz.*

(i) *There exists a boundedly differentiable homeomorphism h of $[a,b]$ onto itself such that $g \circ h$ is metrically differentiable at all $x \in [a,b]$.*

Further, if $\mathrm{md}\,(g,\cdot)$ is bounded (where it exists), then $\mathrm{md}\,(g \circ h,\cdot)$ is bounded everywhere in $[a,b]$. If $\mathrm{md}\,(g,x) \neq 0$ for almost all $x \in [a,b]$, then $\mathrm{md}\,(g \circ h, x) \neq 0$ for almost all $x \in [a,b]$.

(ii) *If g is differentiable almost everywhere, then there exists a boundedly differentiable homeomorphism h of $[a,b]$ onto itself such that $g \circ h$ is differentiable everywhere in $[a,b]$.*

Further, if g' is bounded (where it exists), then $(g \circ h)'$ is bounded everywhere in $[a,b]$. If $g'(x) \neq 0$ for almost all $x \in [a,b]$, then $(g \circ h)'(x) \neq 0$ for almost all $x \in [a,b]$.

By Corollary 2.3 we have that g is metrically differentiable at all $x \in (a,b) \setminus M$ with $\lambda(M) = 0$. By Zahorski's theorem [**19**] (for a proof see, e.g., [**13**, pages 25–27]) there exists a differentiable homeomorphism h of $[a,b]$ onto itself such that for $\mathcal{M} = \{x \in [a,b] : h'(x) = 0\}$ we have:

- $\lambda(\mathcal{M}) = 0$, $h^{-1}(M) \subset \mathcal{M}$,
- h' is bounded and h^{-1} is absolutely continuous.

Now we shall prove that $F = g \circ h$ is metrically differentiable everywhere. If $x \notin \mathcal{M}$, then F is metrically differentiable by Lemma 2.4 (ii). If $x \in \mathcal{M}$, then

$$\operatorname{md}(F,x) = \lim_{t \to 0} \frac{\|g(h(x+t)) - g(h(x))\|}{|h(x+t) - h(x)|} \cdot \frac{|h(x+t) - h(x)|}{|t|} = 0,$$

as g is pointwise Lipschitz and $h'(x) = 0$. Thus F is metrically differentiable at x by Lemma 2.4 (i).

We have that either $\operatorname{md}(F,x) = 0$ or $\operatorname{md}(F,x) = \operatorname{md}(g, h(x)) \cdot |h'(x)|$. Consequently, if $\operatorname{md}(g,\cdot)$ is bounded (where it exists), then $\operatorname{md}(F,\cdot)$ is bounded as well (because $|h'|$ is bounded).

Suppose that $\operatorname{md}(g,x) \neq 0$ for all $x \in [0,1] \setminus N$, with $\lambda(N) = 0$. Then for any $x \notin h^{-1}(N) \cup \mathcal{M}$, we have that $\operatorname{md}(g \circ h, x) \neq 0$. To finish the proof, note that $\lambda(h^{-1}(N) \cup \mathcal{M}) = 0$ because h^{-1} is absolutely continuous (see e.g. [**16**, Theorem 6.1, Chapter VII]). □

Lemma 2.13. *Suppose that X is a Banach space and $g : [a,b] \to X$ is pointwise Lipschitz. Then there exist closed sets $E_{j,k}$, $j,k \in \mathbf{N}$, with* $\operatorname{diam}(E_{j,k}) < 1/j$ *such that*

(i) $\cup E_{j,k} = [a,b]$,

(ii) *we have $\|g(x) - g(y)\| \leq j|x-y|$ for $x \in E_{j,k}$ and $y \in [a,b]$ such that $|x-y| < 1/j$,*

(iii) $V(g, E_{j,k}) < \infty$,

(iv) $\sum_n Osc(g, I_n^{j,k}) < \infty$ *where $I_n^{j,k}$ are all (closed) intervals contiguous to $E_{j,k}$.*

In particular, g is VBG_*.

$$E_j := \{x \in [a,b] \colon \|g(x) - g(z)\| \le j\,|x - z| \text{ if } z \in [a,b] \text{ and } |z - x| < 1/j\}.$$

Clearly each E_j is closed and $\cup E_j = [a,b]$. Write $E_j = \cup_{k \in \mathbf{N}} E_{j,k}$, where each $E_{j,k}$ is closed and $\operatorname{diam}(E_{j,k}) < 1/j$. The conditions (i) and (ii) are clearly satisfied. By (ii), g is Lipschitz on each $E_{j,k}$, which implies (iii).

Let j,k be fixed and $[c,d]$ be an interval contiguous to $E_{j,k}$. Choose $x, y \in [c,d]$ such that $\mathrm{O}sc\,(g,[c,d]) = \|g(x) - g(y)\|$. By (ii) we have

$$\mathrm{O}sc\,(g,[c,d]) \le \|g(x) - g(c)\| + \|g(y) - g(c)\| \le 2j(d-c),$$

which immediately implies (iv). □

Lemma 2.14. *Suppose that X is a Banach space and $f : [a,b] \to X$. Further, suppose that for each $x \in (a,b)$ there exists an open interval U such that $x \in U$, $\overline{U} \subset (a,b)$, and an increasing homeomorphism h_U of $\overline{U}$ onto itself such that $f \circ h_U$ is pointwise Lipschitz on $\overline{U}$, and $\operatorname{md}(f \circ h_U, y) \ne 0$ for almost all $y \in U$. Then there exists an increasing homeomorphism h of $[a,b]$ onto itself such that $f \circ h$ is pointwise Lipschitz on (a,b), $\operatorname{md}(f \circ h, x) \ne 0$ for almost all $x \in (a,b)$, and*

$$\|f(x) - f(h(x))\| \le \operatorname{dist}(x, \{a,b\}) \tag{2.9}$$

for all $x \in (a,b)$.

Proof. For an interval $J = [c,d] \subset (a,b)$, choose an increasing homeomorphism w_J of J onto J such that

(i) $f \circ w_J$ is pointwise Lipschitz on J,

(ii) $\operatorname{md}(f \circ w_J, x) \ne 0$ for almost all $x \in J$,

(iii) $\|f(x) - (f \circ w_J)(x)\| \le \operatorname{dist}(x, \{a,b\})$ for all $x \in J$,

whenever such a homeomorphism exists.

Now observe that, for each $x \in (a,b)$, we can choose an open interval $I_x \subset (a,b)$ containing x such that w_J is defined for each $J = [c,d] \subset I_x$. Indeed, for each x, choose $U_x := U$ by the assumptions of the lemma

$x \in I_x$, and

$$\mathrm{O}sc\,(f, I_x) \leq \mathrm{dist}\,(I_x, \{a, b\}).$$

Note that, for any $J = [c, d] \subset I_x$, there exists an increasing linear homeomorphism l_J from J onto $h_x^{-1}(J)$. Define $w_J := h_x \circ l_J$ and observe that it satisfies (i)–(iii).

Now we will show that w_I is defined for each $I = [c, d] \subset (a, b)$. To this end, find a partition $\{t_0 = c < t_1 < \cdots < t_n = d\}$ of $[c, d]$ such that each interval $[t_{i-1}, t_i]$ is contained in an interval I_x (for some $x \in [c, d]$). We can choose any partition such that $\max\{t_i - t_{i-1} : i = 1, \ldots, n\}$ is smaller than the Lebesgue number of the open cover $\mathcal{C} := \{I_x : x \in [c, d]\}$ of $[c, d]$. (We could also proceed by choosing a minimal finite subcover of $\mathcal{C}$.) Now define $w(x) := w_{[t_{i-1}, t_i]}(x)$ for $1 \leq i \leq n$, $x \in [t_{i-1}, t_i]$, and observe that w is an increasing homeomorphism of I onto itself having the properties (i)–(iii).

Finally, choose points $\{z_k\}_{k \in \mathbf{Z}}$ with $z_k < z_{k+1}$ for $k \in \mathbf{Z}$, $\lim_{k \to \infty} z_k = b$, $\lim_{k \to -\infty} z_k = a$, and put $h(x) = w_{[z_{k-1}, z_k]}(x)$ for each integer k and $x \in [z_{k-1}, z_k]$. Then h is clearly an increasing homeomorphism of $[a, b]$ onto itself, satisfying the conclusion of our lemma. □

Remark 2.15. Let X be a Banach space, $f : [0, 1] \to X$, and $A \subset [0, 1]$. Following [**16**], we say that f is VB_* on A provided the set of the sums $\sum_i \mathrm{O}sc\,(f, [a_i, b_i])$, where $([a_i, b_i])$ is a finite sequence of nonoverlapping intervals with $a_i, b_i \in A$, is bounded.

Since the proofs of [**16**, Theorem 7.1, Chap VII] and [**16**, Theorem 8.5, Chap VII] work also for X-valued functions, we obtain that f is VBG_*, if and only if there exists a sequence of (arbitrary) sets (A_n) such that $[0, 1] = \cup_n A_n$ and f is VB_* on each A_n.

Using this remark, we can easily prove the following version of a lemma from [**19**].

Lemma 2.16. *Let X be a Banach space and let $f : [0, 1] \to X$ be continuous. Suppose that $\tau(f, x)$ exists for all $x \in [0, 1]$ except a countable set. Then f is VBG_*.*

X is separable. Therefore, we can choose a sequence of unit vectors $(x_i^*)_{i\in\mathbf{N}} \subset X^*$ which is a norming sequence for X (i.e., $\|x\| = \sup_{i\in\mathbf{N}} x_i^*(x)$ for every $x \in X$). Denote $C := \{x \in [0,1] : \tau(f,x) \text{ does not exist}\}$ and, for $n, m \in \mathbf{N}$,

$$A_{mn} = \Big\{ x \in [0,1] : \ 0 < \frac{1}{4}\|f(y)-f(x)\| < \operatorname{sign}(y-x)\cdot x_m^*(f(y)-f(x)) \quad \text{whenever } y \in [0,1], 0 < |y-x| < 1/n \Big\}.$$

We have $C \cup (\cup_{m,n} A_{mn}) = [0,1]$. Indeed, let $x \in [0,1] \setminus C$ be given. Since $\tau(f,x)$ exists, we can choose $n \in \mathbf{N}$ such that

$$\left\| \tau(f,x) - \operatorname{sign}(y-x)\, \frac{f(y)-f(x)}{\|f(y)-f(x)\|} \right\| < 1/2,$$

whenever $y \in [0,1]$, $0 < |y-x| < 1/n$. Since $\|\tau(f,x)\| = 1$, we can choose $m \in \mathbf{N}$ such that $x_m^*(\tau(f,x)) > 3/4$. Now it is easy to see that

$$0 < \frac{1}{4}\|f(y)-f(x)\| < \operatorname{sign}(y-x)\cdot x_m^*(f(y)-f(x)),$$

whenever $y \in [0,1]$, $0 < |y-x| < 1/n$, and therefore $x \in A_{mn}$. For each $k \in \mathbf{N}$, choose $A_{mnk} \subset A_{mn}$ such that $\operatorname{diam}(A_{mnk}) < 1/n$ and $A_{mn} = \cup_k A_{mnk}$.

Since C is countable, by Remark 2.15 it is sufficient to show that f is VB_* on A_{mnk} for each $m, n, k \in \mathbf{N}$.

Let $[a_i, b_i]$, $i \in F$, be a finite system of nonoverlapping intervals with the endpoints in A_{mnk}. Take $c_i, d_i \in [a_i, b_i]$, $c_i \le d_i$, such that $\|f(d_i) - f(c_i)\| = \mathrm{O}sc\,(f, [a_i, b_i])$. Observe that the definition of A_{mnk} easily gives that $x_m^*(f(b_i)) > x_m^*(f(a_i))$ and that

$$\{ [x_m^*(f(a_i)), x_m^*(f(b_i))] : i \in F \}$$

is a system of nonoverlapping intervals. Moreover, $x_m^*(f(c_i)) \in [x_m^*(f(a_i)), x_m^*(f(b_i))]$, and $x_m^*(f(d_i)) \in [x_m^*(f(a_i)), x_m^*(f(b_i))]$. Con-

$$
\begin{aligned}
\sum_{i\in F} \mathrm{Osc}\,(f,[a_i,b_i]) &= \sum_{i\in \mathbf{N}} \|f(d_i)-f(c_i)\| \\
&\le \sum_{i\in F}(\|f(a_i)-f(d_i)\|+\|f(a_i)-f(c_i)\|) \\
&\le 4\sum_{i\in F}\big(x_m^*(f(d_i)-f(a_i))+x_m^*(f(c_i)-f(a_i))\big) \\
&\le 8\sum_{i\in F}\big(x_m^*(f(b_i)-f(a_i))\big) \le 8\,\mathrm{Osc}\,(x_m^*\circ f,[0,1]),
\end{aligned}
$$

which completes the proof. □

3. C^1-parametrizations. For $f:[0,1]\to X$, define B_f as the set of all points $x\in[0,1]$ such that there is no neighborhood U of x such that f is either constant or tangentially smooth on U. Clearly B_f is closed and $\{0,1\}\subset B_f$.

Theorem 3.1. *Let X be a Banach space, and let $f:[0,1]\to X$. Then there is a homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ is C^1 if and only if f is continuous, has bounded variation, and $\mathcal{H}^1(f(B_f))=0$.*

Proof. To prove the necessity of our condition, note that from the existence of h, it easily follows that f is continuous and has bounded variation (observe that $f\circ h$ is Lipschitz by [**12**, Section 2.2.7]). Notice that $f(B_f)=(f\circ h)(B_{f\circ h})$, and denote $g:=f\circ h$. Since g is C^1, we easily obtain

$$B_g\subset\{x\in[0,1]\colon g'(x)=0\}\cup\{0,1\}=:D.$$

Now by the Morse-Sard theorem [**12**, Theorem 3.4.3] (or alternatively by Lemma 2.2) we obtain that $\mathcal{H}^1(g(D))=0$, which implies $\mathcal{H}^1(f(B_f))=\mathcal{H}^1(g(B_g))=0$.

Now suppose that our condition is satisfied. First observe that B_f is nowhere dense in $[0,1]$. Indeed, if not, then there exists a nonempty interval $(c,d)\subset B_f$. Then $\mathcal{H}^1(f((c,d)))=0$, but, since clearly f is not

Let $I_i = (c_i, d_i)$, $(i \in \mathcal{I} \subset \mathbf{N})$ be all (pairwise different) components of the (open) set $[0,1] \setminus B_f$. Let $U = \cup\{I_i : f \text{ is constant on } I_i\}$. For $x \in [0,1]$ define $k(x) = v_f(x) + \lambda(U \cap (0,x))$ and $C := \bigvee_0^1 f + \lambda(U)$. If $I_i \cap U = \varnothing$, then clearly f is tangentially smooth on I_i and therefore v_f is strictly increasing on I_i. Therefore $k : [0,1] \to [0,C]$ is a homeomorphism (because $(c,d) \cap (\cup_i I_i) \neq \varnothing$ for any $0 \le c < d \le 1$). Clearly $f \circ k^{-1}$ is Lipschitz.

We will show that $\lambda(k(B_f)) = 0$. Note that

$$\lambda\left(k\left(\bigcup_i I_i\right)\right) = \sum_{I_i \subset U} |I_i| + \sum_{I_i \cap U = \varnothing} |k(d_i) - k(c_i)|$$
$$= \lambda(U) + \sum_i \bigvee_{c_i}^{d_i} f = \lambda(U) + \bigvee_0^1 f,$$

where the last equality follows from Lemma 2.7. Since $k(B_f) = [0,C] \setminus k(\cup_i I_i)$, it follows that $\lambda(k(B_f)) = 0$. By a theorem of Zahorski (see [**13**, pages 25–27]) there exists a continuously differentiable homeomorphism l of $[0,C]$ onto itself such that l^{-1} is absolutely continuous and $l'(x) = 0$ if and only if $x \in l^{-1}(k(B_f))$. Now we shall show that $\psi = f \circ k^{-1} \circ l$ is continuously differentiable.

If $I_i \subset U$, then ψ is constant on $(l^{-1} \circ k)(I_i)$. If $I_i \cap U = \varnothing$, then $f \circ k^{-1}$ is C^1 on $k(I_i)$ by Lemma 2.5 and thus ψ is C^1 on $(l^{-1} \circ k)(I_i)$. Now consider $x \in B^* := l^{-1}(k(B_f))$. Then, for any $y \in [0,C] \setminus \{x\}$, we have

$$\frac{(f \circ k^{-1} \circ l)(y) - (f \circ k^{-1} \circ l)(x)}{y - x} = \frac{(f \circ k^{-1})(l(y)) - (f \circ k^{-1})(l(x))}{l(y) - l(x)} \cdot \frac{l(y) - l(x)}{y - x},$$

and because $f \circ k^{-1}$ is Lipschitz, it follows that $\psi'(x) = 0$. Thus ψ is differentiable on $[0,C]$. The continuity of ψ' at $x \in B^*$ follows for example from the equality $\psi'(y) = (f \circ k^{-1})'(l(y)) \cdot l'(y)$ $(y \in [0,C] \setminus B^*)$, the Lipschitz property of $f \circ k^{-1}$, and continuity of l'. Thus $h(x) = (k^{-1} \circ l)(Cx)$ is the desired homeomorphism of $[0,1]$ onto itself. □

Remark 3.2. Theorem 3.1 for $X = \mathbf{R}^n$ was proved in [**17**]. If $X = \mathbf{R}$, then B_f coincides with the set K_f of varying monotonicity (see [**13**, (see also [**13**, Theorem 3.11]).

The proof of Theorem 3.1 yields also the following result concerning the case of an almost everywhere nonzero continuous derivative.

Theorem 3.3. *Let X be a Banach space, and let $f : [0,1] \to X$. Then there is a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^1 and $(f \circ h)'(y) \neq 0$ for almost all $y \in [0,1]$ if and only if f is continuous, has bounded variation, is not constant on any interval, and $\mathcal{H}^1(f(B_f)) = 0$.*

Proof. The necessity follows from Theorem 3.1, and from the fact that the constancy of f on some interval would easily yield a contradiction with the fact that $(f \circ h)' \neq 0$ almost everywhere in $[0,1]$.

Concerning the sufficiency, let U, k, C, l, ψ, h be defined as in the proof of Theorem 3.1. Note that now $U = \varnothing$, and $k = v_f$. Since $U = \varnothing$, Lemma 2.5 yields that $(f \circ k^{-1})'$ is nonzero on $[0,C] \setminus k(B_f)$ and therefore ψ' is nonzero on $[0,C] \setminus l^{-1}(k(B_f))$. Since l^{-1} is absolutely continuous, $\lambda(l^{-1}(k(B_f))) = 0$ and thus h has the desired property. □

The following theorem characterizes the curves allowing parametrizations with a nonzero continuous derivative.

Theorem 3.4. *Let X be a Banach space and $f : [0,1] \to X$ be continuous. Then there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^1 with $(f \circ h)'(x) \neq 0$ for all $x \in [0,1]$ if and only if $\tau(f,x)$ exists and is continuous on $[0,1]$.*

Proof. Let h be a homeomorphism such that $g = f \circ h$ has a continuous derivative with $g'(x) \neq 0$ for all $x \in [0,1]$. Then $\tau(g,x) = g'(x)/\|g'(x)\|$ and this function is continuous.

Suppose that $\tau(f,x)$ is continuous for all $x \in [0,1]$. Then f is not constant on any interval. It follows from [**9**, Theorem 3.5] that f is one-sidedly smooth in the sense of [**9**], and thus [**9**, Proposition 3.6] implies

$h(x) = v_f^{-1}(\ell \cdot x)$, $x \in [0,1]$ is the desired homeomorphism. □

4. Parametrizations with bounded derivative. First, we shall prove a theorem about curves allowing parametrizations which are boundedly metrically differentiable.

Theorem 4.1. *Let X be a Banach space, and let $f : [0,1] \to X$ be continuous. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is metrically differentiable at all $x \in [0,1]$, and $\operatorname{md}(f \circ h, \cdot)$ is bounded on $[0,1]$.*

(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $\operatorname{md}(f \circ h, \cdot)$ exists, and is bounded on $[0,1]$.*

(iii) *f has bounded variation.*

Proof. Trivially, (i) $\implies$ (ii). To see that (ii) $\implies$ (iii), suppose that h as in (ii) is given. By [**12**, Section 2.2.7] we obtain that $f \circ h$ is Lipschitz, and thus $f \circ h$ and also f have bounded variation.

To prove that (iii) $\implies$ (i), define $\varphi : [0,1] \to [0, 1 + v_f(1)]$ by

$$\varphi(x) = x + v_f(x), \quad 0 \le x \le 1.$$

It is easy to see that $h_1(t) := \varphi^{-1}(t(1 + v_f(1)))$ is an increasing homeomorphism of $[0,1]$ onto itself such that $g := f \circ h_1$ is Lipschitz (cf. [**13**, Proof of Lemma 3.2] or [**12**, Section 2.5.16]). By [**14**, Theorem 2] we obtain that g is metrically differentiable almost everywhere, and $\operatorname{md}(g, \cdot)$ is bounded (where it exists) because g is Lipschitz.

By part (i) of Lemma 2.12, there exists a boundedly differentiable homeomorphism h_2 of $[0,1]$ onto itself such that $g \circ h_2$ is metrically differentiable at all $x \in [0,1]$, and $\operatorname{md}(g \circ h_2, \cdot)$ is bounded on $[0,1]$. Thus $h = h_1 \circ h_2$ is the desired homeomorphism. □

Remark 4.2. The proof of Theorem 4.1 shows (since φ^{-1} is clearly Lipschitz) that in (i) and (ii) we could write that h is a Lipschitz homeomorphism. Then h is differentiable almost everywhere in $[0,1]$, and such that $h \circ h_1$ is Lipschitz and differentiable; also $f \circ h \circ h_1$ is boundedly metrically differentiable by Lemma 2.4. Thus, in parts (i) and (ii) we could even write that h is Lipschitz and differentiable.

Theorem 4.1 easily implies the following result.

Theorem 4.3. *Let X be a Banach space and $f : [0,1] \to X$ be continuous. Then there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has a bounded derivative if and only if f has bounded variation and*

$$\mathcal{H}^1(f(\{x \in [0,1] : \tau(f,x) \text{ does not exist}\})) = 0. \tag{4.1}$$

Proof. To prove the necessity of the condition suppose that h is given. Then f has bounded variation by Theorem 4.1 and (4.1) follows from Lemma 2.6.

To prove the sufficiency of our condition, by Theorem 4.1 there exists a homeomorphism h_1 of $[0,1]$ onto itself such that $\mathrm{md}\,(f \circ h_1, \cdot)$ exists and is bounded on $[0,1]$. Let $g := f \circ h_1$. By (2.3) and Lemma 2.8, we obtain that $g'(x)$ exists for almost every x from $[0,1]$.

By part (ii) of Lemma 2.12, there exists a homeomorphism h_2 of $[0,1]$ onto itself such that $(g \circ h_2)'$ exists and is bounded on $[0,1]$. Thus $h = h_1 \circ h_2$ is the desired homeomorphism for which $(f \circ h)'$ exists and is bounded on $[0,1]$. □

Remark 4.4. In the proof of Theorem 4.3, both h_1 (see Remark 4.2) and h_2 could be chosen Lipschitz and differentiable. Thus in Theorem 4.3, we could ask for the homeomorphism h to be Lipschitz and differentiable.

Remark 4.5. The condition (4.1) can be removed in Theorem 4.3 if and only if X has the Radon-Nikodým property.

If X has this property, then (in the proof of Theorem 4.3) $g = f \circ h_1$ is Lipschitz and therefore (see, e.g., [**7**, Theorem 5.21]) almost everywhere differentiable. Thus we do not need (4.1).

Theorem 5.21]) there exists a Lipschitz nowhere differentiable $f : [0,1] \to X$. For this f there exists no such h, since otherwise $f = (f \circ h) \circ h^{-1}$ would be differentiable almost everywhere.

Theorem 4.6. *Let X be a Banach space and $f : [0,1] \to X$ be continuous. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is metrically differentiable at all $x \in [0,1]$, $\mathrm{md}\,(f \circ h, \cdot)$ is bounded on $[0,1]$, and $\mathrm{md}\,(f \circ h, y) \neq 0$ for almost all $y \in [0,1]$.*

(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $\mathrm{md}\,(f \circ h, \cdot)$ exists and is bounded on $[0,1]$, and such that $\mathrm{md}\,(f \circ h, y) \neq 0$ for almost all $y \in [0,1]$.*

(iii) *f has a bounded variation, and is not constant on any interval.*

Proof. Trivially, (i) $\Longrightarrow$ (ii). To see that (ii) $\Longrightarrow$ (iii), let h be as in (ii). By Theorem 4.1, f has bounded variation. Since $\mathrm{md}\,(f \circ h, x) \neq 0$ for almost every x, we obtain that $f \circ h$, and thus also f, is not constant on any interval.

To prove that (iii) $\Longrightarrow$ (i), denote $\varphi(x) = v_f(x)$ for $x \in [0, v_f(1)]$, and $g = f \circ \varphi^{-1}$. By Lemma 2.9 we obtain that $\mathrm{md}\,(g, x) = 1$ for almost all $x \in [0, v_f(1)]$.

By part (i) of Lemma 2.12 we obtain a homeomorphism h_2 of $[0, v_f(1)]$ onto itself such that $g \circ h_2$ is metrically differentiable everywhere in $[0, v_f(1)]$, such that $\mathrm{md}\,(g \circ h_2, \cdot)$ is bounded, and $\mathrm{md}\,(g \circ h_2, x)$ is nonzero for almost all $x \in [0, v_f(1)]$. Thus $h(x) = \varphi^{-1} \circ h_2(v_f(1) \cdot x)$ is the desired homeomorphism of $[0,1]$ onto itself. □

As a corollary of Theorem 4.6, we obtain

Theorem 4.7. *Let X be a Banach space and $f : [0,1] \to X$ be continuous. Then there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has a bounded almost everywhere nonzero derivative if and only if f has bounded variation, f is not constant on any interval,*

$$(4.2) \qquad \mathcal{H}^1(f(\{x \in [0,1] \colon \tau(f,x) \text{ does not exist}\})) = 0.$$

Proof. The necessity of our condition immediately follows from Theorem 4.6 and Theorem 4.3.

To prove the sufficiency of our condition, Theorem 4.6 supplies a homeomorphism h_1 of $[0,1]$ onto itself such that $\mathrm{md}\,(f \circ h_1, \cdot)$ exists, is bounded on $[0,1]$, and nonzero almost everywhere in $[0,1]$. Let $g = f \circ h_1$. Lemma 2.8 together with (2.3) and (4.2) imply that $g'(x)$ exists for almost every x from $[0,1]$, and (2.4) implies that $g'(x) \neq 0$ for almost every x in $[0,1]$.

By part (ii) of Lemma 2.12 we obtain a homeomorphism h_2 of $[0,1]$ onto itself such that $(g \circ h_2)'$ exists everywhere, is bounded, and $(g \circ h_2)' \neq 0$ almost everywhere. Thus $h := h_1 \circ h_2$ is the desired homeomorphism of $[0,1]$ onto itself. $\square$

5. Finite derivative.

Theorem 5.1. *Let X be a Banach space and $f : [0,1] \to X$. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is metrically differentiable at x for all $x \in [0,1]$.*

(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $\mathrm{md}\,(f \circ h, x)$ exists for all $x \in [0,1]$.*

(iii) f is continuous, and VBG_*.

Proof. Trivially, (i) $\Longrightarrow$ (ii). To see that (ii) $\Longrightarrow$ (iii), let $g = f \circ h$. Using Lemma 2.13, we obtain that g, and thus also f, is continuous and VBG_*.

To prove that (iii) $\Longrightarrow$ (i), by Lemma 2.13 we obtain a homeomorphism θ so that $g = f \circ \theta$ is pointwise Lipschitz.

By part (i) of Lemma 2.12 there exists a homeomorphism h_1 of $[0,1]$ onto itself such $g \circ h_1$ is metrically differentiable at all $x \in [0,1]$. Thus,

□

Remark 5.2. The proof of Theorem 5.1 yields that the homeomorphism h in (i) and (ii) can be taken Lipschitz (see Lemma 2.10 and Lemma 2.12). Then h is differentiable almost everywhere and thus Lemma 2.12 yields a boundedly differentiable homeomorphism $\tilde{h}$ so that $h \circ \tilde{h}$ is differentiable and Lipschitz. Thus, we could require h in (i) and (ii) to be differentiable and Lipschitz.

As a corollary of Theorem 5.1, we obtain:

Theorem 5.3. *Let X be a Banach space and $f : [0,1] \to X$. There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is differentiable if and only if f is continuous, VBG_*, and*

$$\mathcal{H}^1(f(\{x \in [0,1] \colon \tau(f,x) \text{ does not exist}\})) = 0. \tag{5.1}$$

Proof. The necessity of our condition immediately follows by Theorem 5.1 and Lemma 2.6.

For sufficiency, Theorem 5.1 supplies a homeomorphism h_1 of $[0,1]$ onto itself such that $\mathrm{md}\,(f \circ h_1, x)$ exists for all $x \in [0,1]$. Let $g = f \circ h_1$. Lemma 2.8 with (5.1) and (2.3) yield that $g'(x)$ exists for almost every $x \in [0,1]$. By part (ii) of Lemma 2.12 there exists a homeomorphism h_2 of $[0,1]$ onto itself such that $(g \circ h_2)'(x)$ exists for all $x \in [0,1]$. Thus $h = h_1 \circ h_2$ is the desired homeomorphism of $[0,1]$ onto itself. □

Remark 5.4. In the proof of Theorem 5.3, both h_1 (see Remark 5.2) and h_2 can be chosen differentiable and Lipschitz. Thus, in Theorem 5.3, we can require the homeomorphism h to be differentiable and Lipschitz.

Theorem 5.5. *Let X be a Banach space and $f : [0,1] \to X$. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is metrically differentiable at all $x \in [0,1]$, and* $\mathrm{md}\,(f \circ h, y) \neq 0$

(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that* $\mathrm{md}\,(f\circ h, x)$ *exists for all* $x \in [0,1]$ *with* $\mathrm{md}\,(f\circ h, y) \neq 0$ *for almost all* $y \in [0,1]$.

(iii) f *is continuous,* VBG_**, and is not constant on any interval.*

Proof. Trivially, (i) $\Longrightarrow$ (ii). To prove that (ii) $\Longrightarrow$ (iii), let h be as in (ii) and $g := f \circ h$. By Theorem 5.1, f is continuous and VBG_*. If g is constant on some interval, then $\mathrm{md}\,(g, \cdot) = 0$ on that interval, and we have a contradiction. Thus g (and also f) is not constant on any interval.

To show that (iii) $\Longrightarrow$ (i), we can assume that f is pointwise Lipschitz (even metrically differentiable) on $[0,1]$ (by Theorem 5.1 there exists a homeomorphism h such that $f \circ h$ is metrically differentiable; now take $f \circ h$ instead of f). We say that a point $x \in (0,1)$ is *regular* if there exists an open interval U such that $x \in U$, $\overline{U} \subset (0,1)$, and an increasing homeomorphism h_U of $\overline{U}$ onto itself such that $f \circ h_U$ is pointwise Lipschitz on $\overline{U}$ and $\mathrm{md}\,(f \circ h_U, y) \neq 0$ for almost all $y \in U$.

Let $\Omega = \Omega_f$ be the set of all regular points, and $P = [0,1] \setminus \Omega$. Then Ω is open and dense. The openness of Ω is clear. To prove that Ω is dense in $[0,1]$, let $[c,d] \subset [0,1]$, and $E_{j,k}$ be the (closed) sets from Lemma 2.13 applied to $[c,d]$. By the Baire category theorem, there exist $j,k \in \mathbf{N}$ such that $[a,b] \subset E_{j,k} \cap (c,d)$ for some $a < b$. Then f is j-Lipschitz on $[a,b]$. Theorem 4.6 (the implication (iii) $\Longrightarrow$ (i)) yields a homeomorphism of $[a,b]$ onto itself such that $f \circ h$ is pointwise Lipschitz (even metrically differentiable) with $\mathrm{md}\,(f \circ h, x)$ being nonzero almost everywhere in $[a,b]$. Thus, $(a,b) \subset \Omega$, and Ω is dense in $[0,1]$.

Write $\Omega = \cup_i (a_i, b_i)$, where (a_i, b_i) are all open components of the set Ω. An application of Lemma 2.14 to f restricted to $[a_i, b_i]$ gives an increasing homeomorphisms h_i of $[a_i, b_i]$ onto itself such that $f \circ h_i$ is pointwise Lipschitz in (a_i, b_i), $\mathrm{md}\,(f \circ h_i, x) \neq 0$ for almost all $x \in (a_i, b_i)$, and

$$\|f(x) - f(h_i(x))\| \leq \mathrm{dist}\,(x, \{a_i, b_i\}), \quad x \in (a_i, b_i). \tag{5.2}$$

Define $\tilde{h}(x) = h_i(x)$ for $x \in (a_i, b_i)$ and $\tilde{h}(x) = x$ for $x \in P$. If $y \in P$, then (5.2) easily implies that $\|f(x) - f(\tilde{h}(x))\| \leq |y - x|$ for each

Lipschitz at y as well. Consequently, $f \circ h$ is pointwise Lipschitz on $[0, 1]$.

We will prove $P = \{0, 1\}$. Since it is easy to see that $\Omega_{f\circ\tilde{h}} = \Omega_f$, without any loss of generality we can suppose that $\operatorname{md}(f, x) \neq 0$ for almost all $x \in \Omega$ (otherwise take $f \circ \tilde{h}$ instead of f). Now, for a contradiction suppose that $P \cap (0, 1) \neq \varnothing$. It is obvious that $P \cap (0, 1)$ has no isolated points.

Let $E_{j,k}$ be the (closed) sets from Lemma 2.13 applied to f on $[0, 1]$. Since $P \cap (0, 1)$ is a Baire space, at least one set $E_{j,k}$ is not nowhere dense in $P \cap (0, 1)$. Therefore, there exists an interval (c, d) together with $n \in \mathbf{N}$ such that $(c, d) \cap P \neq \varnothing$ and for all $x \in P \cap [c, d]$ we have that

$$\|f(x) - f(y)\| \leq n \cdot |x - y| \quad \text{for all } y \in [c, d]. \tag{5.3}$$

Since $P \cap (0, 1)$ has no isolated points, we can clearly assume that $c, d \in P$.

Let (c_i, d_i) ($i \in \mathcal{I} \subset \mathbf{N}$) be the components of $\Omega \cap (c, d)$. For each $i \in \mathcal{I}$, let $v = v_i = (f(d_i) - f(c_i))/\|f(d_i) - f(c_i)\|$ if $f(d_i) \neq f(c_i)$, and $v \in X$ be an arbitrary unit vector if it is not the case. Further, put $e_i := (c_i + d_i)/2 + \|f(d_i) - f(c_i)\| / 2n$ and observe that $c_i \leq e_i \leq d_i$ by (5.3). Now define (clearly uniquely) g on each $[c_i, d_i]$ so that g is continuous on $[c_i, d_i]$, $g(c_i) = f(c_i)$,

- $g'(x) = nv$ if $c_i < x < e_i$, and
- $g'(x) = -nv$ if $e_i < x < d_i$.

Clearly, $g(d_i) = f(d_i)$.

Thus, defining $g(x) := f(x)$ for $x \in P \cap [c, d]$, we easily see (using (5.3)) that $g : [c, d] \to X$ is n-Lipschitz and not constant on any interval (since P is nowhere dense). Thus v_g (where $v_g(x) := \bigvee_c^x g$) is strictly increasing on $[c, d]$. Further observe that v_g is clearly linear on each (c_i, d_i) with the slope n.

Define $\varphi = v_g^{-1} : [0, v_g(d)] \to [c, d]$. Denote $F := f \circ \varphi$, $\Pi := v_g(P \cap [c, d])$, and $(\alpha_i, \beta_i) := v_g((c_i, d_i))$ for $i \in \mathcal{I}$. Clearly

$$\varphi \text{ is linear on each } (\alpha_i, \beta_i) \text{ with the slope } 1/n. \tag{5.4}$$

$$(5.5) \qquad F(x) = g \circ \varphi(x) \quad \text{for} \quad x \in \Pi.$$

Lemma 2.9 (applied to g on $[c,d]$) implies that $g \circ \varphi$ is 1-Lipschitz and $\operatorname{md}(g \circ \varphi, x) = 1$ for almost all $x \in [0, v_g(d)]$. Consequently, (5.5) gives that F is 1-Lipschitz on Π.

We claim that F is pointwise Lipschitz. Using (5.4) and properties of f, we clearly obtain that F is pointwise Lipschitz at all points of $\cup_i(\alpha_i, \beta_i)$ and $\operatorname{md}(F,x) \neq 0$ for almost all $x \in \cup_i(\alpha_i, \beta_i)$. Let $x \in \Pi$ and $y \in (\alpha_i, \beta_i)$ (for definiteness, say $x < y$). Using (5.3), (5.4) and 1-Lipschitzness of F on Π, we obtain

$$(5.6) \qquad \begin{aligned} \|F(y) - F(x)\| &\le \|F(y) - F(\alpha_i)\| + \|F(\alpha_i) - F(x)\| \\ &\le n\,|\varphi(y) - \varphi(\alpha_i)| + \alpha_i - x \\ &\le y - \alpha_i + \alpha_i - x = y - x. \end{aligned}$$

Since F is 1-Lipschitz on Π, and (5.6) holds, we obtain that F is pointwise Lipschitz at all points of Π.

Because $F = g \circ \varphi$ on Π, $\operatorname{md}(g \circ \varphi, x) = 1$ for almost all $x \in \Pi$ and $\Pi \cap (0, v_g(d))$ has no isolated points, we clearly have that $\operatorname{md}(F,x) = 1$ for almost all $x \in \Pi$. Putting $h_{(c,d)}(x) = \varphi(v_g(d) \cdot (x-c)/(d-c))$, we see that $(c,d) \subset \Omega$, which is a contradiction.

Thus, $P = \{0,1\}$ and consequently $\tilde{f} := f \circ \tilde{h}$ is pointwise Lipschitz on $[0,1]$ with $\operatorname{md}(\tilde{f}, x) \neq 0$ for almost all $x \in [0,1]$. By Lemma 2.12 (i) we obtain a homeomorphism h_1 of $[0,1]$ onto itself such that $\tilde{f} \circ h_1$ is metrically differentiable on $[0,1]$ with $\operatorname{md}(\tilde{f} \circ h_1, y) \neq 0$ for almost all $y \in [0,1]$. Now it is clearly sufficient to put $h := \tilde{h} \circ h_1$. □

Theorem 5.5 easily implies the following theorem.

Theorem 5.6. *Let X be a Banach space and $f : [0,1] \to X$. There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is differentiable with $(f \circ h)'(x) \neq 0$ for almost every $x \in [0,1]$ if and only if f is VBG_*, is not constant on any interval, and*

$$(5.7) \qquad \mathcal{H}^1(f(\{x \in [0,1] \colon \tau(f,x) \text{ does not exist}\})) = 0.$$

Lemma 2.6.

Now suppose that the conditions hold. By Theorem 5.5 we obtain a homeomorphism h_1 of $[0,1]$ onto itself such that $\operatorname{md}(f \circ h_1, \cdot)$ exists everywhere in $[0,1]$ and is nonzero almost everywhere. Let $g = f \circ h_1$. Using Lemma 2.8 with (2.3) and (5.7), we obtain that $g'(x)$ exists and is nonzero for almost all $x \in [0,1]$. By Lemma (2.12) (ii) there is a homeomorphism h_2 of $[0,1]$ onto itself such that $g \circ h_2$ is differentiable everywhere with $(g \circ h_2)'$ being nonzero almost everywhere. Thus $h = h_1 \circ h_2$ is the desired homeomorphism. □

Remark 5.7. Quite similar reasoning (using now also the result of [**3**] on almost everywhere differentiability of pointwise Lipschitz mappings) as in Remark 4.5 yields that the conditions (5.1) and (5.7) can be removed in Theorems 5.3 and 5.6 if and only if X has the Radon-Nikodým property.

Theorem 5.8. *Let $f : [0,1] \to X$ be continuous, and suppose that $\tau(f,x)$ exists for all $x \in [0,1]$ except a countable set. Then there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is differentiable, and $(f \circ h)'(x) \neq 0$ for almost all $x \in [0,1]$.*

Proof. It is obvious that f is not constant on any interval. It follows by Lemma 2.16 that f is VBG_*, and Theorem 5.6 yields the required homeomorphism, since (5.7) clearly holds. □

Remark 5.9. (i) Zahorski proved Theorems 5.6 and 5.8 for $X = \mathbf{R}^n$ (cf. Remark 5.7) in 1943. Since the manuscript was damaged during the war, Zahorski found another (much more complicated) proof (published in [**19**]) in 1946 (see [**19**, page 8]).

(ii) For a characterization of those f, which allow a parametrization having an everywhere nonzero derivative, see [**8**] (the case $X = \mathbf{R}^n$) and [**11**] (the general case).

6. Parametrizations using differentiable homeomorphisms. Bruckner [**5**, pages 89, 90] asked (among other questions) for which onto itself such that $f \circ h$ has a finite (bounded, or summable) derivative on $[0,1]$. In Remarks 4.4 and 5.4 above, we already answered these three questions. Indeed, in these remarks, we have proved the following propositions.

Proposition 6.1. *Let X be a Banach space, $f : [0,1] \to X$. The following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is boundedly differentiable.*

(ii) *There exists a differentiable and Lipschitz homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is boundedly differentiable.*

Proposition 6.2. *Let X be a Banach space, $f : [0,1] \to X$. The following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is differentiable.*

(ii) *There exists a differentiable and Lipschitz homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is differentiable.*

The case of finite summable derivatives follows easily from Proposition 6.1 because a bounded derivative is certainly summable. Thus, we can fill the first three empty spaces in the last column of Table 1 in [**5**, page 90] (in the same way as it is done in the second column).

Now we observe that the case of continuous differentiability is different, as the following simple lemma shows. (Let us remark that the classical Pompeiu function (see [**15**], cf. [**5**, page 24]) is a strictly increasing differentiable function whose derivative vanishes on a dense subset of its interval of definition.)

Lemma 6.3. *Let $f : [0,1] \to \mathbf{R}$ be a continuous strictly increasing function whose derivative (exists and) vanishes on a dense subset of $[0,1]$. Then the following hold.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has a continuous derivative on $[0,1]$.*

itself such that $f \circ h$ has a continuous derivative on $[0,1]$.

Proof. If we denote $[c,d] := f([0,1])$ and set $h(x) = f^{-1}(c+x(d-c))$, then $f \circ h$ is linear on $[0,1]$.

To prove (ii), suppose that such h is given. Since $f \circ h$ is strictly monotone and $(f \circ h)'$ is continuous, there exists an interval $(a,b) \subset (0,1)$ such that $(f \circ h)'(x) \neq 0$ for each $x \in (a,b)$. Denote $(\alpha,\beta) := h((a,b))$, choose $z \in (\alpha,\beta)$ with $f'(z) = 0$ and put $w := h^{-1}(z)$. Since h is pointwise Lipschitz, clearly $(f \circ h)'(w) = 0$, and that is a contradiction. $\square$

Finally note that we know no (interesting) characterization of those $f : [0,1] \to \mathbf{R}$ for which there exists a differentiable homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has a continuous derivative on $[0,1]$.

7. Almost everywhere differentiability. Bari [**2**, pages 637–640] proved that for each continuous $f : [0,1] \to \mathbf{R}$ there exists a homeomorphism h of $[0,1]$ onto itself such that $(f \circ h)'(x) = 0$ almost everywhere. The same result for continuous $f : [0,1] \to \mathbf{R}^n$ was (independently) mentioned without a proof by Tolstov [**17**, page 152]. Below, we present a simple proof which provides a theorem giving a generalization and improvement of these results.

Our theorem is an easy consequence of the following lemma.

Lemma 7.1. *Let $\omega : [0,\infty] \to [0,\infty]$ be nondecreasing, $\omega(0) = 0$, $\lim_{t\to 0+} \omega(t) = 0$ and let $A \subset [0,1]$ be a first category set. Then there exists a nonatomic Radon measure on $\mathbf{R}$ such that* supp $\mu = [0,1]$, $\mu(\mathbf{R}) > 1$ *and*

$$\lim_{t\to 0+} \frac{\omega(\mu((x-t,x+t)))}{t} \longrightarrow 0 \quad \textit{for each } x \in A. \tag{7.1}$$

Proof. Let $(F_n)_{n=1}^{\infty}$ be a sequence of closed nowhere dense subsets of $[0,1]$ such that $A \subset \cup_{n\in\mathbf{N}} F_n$.

and then easily inductively define $\beta_n > 0$, $n \in \mathbf{N}$, with $\sum_{j\geq n} \beta_j < \alpha_n$.

Let r_k, $k \in \mathbf{N}$, be all rational numbers from $(0,1)$. Now we can clearly choose sets C_n ($n \in \mathbf{N}$) such that each C_n is homeomorphic to the Cantor set and

$$C_k \subset ((r_k - 1/k, r_k + 1/k) \cap (0,1)) \setminus \bigcup_{n\in\mathbf{N}} F_n. \tag{7.2}$$

Denote $\rho_{n,k} := \operatorname{dist}(F_n, C_k) > 0$. Now, for each $k \in \mathbf{N}$, choose μ_k such that

$$\mu_k \text{ is a nonzero nonatomic Radon measure on } \mathbf{R} \text{ with } \operatorname{supp} \mu_k \subset C_k, \tag{7.3}$$

$$\mu_1(\mathbf{R}) = 1, \tag{7.4}$$

$$\begin{aligned} &\mu_k(\mathbf{R}) < 2^{-k}\beta_p \quad \text{for each} \quad k \geq 2 \quad \text{and} \\ &1 \leq p < 1 + \max\{(\rho_{n,k})^{-1} : 1 \leq n < k\}, \end{aligned} \tag{7.5}$$

and put $\mu := \sum_{k\in\mathbf{N}} \mu_k$. Clearly μ is a nonatomic Radon measure with $1 < \mu(\mathbf{R}) < 1 + \alpha_1$ and (7.2) clearly implies $\operatorname{supp} \mu = [0,1]$.

To prove (7.1), consider an arbitrary $x \in A$. Choose $n \in \mathbf{N}$ with $x \in F_n$ and find $p_0 \in \mathbf{N}$ such that $p_0 > (\rho_{n,k})^{-1}$ for each $1 \leq k \leq n$. Denote $V_p := \{y \in \mathbf{R} : (p+1)^{-1} \leq \operatorname{dist}(y, F_n) < p^{-1}\}$ and consider an arbitrary integer $p \geq p_0$.

By (7.3) and the choice of p_0 clearly $\mu_k(V_p) = 0$ for each $1 \leq k \leq n$. If $k > n$, then $\mu_k(V_p) < 2^{-k}\beta_p$. Indeed, if $p < (\rho_{n,k})^{-1}$ then it follows by (7.5) and if $p \geq (\rho_{n,k})^{-1}$, then $\mu_k(V_p) = 0$ by (7.3). Thus we have

$$\mu(V_p) = \sum_{k=2}^{\infty} \mu_k(V_p) < \sum_{k=2}^{\infty} 2^{-k}\beta_p < \beta_p. \tag{7.6}$$

Now for each $0 < t < (p_0)^{-1}$ consider $p \in \mathbf{N}$ for which $(p+1)^{-1} \leq t < p^{-1}$ (clearly $p \geq p_0$). By (7.6) we obtain

$$\mu((x-t, x+t)) \leq \sum_{j=p}^{\infty} \mu(V_j) \leq \sum_{j=p}^{\infty} \beta_j < \alpha_p,$$

implies (7.1). □

Theorem 7.2. *Let X be a normed linear space, $f : [0,1] \to X$ continuous and $A \subset [0,1]$ a first category set. Then there exists a homeomorphism h of $[0,1]$ onto itself such that $(f \circ h)'(x) = 0$ for all $x \in A$.*

Proof. Let ω be the modulus of continuity of f, and let μ be a measure which corresponds to ω by Lemma 7.1. Denote $h(x) = (\mu(\mathbf{R}))^{-1} \cdot \mu((0,x))$, $x \in [0,1]$. Then h is clearly a homeomorphism of $[0,1]$ onto itself. For each $x \in A$ and $t \neq 0$ with $x+t \in [0,1]$, we obtain (using $\mu(\mathbf{R}) > 1$)

$$\|f(h(x+t)) - f(h(x))\| \le \omega(|h(x+t) - h(x)|) \le \omega(\mu((x-|t|, x+|t|))),$$

and thus $(f \circ h)'(x) = 0$ follows from (7.1). □

Since there exists a residual set of Hausdorff dimension zero, we obtain the following improvement of Bari's result mentioned at the beginning of this section.

Corollary 7.3. *Let X be a normed linear space and $f : [0,1] \to X$ be continuous. Then there exists a homeomorphism h of $[0,1]$ onto itself such that $(f \circ h)'(x) = 0$ for all $x \in [0,1]$ except those belonging to a set of Hausdorff dimension zero.*

Remark 7.4. It is easy to show that, in Theorem 7.2, the assumption that A is of the first category cannot be relaxed. Indeed, if $f \circ h$ is differentiable at all points of a second category set, then (proceeding as in the proof of Lemma 2.13) we obtain that $f \circ h$ is Lipschitz on some interval and consequently f has bounded variation on some interval.

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