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GENERALIZED PRESCRIBED SCALAR CURVATURE TYPE EQUATION ON A COMPACT MANIFOLD OF NEGATIVE SCALAR CURVATURE

MOHAMMED BENALILI AND YOUSSEF MALIKI

ABSTRACT. This paper deals with the problem of the so-called generalized prescribed scalar curvature type equation on a compact Riemannian manifold with negative scalar curvature. We give the existence of a positive solution which is the subject of the first theorem. In the second one, we prove the multiplicity of solutions of the subcritical quasilinear elliptic equation.

1. Introduction. Let (M,g) be a Riemannian n-manifold. For $n \geq 3$, if $g' = u^{4/(n-2)}g$, $u \in C^{\infty}(M)$, $u > 0$, on M, is a metric conformal to g, the scalar curvatures R and $\widetilde{R}$ of g and g' respectively satisfy the equation

$$\Delta_g u + \frac{n-2}{4(n-1)} R u = \frac{n-2}{4(n-1)} \widetilde{R} u^{2^*-1}$$

where $2^* = (2n/n-2)$ and $\Delta_g u = -\mathrm{div}_g(\nabla u)$ is the Laplacian of u.

A smooth function f on M will be the scalar curvature of a conformal metric g' if there exists a function $u \in C^{\infty}(M)$, $u > 0$, solution of the equation

$$\Delta_g u + \frac{n-2}{4(n-1)} R u = f u^{2^*-1}. \tag{1}$$

Such equation has been intensively studied in the past two decades: as examples, we can refer to the works of Aubin [**1**], Bahri-Coron [**2**], Escobar-Schoen [**4**], Hebey [**6**], Kazdan-Warner [**7**], Schoen [**9**] and Druet [**3**].

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In the case where R is a negative constant and f is a changing sign function, Rauzy [**8**] states the following results:

Letting f be a C^∞function on M, we denote by $f^- = -\inf(f,0)$, $f^+ = \sup(f,0)$, $H_1^2(M)$ the standard Sobolev space and

$$A = \left\{ u \in H_1^2(M), u \geq 0, u \not\equiv 0 \quad \text{such that} \quad \int_M f^- u \, dv_g = 0 \right\}.$$

Put

$$\lambda_f = \inf_{u \in A} \frac{\int_M |\nabla u|^2 \, dv_g}{\int_M u^2 \, dvg}$$

and

$$\lambda_f = +\infty \quad \text{if} \quad A = \phi.$$

Theorem $1'$ **(Critical Case).** *There is a constant $C > 0$ which depends only on $f^- / \int_M f^- dv_g$ such that if $f \in C^\infty$ on M fulfills the following conditions*:

($1'$) $|R| < (4(n-1))/(n-2)\lambda_f$

($2'$) $(\sup f^+ / \int_M f^- dv_g) < C$.

Equation (1) *admits a positive solution.* (R *is a negative constant and f is a changing sign function*).

Theorem $2'$ **(Subcritical Case).** *For every f C^∞ function on M, there exists a constant $C > 0$ which depends only on $f^- / (\int_M f^- dv_g)$ such that if f satisfies the following conditions*

($1''$) $|R| < (4(n-1)/n - 2\lambda_f)$

($2''$) $\sup f^+ / (\int f^- dv_g) < C$

($3''$) $\sup f > 0$.

Then the equation $\Delta_g u + Ru = f u^{q-1}$, $q \in \,]2, 2^*[$ (R is strictly negative and f is a changing sign function) admits two nontrivial distinct solutions.

In this work we try to extend the results cited above to the quasilinear elliptic equation. To achieve this task, we let (M, g) be a compact

Riemannian manifold of dimension $n \geq 3$, $p \in (1,n)$ and let $H_1^p(M)$ be the Sobolev space defined as the subspace of functions from $L^p(M)$ whose gradients are also in $L^p(M)$ endowed with the norm $\|u\|_{1,p} = \|\nabla u\|_p + \|u\|_p$, where $\|.\|_p$ denotes the norm in $L^p(M)$.

We consider the following generalization of equation (1)

$$\Delta_p u + au^{p-1} = fu^{p^*-1}, \tag{2}$$

where $p^* = np/(n-p)$ and $\Delta_p u = -\mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u)$ is the p-Laplacian operator on the manifold M, $u \in H_1^p(M)$ is a positive function, f is a changing sign C^∞ function on M and a is a negative constant. Following the terminology used in [**3**], we refer to equation (2) as the *generalized scalar curvature type equation.* Merely speaking, we state

Theorem 1 (Critical Case). *There is a constant $C > 0$ which depends only on $f^-/(\int f^- dv_g)$ such that if $f \in C^\infty$ on M fulfills the following conditions*:

(i) $|a| < \lambda_f$

(ii) $\sup f^+/(\int f^- dv_g) < C$

(iii) $\sup f > 0$.

Then equation (2) *admits a positive solution of class $C^{1,\alpha}(M)$, for some $\alpha \in (0,1)$.*

Theorem 2 (Subcritical Case). *For every C^∞ function f on M, there exists a constant $C > 0$ which depends only on $f^-/(\int f^- dv_g)$ such that if f satisfies the following conditions*:

(i) $|a| < \lambda_f$

(ii) $(\sup f^+/\int f^- dv_g) < C$

(iii) $\sup f > 0$.

Then the equation

$$\Delta_p u + au^{p-1} = fu^{q-1}, \quad q \in]p, p^*[\tag{3}$$

admits two nontrivial distinct positive solutions of class $C^{1,\alpha}(M)$, for some $\alpha \in (0,1)$.

2. The critical case. In this section we are going to prove Theorem 1. For this task we use the following theorem from minimization theory.

Theorem 3. *Let V be a reflexive Banach space, and let X be a weakly closed subset of V. Let $f : X \to R \cup \{+\infty\}$ be such that*

1. *f is coercive*
2. *f is weakly lower semi-continuous, that is, $u_n \in X, u_n \to u$ implies $f(u) \leq \liminf f(u_n)$.*

Then

a) $\beta = \inf_{x \in X} f > -\infty$

b) *There exists $x_0 \in X$ such that $f(x_0) = \beta$.*

If f is a Gateaux differentiable in x_0, then $d_G f(x_o) = 0$.

Since it is possible to solve equation (2) for f or αf where α is a constant, we consider the functional

$$F_q(u) = \|\nabla u\|_p^p + a\, \|u\|_p^p - \int_M f u^q dv_g, \quad q \in]p, p^*[\,,$$

and, first, we show that F_q is weakly lower semi-continuous. Letting $\overline{B_{H_1^p(M)}(0,\rho)} = \left\{ u \in H_1^p(M) : \; u \geq 0, \; \|u\|_{1,p} \leq \rho \right\}$, we state

Lemma 1. *There exists $\rho > 0$ such that F_{p^*} is weakly lower semi-continuous on the closed ball $\overline{B_{H_1^p(M)}(0,\rho)}$.*

Proof. Let $(u_j) \subset H_1^p(M)$ with $u_j \to u$ weakly in $H_1^p(M)$ and $\|u_j\|_{1,p} \leq \rho$, up to a subsequence we have

$$u_j \to u \quad \text{in} \quad L^s(M) \quad \text{for} \quad s < p^* \tag{4}$$

$$\|u\|_{1,p} \leq \liminf \|u_j\|_{1,p} \tag{5}$$

$$u_j(x) \to u(x) \quad \text{for a.e.} \quad x \in M. \tag{6}$$

It suffices to show that

$$\int_M |\nabla u_j|^p \, dv_g - \int_M |\nabla u|^p \, dv_g - \int_M f(u_j^{p^*} - u^{p^*}) \, dv_g \geq o(1). \tag{7}$$

Thanks to the Bresis-Lieb lemma, we have

$$\int_M |\nabla u_j|^p \, dv_g - \int_M |\nabla u|^p \; dv_g = \|u_j - u\|_{1,p}^p + o(1)$$

and

$$\int_M f(u_j^{p^*} - u^{p^*}) \, dv_g = \int_M f \, |u_j - u|^{p^*} \; dv_g + o(1).$$

On the other hand, the Sobolev inequality gives us

$$\int_M f \, |u_j - u|^{p^*} \; dv_g \leq \sup_M f(x) \max\left[K(n,p)^p + \varepsilon, A\right]^{p^*/p} \|u_j - u\|_{1,p}^{p^*}$$

where ε is any positive number and $K(n,p)$ and A are the best constants in the Sobolev imbedding. So the righthand side of the inequality (7) is greater or equals to

$$\begin{aligned} &\|u_j - u\|_{1,p}^p \left(1 - \sup_M f(x) \max\left[K(n,p)^p + \varepsilon, A\right]^{p^*/p} \|u_j - u\|_{1,p}^{p^*-p}\right) \\ &\quad + o(1) \\ &\quad \geq \|u_j - u\|_{1,p}^p \left(1 - \sup_M f(x) \max\left[K(n,p)^p + \varepsilon, A\right]^{p^*/p} 2^{p^*-p} \right. \\ &\qquad\qquad\qquad \left. \times \; \max(\|u_j\|_{1,p}, \|u\|_{1,p})^{p^*-p}\right) + o(1). \end{aligned}$$

We choose the radius of the ball $\overline{B_{H_1^p(M)}(0,\rho)}$ small enough so that it satisfies our claim. □

As in [**8**], we define the quantities

$$\lambda_{f,\eta,q} = \inf_{u \in A(\eta,q)} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}$$

with

$$A\,(\eta,q) = \left\{u \in H_1^p(M) : u \geq 0, \; \|u\|_q^q = 1, \right. \\ \left. \text{and} \int_M f^- u^q \, dv_g = \eta \int_M f^- \, dv_g \right\} \tag{8}$$

for a real $\eta > 0$, $p < q < p^*$, and

$$\lambda'_{f,\eta,q} = \inf_{u \in A'(\eta,q)} \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \tag{9}$$

with

$$A'(\eta,q) = \Big\{ u \in H_1^p(M) : u \geq 0, \|u\|_q^q = 1 \\ \text{and } \int_M f^- u^q \, dv_g \leq \eta \int_M f^- \, dv_g \Big\}.$$

The following facts which are proven in [**8**], for $p = 2$, remain valid in the general case $p \in (1, n)$: $\lambda'_{f,\eta,q}$ is a decreasing function with respect to η, and it is bounded by λ_f for any fixed $\eta > 0$ and $\lambda_{f,\eta,q} = \lambda'_{f,\eta,q}$, so $\lambda_{f,\eta,q}$ is also a decreasing function with respect to η, and bounded by λ_f, for any fixed $\eta > 0$.

Also the following lemmata which are established in [**8**], for $p = 2$, are still valid in the case $p \in (1, n)$.

Lemma 2. *For any $q \in]p, p^*[$, $\lambda_{f,\eta,q}$ goes to λ_f whenever η goes to zero.*

Lemma 3. *Let $\varepsilon > 0$. There exists η_o such that for any $\eta < \eta_o$, there is q_η such that $\lambda_{f,\eta,q} \geq \lambda_f - \varepsilon$, for any $q > q_\eta$.*

Lemma 4. *There exists η_o such that for any $\eta < \eta_o$, there is q_η such that for any $q > q_\eta$, $\lambda_{f,\eta,q} > |a|$.*

Denote by $K(n,p)$ and A, as mentioned before, the best constants in the Sobolev imbedding and $\varepsilon > 0$ any fixed real number

Keeping in mind the results given in the above lemmata, we prove the following

Lemma 5. *Suppose that $\sup_M f^+ / (\int f^- dv_g) < (\eta\mu / 8|a|)$, where $\mu = \inf(|a|, (\delta/A + (|a| + \delta)(K(n,p)^p + \varepsilon)))$.*

Then for a fixed $\rho > 0$, there exists $\xi > 0$ and such that for every $u \in H_1^p(M)$ with $\|u\|_{1,p} = \rho$, we have $F_q(u) > \xi \|u\|_{1,p}$ where $q \in]p, p^*]$.

Proof. We adapt to our setting the proof in [**8**]. Let $u \in H_1^p(M)$ with $\|u\|_q^q = k$ and $k^{1-(p/q)} \geq 2|a|/\eta \int f^- dv_g$.

Putting

$$G_q(u) = \|\nabla u\|_p^p + a\,\|u\|_p^p + \int_M f^- u^q\,dv_g,$$

we get

$$G_q(u) \geq a\,\|u\|_p^p + \int_M f^- u^q\,dv_g.$$

So if

$$\int_M f^- u^q dvg \geq \eta k \int_M f^-\,dv_g$$

then

$$\begin{aligned} G_q(u) &\geq a\,\|u\|_p^p + \eta k \int_M f^-\,dv_g \\ &\geq a k^{p/q} + \eta k \int_M f^- dv_g \\ &= k^{p/q}\,|a| \left[-1 + \frac{\eta \int_M f^- dv_g}{|a|} k^{1-(p/q)}\right] \\ &\geq k^{p/q}\,|a|\,. \end{aligned} \tag{10}$$

And in the case $\int_M f^- u^q\,dv_g < \eta k \int_M f^-\,dv_g$, by (9), we get

$$G_q(u) \geq (\lambda_{f,\eta,q} + a)\,\|u\|_p^p + \int_M f^- u^q\,dv_g$$

and by Lemma 3, we can choose η_o and q_η such that for every $q < \eta_o$ and $q > q_\eta$ we have $\delta = \lambda_{f,\eta,q} + a > 0$. The lower boundedness of $G_q(u)$ will be obtained as follows

$$\begin{aligned} G_q(u) &\geq \delta\,\|u\|_p^p + \int_M f^- u^q\,dv_g \\ &= \delta_1\,\|u\|_p^p + \frac{\delta_2}{|a|}\Big(\,\|\nabla u\|_p^p + \int_M f^- u^q\,dvg - G_q(u)\Big) \\ &\qquad + \int_M f^- u^q\,dv_g \end{aligned}$$

where δ_1, δ_2 are two positive constants such that $\delta_1 + \delta_2 = \delta$.

So

$$\left(1+\frac{\delta_2}{|a|}\right)G_q(u) \geq \delta_1 \|u\|_p^p + \frac{\delta_2}{|a|}\|\nabla u\|_p^p + \left(1+\frac{\delta_2}{|a|}\right)\int_M f^- u^q \, dv_g.$$

Letting $\delta_1 = A/(|a|(K(n,p)^p+\varepsilon))\delta_2$ where $K(n,p)$ and A are, as mentioned before, the best constants in the Sobolev imbedding and $\varepsilon > 0$ is any fixed real number. We have

$$\begin{aligned}\left(1+\frac{\delta_2}{|a|}\right)&G_q(u)\\ &\geq \frac{\delta_2}{|a|\left(K(n,p)^p+\varepsilon\right)}\left(\left(K(n,p)^p+\epsilon\right)\|\nabla u\|_p^p + A\|u\|_p^p\right)\\ &\quad + \left(1+\frac{\delta_2}{|a|}\right)\int_M f^- u^q \, dv_g\\ &\geq \frac{\delta_2}{|a|\left(K(n,p)^p+\varepsilon\right)}\|u\|_q^{p/q},\end{aligned}$$

then we obtain

$$G_q(u) \geq \frac{\delta}{A + (|a|+\delta)\left(K(n,p)^p+\epsilon\right)} k^{p/q}.$$

Letting $\mu = \inf(|a|, (\delta/A + (|a|+\delta)(K(n,p)^p+\varepsilon)))$, we get

$$\begin{aligned}F_q(u) &\geq G_q(u) - \int_M f^+ u^q dv_g\\ &\geq \mu k^{p/q} - \left(\sup_M f^+\right) k\\ &= \frac{1}{2}\mu k^{p/q} + k^{p/q}\left[\frac{\mu}{2} - \left(\sup_M f^+\right) k^{1-(p/q)}\right]\\ &\geq \frac{1}{2}\mu k^{p/q}\end{aligned}$$

provided that $k \leq (\mu/2 \sup_M f^+)^{q/(q-p)}$.

Now, since by assumption we have assumed that $(\sup_M f^+/\int f^- dv_g) < (\eta\mu/8t|a|)$ $(C = (8\mu/\eta|a|))$, it follows that $F_q(u) \geq (1/2)\mu k^{p/q}$ provided that $k \leq (2|a|/\sup_M f^+))^{q/(q-p)} 2^{q/(q-p)}$. Moreover, if we let

$q \to p^*$, μ does not go to zero, since by Lemma 4 we can choose η_o such that for any $\eta < \eta_o$ there exists q_η such that for every $q > q_\eta$, $\delta = a + \lambda_{f,\eta,q} > 0$ and $F_{p^*}(u) = \lim_{q\to p^*} F_q(u) \geq (1/2)\mu k^{n/(n-p)}$ provided that $k \leq 2^{n/p}(2|a|/\eta \int f^-)^{n/p}$. And since k is chosen in the beginning of the proof as $k \geq (2|a|/\eta \int f^-)^{q/(q-p)}$ that means that, at the limit, k belongs to the interval

$$I = \left[\left(\frac{2\,|a|}{\eta \int f^-} \right)^{n/p}, 2^{n/p} \left(\frac{2\,|a|}{\eta \int f^-} \right)^{n/p} \right].$$

Finally, fix $\rho > 0$, and let $u \in H_1^p(M)$ with $\|u\|_{1,p} = \rho$ and $\xi_1 > 0$ such that $k^{n/(n-p)} \geq \xi_1 \rho$; it follows that

$$F_{p^*}(u) \geq \frac{1}{2}\xi \,\|u\|_{1,p} \quad \text{with} \quad \xi = \frac{1}{2}\mu\xi_1,$$

and Lemma 5 is proven. □

Lemma 6. *For each $t > 0$ small enough, $\inf_{\|u\|_{1,p}\leq t} F_q(u) < 0$, $q \in \,]p, p^*]$.*

In fact, $F_q(t) = t^p(a \operatorname{vol}(M) - t^{q-p} \int_M f\, dv_g)$, where $\operatorname{vol}(M)$ denotes the volume of M, and there is $t_o > 0$ small enough such that $\inf_{\|u\|_{1,p}\leq t} F_q(u) < 0$ for each $t \in \,]0, t_o[$.

2.1 *Proof of Theorem* 1.

Proof. By Lemmas 1, 5 and 6, there exists $u_1 \in \overline{B_{H_1^p(M)}(o,\rho)}$ such that $F_q(u_1) = \min_{\|u\|_{1,p}\leq\rho} F_q(u_1) < 0$, for ρ small enough and it has to be $\|u\|_{1,p} < \rho$, otherwise we get by Lemma 5 $F_q(u_1) > 0$. In particular, u_1 is a weak solution of equation(2).

It remains to show that the solution of equation (2) is regular and positive. To do so, we use the following theorems adapted by Druet [**3**] for the context of manifolds from those of Tolksdorf [**10, 11**], Guedda-Veron [**5**] and Vasquez [**12**] when dealing with Euclidian context. □

Theorem 4 ($C^{1,\alpha}$-regularity). *Let (M, g) be a compact Riemannian n-manifold, $n \geq 2$, $p \in (1, n)$, and let $h \in C^0(M \times R)$ be such that, for* *all* $(x,r) \in M \times R$,

$$|h(x,r)| \leq C|r|^{p^*-1} + D$$

where C *and* D *are positive constants.*

If $u \in H_1^p(M)$ *is a solution of* $\Delta_p u + h(x,u) = 0$, *then* $u \in C^{1,\alpha}(M)$.

Theorem 5 (Strong maximum principle). *Let* (M,g) *be a compact Riemannian* n*-manifold,* $p \in (1,n)$, *and let* $u \in C^1(M)$ *be such that*

$$\Delta_p u + f(.,u) \geq 0 \quad on \quad M,$$

f *such that*

$$\begin{cases} f(x,s) < f(x,r) & \forall x \in M, \ \forall\ 0 \leq r < s \\ |f(x,s)| \leq C\left(K + |r|^{p-2}\right)|r| & \forall (x,r) \in M \times R,\ C > 0, \end{cases}$$

where C *and* K *are positive constants.*

If $u \geq 0$ *on* M *and* u *does not vanish identically, then* $u > 0$ *on* M.

3. The subcritical case.

3.1 *The first subcritical solution.* Letting $B_{k,q} = \{u \in H_1^p(M),\ u \geq 0,\ \|u\|_q^q = k\}$ and $\mu_{k,q} = \inf_{u \in B_{k,q}} F_q(u)$, we have

Proposition 1. *The subcritical equation* (3) *admits a positive solution* v *with* $F_q(v) < 0$.

Proof. Let $(u_j) \subset B_{k,q}$ be a minimizing sequence,

$$\lim_{j \to +\infty} F_q(u_j) = \mu_{k,q}$$

so for j large enough,

$$F_q(u_j) \leq \mu_{k,q} + 1,$$

$$\begin{aligned}\|\nabla u_j\|_p^p &\leq -a\,\|u_j\|_p^p + \sup_M |f(x)|\,\|u_j\|_q^q + \mu_{k,q} + 1 \\ &= -ak^{\frac{p}{q}} + k\sup_M |f(x)| + \mu_{k,q} + 1 < +\infty\end{aligned}$$

and

$$\|u_j\|_p^p = k^{p/q}$$

then (u_j) is bounded in $H_1^p(M)$ and there exists a subsequence still denoted (u_j) which converges weakly to u in $H_1^p(M)$.

The compactness of the imbedding of $H_1^p(M)$ in $L^q(M)$ and the uniqueness of the weak limit guarantee the existence of subsequence which converges strongly to u in $L^q(M)$.

Then u fulfills

$$\|u\|_q^q = k$$

and

$$\int |\nabla u|^{p-2}\,\nabla u.\nabla v + a\int u^{p-1}v - \frac{q}{2}\int fu^{q-1}v = \mu_{k,q}\int u^{q-1}v.$$

The regularity Theorem 4 shows us that $u \in C^{1,\alpha}(M)$ and the strong maximum principle, Theorem 5, asserts that $u > 0$.

To show that $F_q(u) < 0$, it suffices to remark that $F_q(u) \leq F_q(k^{1/q}) = k^{p/q}(\text{avol}\,(M) - k^{1-(p/q)}\int_M f)$, and we choose k small enough such that $F_q(k^{1/q}) < 0$. Then $F_q(u) < 0$. □

3.2 *The second subcritical solution.* In this section we seek for a second subcritical solution to the equation (2); to achieve this task we use

Theorem 6 (The Mountain Pass theorem). *Suppose that* $f \in C^1(V), V$ *Banach space. Assume*

1. $f(0) = 0$.
2. *There exists* $r > 0$ *such that* $f(u) > \alpha > 0$ *for all* $\|u\| = r$.
3. *There exists* $\check{u} \in V$ *such that* $\|\check{u}\| > r$ *and* $f(\check{u}) < 0$.

Then setting

$$\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0,\ f(\gamma(1)) < 0\}$$

and

$$\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$$

we have $\beta \geq \alpha > 0$, there exists a $(PS)_\beta$ sequence and, if $(PS)_\beta$ holds, β is a critical level for f.

First we show that F_q, $q \in]p, p^*[$ satisfies the Palais-Smale condition.

Lemma 7. *Each Palais-Smale sequence for the functional F_q is bounded.*

Proof. We argue by contradiction. Suppose that there exists a sequence $\{u_j\}$ such that $F_q(u_j)$ tends to a finite limit c, $F'_q(u_j)$ goes to zero and u_j to infinite in the $H_1^p(M)$-norm. More explicitly we have for each $v \in H_1^p(M)$

$$\int_M |\nabla u_j|^p \, dv_g + a \int_M u_j^p \, dv_g - \int_M f u_j^q \, dv_g \longrightarrow c$$

and

$$\int_M |\nabla u_j|^{p-2} \langle \nabla u_j, \nabla v \rangle \, dv_g + a \int_M u_j^{p-1} v \, dv_g - \frac{q}{p} \int_M f u_j^{q-1} v \, dv_g \longrightarrow 0$$

so for any $\varepsilon > 0$ there exists a positive integer N such that for every $j \geq N$ we have

$$\left| \int_M |\nabla u_j|^p \, dv_g + a \int_M u_j^p \, dv_g - \int_M f u_j^q \, dv_g - c \right| \leq \varepsilon$$

and

$$\left| \int_M |\nabla u_j|^{p-2} \langle \nabla u_j, \nabla v \rangle \, dv_g + a \int_M u_j^{p-1} v \, dv_g - \frac{q}{p} \int_M f u_j^{q-1} v \, dv_g \right| \leq \varepsilon.$$

In the particular case where $v = u_j$, we get

$$\left| \int_M |\nabla u_j|^p \, dv_g + a \int_M u_j^p \, dv_g - \int_M f u_j^q \, dv_g - c \right| \leq \varepsilon$$

and

$$\left|\int_M |\nabla u_j|^p \, dv_g + a\int_M u_j^p \, dv_g - \frac{q}{p}\int_M f u_j^q \, dv_g\right| \le \varepsilon. \tag{11}$$

Then, we obtain

$$\left|\int_M |\nabla u_j|^p \, dv_g + a\int_M u_j^p \, dv_g - qc\right| \le \frac{p+q}{q-p}\varepsilon \tag{12}$$

and

$$\left|(q-p)\int_M f u_j^q \, dv_g - pc\right| \le 2p\varepsilon. \tag{13}$$

By Lemma 5, we can choose k to be an $L^q(M)$-norm such that

$$\inf_{\|u\|_q^q = k} F_q(u) > 0.$$

Letting $v_j = k^{1/q}(u_j/\|u_j\|_q)$, we obtain from (12) and (13) that

$$\left|(q-p)\int_M f v_j^q \, dv_g - \frac{pck^{p/q}}{\|u_j\|_q^p}\right| \le 2p\varepsilon \frac{k^{p/q}}{\|u_j\|_p^p}. \tag{14}$$

Since $\|v_j\|_q$ is a bounded sequence, it follows that $\{v_j\}$ is bounded in $H_1^p(M)$. If $\|u_j\|_q$ goes to infinity, it follows from (11) and (14) that $F_q(v_j)$ tends to zero. And since $\|v_j\|_q^q = k$, we have

$$\inf_{\|u\|_q^q = k} F_q(u) \le F_q(v_j)$$

so

$$\inf_{\|u\|_q^q = k} F_q(u) \le 0,$$

hence a contradiction. Then the sequence $\{u_j\}$ is bounded in $H_1^p(M)$. Now, since $q < p^*$, the Sobolev injections are compact, so the Palais-Smale condition is satisfied. □

3.3 *Proof of Theorem* 2. To prove Theorem 2, we use the Pass Mountain theorem. By Lemma 5, there exists $\rho > 0$ such that

$F_q(u) > \eta$ for any $\|u\|_{1,p} = \rho$. Let $u \in C^1(M)$ such that $\int_M f u^q \, dv_g > 0$ and $\|u\|_q^q = 1$. It follows

$$\lim_{t\to\infty} F_q(tu) = \lim_{t\to\infty} t^p \left[\|\nabla u\|_p^p + a\,\|u\|_p^p - t^{q-p} \int_M f u^q \, dv_g \right] = -\infty;$$

then there exists $t_o > 0$ with $\sup_{t \geq t_o} F_q(tu) < 0$. Put $w = t_1 u$ with $t_1 > \rho$.

Proof. (Theorem 2). Now setting

$$\Gamma = \left\{ \gamma \in C\left([0,1], H_1^p(M)\right) : \gamma(0) = 0, \gamma(1) = w \right\},$$

and

$$\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F_q\left(\gamma(t)\right),$$

by the Pass-Mountain theorem we have $\beta \geq \alpha > 0$, and there exists a Palais Smale sequence at level β. Since by Lemma 7, the Palais-Smale condition at level β is a critical level for F_q, then the subcritical equation admits a solution $u_2 \in H_1^p(M)$. By the regularity Theorem 4, and the Strong maximal principle, Theorem 5, we have $u_2 \in C^{1,\alpha}(M)$ and $u_2 > 0$ and Proposition 1 completes the proof. □

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University Abou-BekrBelkaïd, Faculty of Sciences, Dept. Math., B.P.119, Tlemcen, Algeria
Email address: m_benalili@mail.univ-tlemcen.dz

University Abou-BekrBelkaïd, Faculty of Sciences, Dept. Math., B.P.119, Tlemcen, Algeria
Email address: m-youssef@mail.univ-tlemcen.dz, malyouc@yahoo.fr