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THE MULTIPLICITY OF SPECTRA OF A VECTORIAL STURM-LIOUVILLE DIFFERENTIAL EQUATION OF DIMENSION TWO AND SOME APPLICATIONS

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ABSTRACT. We consider problems related to the multiplicity of the eigenvalues of the following vectorial Sturm-Liouville differential equation on the interval $[0,1]$ with respect to certain homogeneous boundary conditions at the points $x=0$ and $x=1$:

$$(E)\quad \begin{cases} (p\vec{u}\,')'(x)+(\lambda r(x)E_2-Q(x))\vec{u}(x)=0; \\ \vec{u}(0)\cos\alpha+\vec{u}\,'(0)\sin\alpha=\vec{u}(1)\cos\beta+\vec{u}\,'(1)\sin\beta=0, \end{cases}$$

where $0\le\alpha,\ \beta<2\pi$, $p(x)$ and $r(x)$ are positive scalar-valued functions and have absolutely continuous first derivatives with measurable second derivatives on $0\le x\le 1$, E_2 denotes a 2×2 identity matrix, $Q(x)$ is a continuous 2×2 real symmetric matrix-valued functions on $0\le x\le 1$, and $\vec{u}(x)$ is an R^2-valued function. We present that under certain assumptions on the scalar-valued functions $p(x), r(x)$ and the matrix-valued function $Q(x)$, (E) can only possess finitely many eigenvalues which have multiplicity 2 and find a lower bound m_Q, such that the eigenvalues of (E) with index exceeding m_Q are all simple. These results are applied to find some sufficient conditions which ensure that the spectra of the following two potential equations, $i=1,2$:

$$(q_i)\quad \begin{cases} (pu')'(x)+(\lambda r(x)-q_i(x))u(x)=0; \\ u(0)\cos\alpha+u'(0)\sin\alpha=u(1)\cos\beta+u'(1)\sin\beta=0 \end{cases}$$

have finitely many elements in common, and we obtain an estimate of the number of elements in the intersection of two spectra.

1. Introduction. As a generalization of the scalar Sturm-Liouville equations, vectorial Sturm-Liouville equations were recently found to

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be important in the study of particle physics, see [**9**]. In this paper, we concentrate mainly on the multiplicity of eigenvalues of a vectorial Sturm-Liouville differential equation of dimension two. It was known, see [**6**], that if the functions $r(x)$ and $p(x)$ are taken as positive and continuous on $[0,1]$, furthermore, they are assumed to have absolutely continuous first derivatives with measurable second derivatives and $q(x)$ is assumed to be measurable on $[0,1]$ as well, then the potential equation

$$\begin{cases} (pu')'(x) + (\lambda r(x) - q(x))u(x) = 0; \\ u(0)\cos\alpha + u'(0)\sin\alpha = 0, \\ u(1)\cos\beta + u'(1)\sin\beta = 0, & 0 \le \alpha, \beta < \pi, \end{cases} \tag{1.1}$$

possesses discrete spectra, that all eigenvalues are real, simple, i.e., of multiplicity one, and that the kth eigenfunction has exactly $k-1$ sign-changed zeros in the interval $0 < x < 1$, and the functions form a complete orthogonal system in the Hilbert space

$$L_r^2[0,1] = \Big\{ f : [0,1] \longrightarrow \mathbf{C}, \int_0^1 |f|^2 r \, dx < \infty \Big\},$$

where we use the inner product

$$(f,g) = \int_0^1 r f \bar{g} \, dx, \quad \text{for any} \quad f, g \in L_r^2[0,1].$$

But for the following two-dimensional vectorial Sturm-Liouville differential equation

$$(p\vec{u}\,')'(x) + (\lambda r(x) E_2 - Q(x))\,\vec{u}(x) = 0, \tag{1.2}$$

subject to the boundary conditions

$$\vec{u}(0)\cos\alpha + \vec{u}\,'(0)\sin\alpha = \vec{u}(1)\cos\beta + \vec{u}\,'(1)\sin\beta = 0, \tag{1.3}$$

where E_2 is a 2×2 identity matrix, $Q(x) = \begin{pmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{pmatrix}$, $0 \le \alpha$, $\beta < \pi$, $p_i(x)$, $i = 1,2$, are assumed to be continuous functions defined on $0 \le x \le 1$ and $r_1(x)$ is C^1-function on $[0,1]$, and the functions $r(x)$ and $p(x)$ are taken as positive scalar-valued functions and have absolutely continuous first derivatives with measurable second derivatives on $0 \le x \le 1$, the multiplicity of the eigenvalues and properties of the eigenfunctions are not well understood except for the case when $p(x) = r(x) = 1$ and $\sin\alpha = \sin\beta = 0$ in (1.2) and (1.3), see [**8**]. One purpose of this paper is to investigate the multiplicity problems of eigenvalues of the equation (1.2) and (1.3). In this work, using the theory of matrix differential equations and the ideas and methods in the paper [**8**], with some modification of course, we obtain the following theorems.

Theorem 1. *If* $Q(x) = \left(\begin{smallmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{smallmatrix}\right)$, *and*

$$\int_0^1 \frac{r_1(x)}{\sqrt{r(x)p(x)}}\, dx \neq 0,$$

then except finitely many eigenvalues, all eigenvalues of (1.2) *and* (1.3) *are simple.*

Let $\sigma(q)$ denote the spectrum of the potential equation (1.1) with the potential function $q(x)$.

Theorem 2. *Suppose* $q_1(x)$ *and* $q_2(x)$ *are two real-valued continuous functions on* $0 \le x \le 1$. *If*

$$\int_0^1 \frac{q_1(x) - q_2(x)}{\sqrt{r(x)p(x)}}\, dx \neq 0,$$

then $\sigma(q_1)$ *and* $\sigma(q_2)$ *only have finitely many elements in common.*

Theorem 3. *Suppose* $Q(x) = \left(\begin{smallmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{smallmatrix}\right)$, *and*

$$\int_0^1 \frac{r_1(x)}{\sqrt{r(x)p(x)}}\, dx \neq 0.$$

If λ_n *is an eigenvalue of* (1.2) *and* (1.3) *satisfying the following condition,*

$$\lambda_n > \Lambda_{*1} \quad \text{if} \ \sin\alpha = 0; \qquad \lambda_n > \Lambda_{*2} \quad \text{if} \ \sin\alpha \neq 0,$$

then λ_n *is a simple eigenvalue.*

It is worth pointing out that Theorem 3 plays an important role in the estimate of some related eigenvalue problems, see Theorems 4 and 5.

2. The multiplicity of spectra of a vectorial Sturm-Liouville differential equation of dimension two. To study the eigenvalue problems (1.2) and (1.3), we now apply the Liouville transformation to reduce the equation (1.2) and (1.3) to the Liouville normal form. Let

$$t=\int_0^x \sqrt{\frac{r(s)}{p(s)}}\,ds, \qquad \rho(x)=(p(x)r(x))^{1/4}, \qquad \vec{y}(t)=\rho(x)\,\vec{u}(x), \tag{2.1}$$

so that the equation (1.2) and (1.3) is transformed into a vectorial Sturm-Liouville equation of the form

$$\vec{y}\,''(t)+(\lambda E_2-\widetilde{Q}(t))\,\vec{y}(t)=0 \tag{2.2}$$

on the interval $[0,c]$ subject to the boundary conditions

$$A_0\,\vec{y}(0)+B_0\,\vec{y}\,'(0)=A_c\,\vec{y}(c)+B_c\,\vec{y}\,'(c)=0, \tag{2.3}$$

where

$$\begin{aligned}
c&=\int_0^1 \sqrt{\frac{r(s)}{p(s)}}\,ds,\\
\widetilde{Q}(t)&=\frac{1}{r(x)}\,Q(x)+\left\{\frac{p(x)\rho''(x)}{r(x)\rho(x)}+\frac{\rho'(x)}{2\rho(x)}\left(\frac{p(x)}{r(x)}\right)'\right\}E_2,\\
A_0&=\frac{\cos\alpha}{\sqrt[4]{r(0)p(0)}}-\frac{r'(0)p(0)+r(0)p'(0)}{4r(0)p(0)\sqrt[4]{r(0)p(0)}}\,\sin\alpha,\\
B_0&=\sqrt[4]{\frac{r(0)}{p^3(0)}}\,\sin\alpha,\\
A_c&=\frac{\cos\beta}{\sqrt[4]{r(1)p(1)}}-\frac{r'(1)p(1)+r(1)p'(1)}{4r(1)p(1)\sqrt[4]{r(1)p(1)}}\,\sin\beta,\\
B_c&=\sqrt[4]{\frac{r(1)}{p^3(1)}}\,\sin\beta.
\end{aligned} \tag{2.4}$$

Remark 1. The Liouville transformation implies that a real value λ is an eigenvalue of problem (1.2) and (1.3) if and only if λ is an eigenvalue of problem (2.2) and (2.3), and the multiplicity of λ as an eigenvalue of problem (1.2) and (1.3) is equal to the multiplicity of λ as an eigenvalue of problem (2.2) and (2.3).

Denote

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{p_1}(t) & -\widetilde{r_1}(t) \\ -\widetilde{r_1}(t) & \widetilde{p_2}(t) \end{pmatrix}, \tag{2.5}$$

from (2.4) we see that

$$\widetilde{r_1}(t) = \frac{r_1(x)}{r(x)}. \tag{2.6}$$

We consider the following matrix differential equations:

$$\begin{aligned} &Y''(t) + (\lambda E_2 - \widetilde{Q}(t))Y(t) = 0, \\ &Y(0) = B_0 E_2,\ Y'(0) = -\,A_0 E_2, \end{aligned} \tag{2.7}$$

where $Y(t)$ is a 2×2 matrix-valued function. We use $Y(t, \lambda)$ to denote the solution of the initial value problem (2.7). Then λ_* is an eigenvalue of problem (1.2) and (1.3), and hence also of problem (2.2) and (2.3), if and only if $A_c Y(c, \lambda_*) + B_c Y'(c, \lambda_*)$ is a singular matrix. The multiplicity of λ_* as an eigenvalue of problem (1.2) and (1.3) is equal to the dimension of the null space of $A_c Y(c, \lambda_*) + B_c Y'(c, \lambda_*)$. By the theory of self-adjoint extensions of vectorial differential operators, we know that the boundary conditions (1.3) are separated self-adjoint boundary conditions. It is known that problem (1.2) and (1.3) has a countably infinite number of real eigenvalues which are bounded from below, see [**4, 6**]. Counting the multiplicity of eigenvalues, we arrange all eigenvalues of problem (1.2) and (1.3) as an ascending sequence

$$-\infty < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots,\ \lim_{m \to \infty} \lambda_m = \infty.$$

The multiplicity of an eigenvalue λ_* of problem (1.2) and (1.3) is at most two, and the multiplicity of λ_* is 2 if and only if $A_c Y(c, \lambda_*) + B_c Y'(c, \lambda_*)$ is a zero matrix.

The matrix-valued solution $Y(t,\lambda)$ of the equation (2.7) can now be recast as a Volterra integral equation, see [**1**]:

$$
\begin{aligned}
Y(t,\lambda) &= B_0\cos(\sqrt{\lambda}t)E_2 - \frac{A_0\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,E_2 \\
&\quad + \int_0^t \frac{\sin(\sqrt{\lambda}(t-s))}{\sqrt{\lambda}}\,\widetilde{Q}(s)Y(s,\lambda)\,ds.
\end{aligned}
\tag{2.8}
$$

By Gronwall's inequality, (2.8) implies that for enough large positive λ, the asymptotic formula of $Y(t,\lambda)$ is

$$
\begin{aligned}
&Y(t,\lambda) \\
&= \begin{cases} B_0\cos(\sqrt{\lambda}t)E_2 - \dfrac{A_0\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,E_2 + O\left(\dfrac{1}{\sqrt{\lambda}}\right) & \text{for } \sin\alpha\neq 0; \\ -\dfrac{A_0\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,E_2 + O\left(\dfrac{1}{\lambda}\right) & \text{for } \sin\alpha = 0. \end{cases}
\end{aligned}
\tag{2.9}
$$

First we give an important lemma.

Lemma 1. *If* $Q(x) = \begin{pmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{pmatrix}$, *and* λ_* *is an eigenvalue of* (2.2) *and* (2.3) *which is multiplicity two, let* $Y(t,\lambda_*) = (\overrightarrow{y_1}(t), \overrightarrow{y_2}(t))$, *where* $\overrightarrow{y_1}(t) = \begin{pmatrix} y_{11}(t) \\ y_{21}(t) \end{pmatrix}$ *and* $\overrightarrow{y_2}(t) = \begin{pmatrix} y_{12}(t) \\ y_{22}(t) \end{pmatrix}$ *are two solutions of* (2.2) *and* (2.3) *corresponding to* λ_*. *There holds*

$$
\int_0^c \frac{r_1(x)}{r(x)}\, det\, Y(t,\lambda_*)\,dt = 0. \tag{2.10}
$$

Proof. Since $\overrightarrow{y_1}(t)$ and $\overrightarrow{y_2}(t)$ are two solutions of (2.2) and (2.3) corresponding to λ_*, $\overrightarrow{y_1}(t)$ and $\overrightarrow{y_2}(t)$ satisfy the boundary conditions of (2.3), that is,

$$
\begin{cases} A_0\,y_{11}(0) + B_0\,y_{11}'(0) = 0, \\ A_0\,y_{12}(0) + B_0\,y_{12}'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} A_c\,y_{11}(c) + B_c\,y_{11}'(c) = 0, \\ A_c\,y_{12}(c) + B_c\,y_{12}'(c) = 0. \end{cases}
\tag{2.11}
$$

Since $A_0^2 + B_0^2 \neq 0$ and $A_c^2 + B_c^2 \neq 0$, then

$$
y_{11}(0)y_{12}'(0) - y_{11}'(0)y_{12}(0) = y_{11}(c)y_{12}'(c) - y_{11}'(c)y_{12}(c) = 0. \tag{2.12}
$$

In addition, $\overrightarrow{y_i}(t)$ $(i = 1, 2)$ satisfies the equation (2.2), thus

$$y_{11}''(t) + [\lambda_* - \widetilde{p_1}(t)]\, y_{11}(t) + \widetilde{r_1}(t)\, y_{21}(t) = 0, \tag{2.13}$$

$$y_{12}''(t) + [\lambda_* - \widetilde{p_1}(t)]\, y_{12}(t) + \widetilde{r_1}(t)\, y_{22}(t) = 0. \tag{2.14}$$

By (2.13) and (2.14), we get

$$y_{11}''(t)y_{12}(t) - y_{11}(t)y_{12}''(t) = \widetilde{r_1}(t)[y_{11}(t)y_{22}(t) - y_{12}(t)y_{21}(t)],$$

which can be rewritten as

$$\frac{d}{dt}\,[y_{11}'(t)y_{12}(t) - y_{11}(t)y_{12}'(t)] = \widetilde{r_1}(t)[y_{11}(t)y_{22}(t) - y_{12}(t)y_{21}(t)]. \tag{2.15}$$

Employing (2.12) and integrating (2.15) from $t = 0$ to $t = c$, together with (2.6), we obtain (2.10), thus proving Lemma 1. □

Theorem 1. *If* $Q(x) = \begin{pmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{pmatrix}$, *and*

$$\int_0^1 \frac{r_1(x)}{\sqrt{r(x)p(x)}}\, dx \neq 0,$$

then except finitely many eigenvalues, all eigenvalues of (1.2) *and* (1.3) *are simple.*

Proof. Suppose, on the contrary, there exist infinitely many eigenvalues λ_n, $n = 1, 2, \dots$, with multiplicity 2 for problem (1.2) and (1.3), and hence also for problem (2.2) and (2.3). Denote the solution of (2.7) for $\lambda = \lambda_n$ by

$$Y(t, \lambda_n) = \begin{pmatrix} y_{11}(t, \lambda_n) & y_{12}(t, \lambda_n) \\ y_{21}(t, \lambda_n) & y_{22}(t, \lambda_n) \end{pmatrix}.$$

Then, by (2.9), we have

$$\begin{aligned}
&\det Y(t, \lambda_n) \\
&\quad = y_{11}(t, \lambda_n)y_{22}(t, \lambda_n) - y_{12}(t, \lambda_n)y_{21}(t, \lambda_n) \\
&\quad = \begin{cases} \dfrac{B_0^2}{2} + \dfrac{B_0^2 \cos(2\sqrt{\lambda_n}\, t)}{2} + \dfrac{A_0^2 \sin^2(\sqrt{\lambda_n}\, t)}{\lambda_n} \\ \quad - \dfrac{A_0 B_0 \sin(2\sqrt{\lambda_n}\, t)}{\sqrt{\lambda_n}} + O\left(\dfrac{1}{\sqrt{\lambda_n}}\right) & \sin\alpha \neq 0; \\ \dfrac{A_0^2 \sin^2(\sqrt{\lambda_n}\, t)}{\lambda_n} + O\left(\dfrac{1}{\lambda_n^{3/2}}\right) & \sin\alpha = 0. \end{cases}
\end{aligned} \tag{2.16}$$

Equations (2.10) and (2.16) imply that

(i) For $\sin\alpha = 0$, then

$$\int_0^c \frac{r_1(x)}{r(x)} \left\{ \frac{1-\cos(2\sqrt{\lambda_n}\,t)}{2\lambda_n}\, A_0^2 + O\left(\frac{1}{\lambda_n^{3/2}}\right)\right\} dt = 0,$$

which is reduced to

$$\int_0^c \frac{r_1(x)}{r(x)}\, dt = \int_0^c \frac{r_1(x)}{r(x)}\, \cos(2\sqrt{\lambda_n}\,t)\, dt + O\left(\frac{1}{\sqrt{\lambda_n}}\right). \tag{2.17}$$

By (2.17), the Riemann-Lebesgue lemma tells us that if (1.2) and (1.3) have infinitely many eigenvalues of multiplicity two, then $\int_0^c r_1(x)/r(x)dt = 0$. Together with (2.1) and (2.4), a substitution of variable implies that $\int_0^1 r_1(x)/\sqrt{r(x)p(x)}\, dx = 0$, which contradicts the given assumption;

(ii) For $\sin\alpha \neq 0$, then

$$\int_0^c \frac{r_1(x)}{r(x)} \left\{ \frac{B_0^2 + B_0^2\cos(2\sqrt{\lambda_n}\,t)}{2} + \frac{A_0^2\sin^2(\sqrt{\lambda_n}\,t)}{\lambda_n} - \frac{A_0B_0\sin(2\sqrt{\lambda_n}\,t)}{\sqrt{\lambda_n}}\right\} dt + O\left(\frac{1}{\sqrt{\lambda_n}}\right) = 0,$$

which is equivalent to

$$\begin{aligned} \int_0^c \frac{r_1(x)}{r(x)}\, dt = &-\int_0^c \frac{r_1(x)}{r(x)}\, \cos(2\sqrt{\lambda_n}\,t)\, dt \\ &- \frac{2A_0^2}{B_0^2\lambda_n}\int_0^c \frac{r_1(x)\sin^2(\sqrt{\lambda_n}\,t)}{r(x)}\, dt \\ &+ \frac{2A_0}{B_0\sqrt{\lambda_n}}\int_0^c \frac{r_1(x)\sin(2\sqrt{\lambda_n}\,t)}{r(x)}\, dt + O\left(\frac{1}{\sqrt{\lambda_n}}\right), \end{aligned} \tag{2.18}$$

by (2.18) and the Riemann-Lebesgue lemma, if (1.2) and (1.3) have infinitely many eigenvalues of multiplicity two, we get $\int_0^c r_1(x)/r(x)\, dt = 0$, that is, $\int_0^1 r_1(x)/\sqrt{r(x)p(x)}\, dx = 0$, leading to a contradiction also. Thus, if $\int_0^1 r_1(x)/\sqrt{r(x)p(x)}\, dx \neq 0$, then all, except finitely many, eigenvalues of (1.2) and (1.3) are simple. $\square$

An important application of Theorem 1 is to study the intersection of the spectra of two potential equations of the form (1.1). Let $\sigma(q)$ denote the spectrum of the potential equation (1.1) with the potential function $q(x)$.

Theorem 2. *Suppose $q_1(x)$ and $q_2(x)$ are two real-valued continuous functions on $0 \le x \le 1$. If*

$$\int_0^1 \frac{q_1(x) - q_2(x)}{\sqrt{r(x)p(x)}}\, dx \neq 0, \tag{2.19}$$

then $\sigma(q_1)$ and $\sigma(q_2)$ only have finitely many elements in common.

Proof. Denote

$$Q_0(x) = \begin{pmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{pmatrix}, \quad U(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

then

$$\begin{aligned} Q_\theta &= U^*(\theta) Q_0(x) U(\theta) \\ &= \begin{pmatrix} q_1(x)\cos^2\theta + q_2(x)\sin^2\theta & (q_2(x) - q_1(x))\sin\theta\cos\theta \\ (q_2(x) - q_1(x))\sin\theta\cos\theta & q_1(x)\sin^2\theta + q_2(x)\cos^2\theta \end{pmatrix}, \end{aligned}$$

where θ is a real constant such that $\sin\theta\cos\theta \neq 0$. Then for $Q(x) = Q_\theta(x)$ in (1.2) and (1.3), where $r_1(x) = (q_1(x) - q_2(x))\sin\theta\cos\theta$, the sequence of eigenvalues, counting multiplicity, of (1.2) and (1.3), consists of elements of $\sigma(q_1)$ and $\sigma(q_2)$, counting multiplicity. By the assumption that $\int_0^1 (q_1(x) - q_2(x))/\sqrt{r(x)p(x)}\, dx \neq 0$, and it follows from Theorem 1 that, for $Q(x) = Q_\theta(x)$, (1.2) and (1.3) have only finitely many eigenvalues which have multiplicity two, which implies that $\sigma(q_1) \cap \sigma(q_2)$ is a finite set. This completes the proof of Theorem 2. $\square$

Example 1. Consider the intersection of spectra of the following two potential equations

$$\begin{cases} -u''(x) + 64\pi^2 u(x) = \lambda u(x); \\ u'(0) = u'(1) = 0 \end{cases} \tag{q_1}$$

and

$$(q_2) \qquad \begin{cases} -u''(x) - 64\pi^2\, u(x) = \lambda\, u(x); \\ u'(0) = u'(1) = 0. \end{cases}$$

Obviously,

$$\int_0^1 \frac{q_1(x) - q_2(x)}{\sqrt{r(x)p(x)}}\, dx = 128\pi^2 \neq 0.$$

Theorem 2 implies that $\sigma(q_1) \cap \sigma(q_2)$ only has finitely elements. In fact, a straightforward computation tells us

$$\sigma(q_1) = \{(n^2 + 64)\pi^2 \mid n = 0, 1, 2, \dots\};$$
$$\sigma(q_2) = \{(n^2 - 64)\pi^2 \mid n = 0, 1, 2, \dots\},$$

thus

$$\sigma(q_1) \cap \sigma(q_2) = \{80\pi^2, 260\pi^2, 1025\pi^2\}.$$

3. Some estimates related eigenvalue problems. In this section, to be more precise, we are going to find a lower bound m_Q, such that the eigenvalues of (E) with index exceeding m_Q are all simple. These results are applied to find some sufficient conditions which ensure that the spectra of two potential equations of the form (1.1) have finitely many elements in common, and we obtain an estimate of the number of elements in the intersection of two spectra.

Let $Q(x) = \left(\begin{smallmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{smallmatrix}\right)$, where $p_1(x)$ and $p_2(x)$ are continuous functions and $r_1(x)$ is a C^1-function on $0 \leq x \leq 1$, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$ denote the sequence of eigenvalues of (1.2) and (1.3). If $\int_0^1 r_1(x)/\sqrt{r(x)p(x)}\, dx \neq 0$, then, by Theorem 1, there exists an index m_Q such that for $n \geq m_Q$ the eigenvalues λ_n of (1.2) and (1.3) are simple. Hence there is a natural question: how to estimate m_Q? The estimate has a lot of strong backgrounds and useful applications. In this section, we shall try to find a lower bound estimate of the number m_Q. This estimate is obviously applicable to the problem of estimating the number of elements in the intersection of the spectra of two potential equations.

Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ be a 2×2 matrix. Define the maximum norm of A

$$\|A\| = \sup\{|a_{ij}|,\ 1 \leq i, j \leq 2\}.$$

If A and B are 2×2 matrices, then we have

$$\|AB\| \leq 2\|A\| \cdot \|B\|. \tag{3.1}$$

By (2.8) we have

$$\|Y(t,\lambda)\| \leq |B_0| + \frac{|A_0|}{\sqrt{\lambda}} + \frac{2}{\sqrt{\lambda}} \int_0^t \|\widetilde{Q}(s)\| \cdot \|Y(s,\lambda)\| \, ds, \tag{3.2}$$

together with Gronwall's inequality, (3.2) tells us that

$$\begin{aligned} \|Y(t,\lambda)\| &\leq \left(|B_0| + \frac{|A_0|}{\sqrt{\lambda}}\right) \exp\left\{\frac{2}{\sqrt{\lambda}} \int_0^t \|\widetilde{Q}(s)\| ds\right\} \\ &\leq \left(|B_0| + \frac{|A_0|}{\sqrt{\lambda}}\right) \exp\left\{\frac{2}{\sqrt{\lambda}} \int_0^c \|\widetilde{Q}(s)\| \, ds\right\}. \end{aligned} \tag{3.3}$$

In (2.8), denote

$$Y(t,\lambda) = B_0 \cos(\sqrt{\lambda}\, t) E_2 - \frac{A_0 \sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} \, E_2 + G(t,\lambda), \tag{3.4}$$

where

$$G(t,\lambda) = (g_{ij}(t,\lambda))_{i,j=1}^2 = \int_0^t \frac{\sin(\sqrt{\lambda}\,(t-s))}{\sqrt{\lambda}} \, \widetilde{Q}(s) Y(s,\lambda) \, ds. \tag{3.5}$$

Then by (3.3) and (3.5), we have, for $1 \leq i, j \leq 2$, $0 \leq t \leq c$,

$$\begin{aligned} |g_{ij}(t,\lambda)| &\leq \|G(t,\lambda)\| \\ &\leq \frac{2}{\sqrt{\lambda}} \left(|B_0| + \frac{|A_0|}{\sqrt{\lambda}}\right) \exp\left\{\frac{2}{\sqrt{\lambda}} \int_0^c \|\widetilde{Q}(s)\| \, ds\right\} \\ &\quad \times \int_0^t \|\widetilde{Q}(s)\| \, ds \\ &\leq \frac{2}{\sqrt{\lambda}} \left(|B_0| + \frac{|A_0|}{\sqrt{\lambda}}\right) \exp\left\{\frac{2}{\sqrt{\lambda}} \int_0^c \|\widetilde{Q}(s)\| \, ds\right\} \\ &\quad \times \int_0^c \|\widetilde{Q}(s)\| \, ds. \end{aligned} \tag{3.6}$$

Using (3.4) and (3.6), we get

$$\det Y(t,\lambda) = B_0^2\cos^2(\sqrt{\lambda}\,t) + \frac{A_0^2\sin^2(\sqrt{\lambda}\,t)}{\lambda} - \frac{A_0B_0\sin(2\sqrt{\lambda}\,t)}{\sqrt{\lambda}} + h(t,\lambda), \tag{3.7}$$

where

$$h(t,\lambda) = \left(B_0\cos(\sqrt{\lambda}\,t) - \frac{A_0\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}}\right)\operatorname{Trace} G(t,\lambda) + \det G(t,\lambda). \tag{3.8}$$

We shall also need the following inequality, which follows from integration by parts,

$$\left|\int_0^c \frac{r_1(x)}{r(x)}\cos(2\sqrt{\lambda}\,t)\,dt\right| \le \frac{|r_1(1)|}{2\sqrt{\lambda}\,r(1)} + \frac{1}{2\sqrt{\lambda}}\int_0^c\left|\frac{d}{dt}\,\frac{r_1(x)}{r(x)}\right| dt. \tag{3.9}$$

And by (3.6), (3.7) and (3.8), we have

$$\begin{aligned}&\left|\int_0^c \frac{r_1(x)}{r(x)}h(t,\lambda)\,dt\right| \\ &\le \left\{\frac{4}{\sqrt{\lambda}}\left(|B_0| + \frac{|A_0|}{\sqrt{\lambda}}\right)^2\int_0^c \|\widetilde{Q}(s)\|\,ds \exp\left\{\frac{2}{\sqrt{\lambda}}\int_0^c\|\widetilde{Q}(s)\|\,ds\right\}\right. \\ &\quad \left. + \frac{8}{\lambda}\left(|B_0| + \frac{|A_0|}{\sqrt{\lambda}}\right)^2\left(\int_0^c\|\widetilde{Q}(s)\|\,ds\right)^2 \exp\left\{\frac{4}{\sqrt{\lambda}}\int_0^c\|\widetilde{Q}(s)\|\,ds\right\}\right\} \\ &\quad \cdot\int_0^c\frac{|r_1(x)|}{r(x)}\,dt.\end{aligned} \tag{3.10}$$

To precisely estimate the number m_Q, we introduce two equations

$$\begin{aligned}(E_1)\quad f_1(z) := {}& \frac{|r_1(1)|}{2\sqrt{z}\,r(1)} + \frac{1}{2\sqrt{z}}\int_0^c\left|\frac{d}{dt}\,\frac{r_1(x)}{r(x)}\right| dt \\ & + \left\{\frac{8}{\sqrt{z}}\int_0^c\|\widetilde{Q}(s)\|\,ds\,\exp\left\{\frac{2}{\sqrt{z}}\int_0^c\|\widetilde{Q}(s)\|\,ds\right\}\right.\end{aligned} \tag{3.11}$$

$$+\frac{16}{z}\left(\int_0^c \|\widetilde{Q}(s)\|\,ds\right)^2 \exp\left\{\frac{4}{\sqrt{z}}\int_0^c \|\widetilde{Q}(s)\|\,ds\right\}\Bigg\}$$
$$\cdot\int_0^c \frac{|r_1(x)|}{r(x)}\,dt \;=\; \left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right|,$$

(3.12)

$$\begin{aligned}(E_2)\quad & f_2(z)\\ :=\; & \frac{|r_1(1)|}{2\sqrt{z}\,r(1)} + \frac{1}{2\sqrt{z}}\int_0^c \left|\frac{d}{dt}\,\frac{r_1(x)}{r(x)}\right| dt\\ & + \frac{2A_0^2}{B_0^2 z}\int_0^c \frac{|r_1(x)|}{r(x)}\,dt + \frac{2A_0}{B_0\sqrt{z}}\int_0^c \frac{|r_1(x)|}{r(x)}\,dt\\ & + \Bigg\{\frac{8}{B_0^2\sqrt{z}}\left(|B_0| + \frac{|A_0|}{\sqrt{z}}\right)^2 \int_0^c \|\widetilde{Q}(s)\|\,ds\,\exp\left\{\frac{2}{\sqrt{z}}\int_0^c \|\widetilde{Q}(s)\|\,ds\right\}\\ & \quad + \frac{16}{B_0^2 z}\left(|B_0| + \frac{|A_0|}{\sqrt{z}}\right)^2\left(\int_0^c \|\widetilde{Q}(s)\|\,ds\right)^2 \exp\left\{\frac{4}{\sqrt{z}}\int_0^c \|\widetilde{Q}(s)\|\,ds\right\}\Bigg\}\\ & \cdot\int_0^c \frac{|r_1(x)|}{r(x)}\,dt \;=\; \left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right|.\end{aligned}$$

Since $f_1(z)$ and $f_2(z)$ are strictly monotone decreasing, positive functions and satisfy

$$\lim_{z\to 0} f_1(z) = \lim_{z\to 0} f_2(z) = \infty, \quad \lim_{z\to\infty} f_1(z) = \lim_{z\to\infty} f_2(z) = 0,$$

it is easy to verify that, under the assumption that $\int_0^1 r_1(x)/\sqrt{r(x)p(x)}\,dx \neq 0$, equations (E_1), (E_2) have unique positive solutions, denoted by Λ_{*1} and Λ_{*2} respectively.

Remark 2. Employing Newton's iteration method or bisection method, we can solve the approximate solutions $\Lambda_{*1}, \Lambda_{*2}$ of equations (E_1) and (E_2).

From the above discussions, now we are ready to prove the following result.

Theorem 3. *Suppose* $Q(x) = \begin{pmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{pmatrix}$, *where* $p_1(x)$, $p_2(x)$ *are continuous functions and* $r_1(x)$ *is a* C^1*-function on* $0 \le x \le 1$, *and*

$$\int_0^1 \frac{r_1(x)}{\sqrt{r(x)p(x)}}\, dx \neq 0.$$

If λ_n *is an eigenvalue of* (1.2) *and* (1.3) *satisfying the following conditions,*

$$\lambda_n > \Lambda_{*1} \quad \text{if} \quad \sin\alpha = 0; \tag{3.13}$$

$$\lambda_n > \Lambda_{*2} \quad \text{if} \quad \sin\alpha \neq 0, \tag{3.14}$$

then λ_n *is a simple eigenvalue.*

Proof. Suppose λ_n satisfies condition (3.13) or (3.14), but the multiplicity of λ_n is two. If $Y(t,\lambda_n)$ is the same as Lemma 1, by Lemma 1, we have the following identity:

$$\int_0^c \frac{r_1(x)}{r(x)}\, \det Y(t,\lambda_n)\, dt = 0.$$

If $\sin\alpha = 0$, and by (2.4) and (3.7), we have

$$\int_0^c \frac{r_1(x)}{r(x)} \left\{ \frac{A_0^2 \sin^2(\sqrt{\lambda_n}\, t)}{\lambda_n} + h(t,\lambda_n) \right\} dt = 0,$$

which is equivalent to

$$\int_0^c \frac{r_1(x)}{r(x)}\, dt = \int_0^c \frac{r_1(x)}{r(x)}\, \cos(2\sqrt{\lambda_n}\, t)\, dt - \frac{2\lambda_n}{A_0^2} \int_0^c \frac{r_1(x)}{r(x)}\, h(t,\lambda_n)\, dt.$$

By (3.9) and (3.10), this implies

$$\begin{aligned} \left| \int_0^c \frac{r_1(x)}{r(x)}\, dt \right| \le{}& \frac{|r_1(1)|}{2\sqrt{\lambda_n}\, r(1)} + \frac{1}{2\sqrt{\lambda_n}} \int_0^c \left| \frac{d}{dt} \frac{r_1(x)}{r(x)} \right| dt \\ &+ \left\{ \frac{8}{\sqrt{\lambda_n}} \int_0^c \|\widetilde{Q}(s)\|\, ds \exp\left\{ \frac{2}{\sqrt{\lambda_n}} \int_0^c \|\widetilde{Q}(s)\| ds \right\} \right. \\ &\left. + \frac{16}{\lambda_n} \left(\int_0^c \|\widetilde{Q}(s)\|\, ds \right)^2 \exp\left\{ \frac{4}{\sqrt{\lambda_n}} \int_0^c \|\widetilde{Q}(s)\|\, ds \right\} \right\} \\ &\times \int_0^c \frac{|r_1(x)|}{r(x)}\, dt, \end{aligned}$$

since the function $f_1(z)$ is strictly monotone decreasing and Λ_{*1} is a solution of $f_1(z) = |\int_0^c r_1(x)/r(x)\,dt|$, we infer by (3.13),

$$\left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right| \le f_1(\lambda_n) < f_1(\Lambda_{*1}) = \left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right|,$$

which is absurd.

If $\sin\alpha \neq 0$, and by (3.7), we have

$$\begin{aligned}\int_0^c \frac{r_1(x)}{r(x)}\bigg\{B_0^2\cos^2(\sqrt{\lambda_n}\,t) + \frac{A_0^2\sin^2(\sqrt{\lambda_n}\,t)}{\lambda_n} - \frac{A_0B_0\sin(2\sqrt{\lambda_n}\,t)}{\sqrt{\lambda_n}}&\\ + h(t,\lambda_n)\bigg\}\,dt = 0,&\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_0^c \frac{r_1(x)}{r(x)}\,dt\\ &\quad= -\int_0^c \frac{r_1(x)}{r(x)}\cos(2\sqrt{\lambda_n}\,t)\,dt - \frac{2A_0^2}{B_0^2\lambda_n}\int_0^c \frac{r_1(x)}{r(x)}\sin^2(\sqrt{\lambda_n}\,t)\,dt\\ &\qquad + \frac{2A_0}{B_0\sqrt{\lambda_n}}\int_0^c \frac{r_1(x)}{r(x)}\sin(2\sqrt{\lambda_n}\,t)\,dt - \frac{2}{B_0^2}\int_0^c \frac{r_1(x)}{r(x)}h(t,\lambda_n)\,dt,\end{aligned}$$

which, using (3.9) and (3.10), leads to

$$\begin{aligned}\left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right| &\le \frac{|r_1(1)|}{2\sqrt{\lambda_n}\,r(1)} + \frac{1}{2\sqrt{\lambda_n}}\int_0^c \left|\frac{d}{dt}\frac{r_1(x)}{r(x)}\right|dt\\ &\quad + \frac{2A_0^2}{B_0^2\lambda_n}\int_0^c \frac{|r_1(x)|}{r(x)}\,dt + \frac{2A_0}{B_0\sqrt{\lambda_n}}\int_0^c \frac{|r_1(x)|}{r(x)}\,dt\\ &\quad + \bigg\{\frac{8}{B_0^2\sqrt{\lambda_n}}\left(|B_0| + \frac{|A_0|}{\sqrt{\lambda_n}}\right)^2\\ &\qquad\quad \times \int_0^c \|\widetilde{Q}(s)\|\,ds\,\exp\left\{\frac{2}{\sqrt{\lambda_n}}\int_0^c \|\widetilde{Q}(s)\|\,ds\right\}\\ &\qquad\quad + \frac{16}{B_0^2\lambda_n}\left(|B_0| + \frac{|A_0|}{\sqrt{\lambda_n}}\right)^2\left(\int_0^c \|\widetilde{Q}(s)\|\,ds\right)^2\\ &\qquad\quad \times \exp\left\{\frac{4}{\sqrt{\lambda_n}}\int_0^c \|\widetilde{Q}(s)\|\,ds\right\}\bigg\}\cdot\int_0^c \frac{|r_1(x)|}{r(x)}\,dt,\end{aligned}$$

since the function $f_2(z)$ is strictly monotone decreasing and Λ_{*2} is a solution of $f_2(z) = |\int_0^c r_1(x)/r(x)\,dt|$, by (3.14) we infer,

$$\left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right| \le f_2(\lambda_n) < f_1(\Lambda_{*2}) = \left|\int_0^c \frac{r_1(x)}{r(x)}\,dt\right|,$$

which is a contradiction also. Thus, the multiplicity of λ_n satisfying (3.13) or (3.14) is simple. □

It was known, see [**5**], that the eigenvalues of the boundary value problem

$$\begin{cases} (pu')'(x) + (\lambda r(x) - q(x))u(x) = 0 & \\ u(0)\cos\alpha + u'(0)\sin\alpha = 0 & 0 \le \alpha < \pi \\ u(1)\cos\beta + u'(1)\sin\beta = 0 & 0 \le \beta < \pi \end{cases}$$

are the zeros of the transcendental, entire function $\omega(\lambda)$, where for large positive λ,

(3.15)

$$\omega(\lambda) = \begin{cases} -\sin\alpha\sin\beta\,\dfrac{p(0)\sqrt[4]{r(1)p(1)}}{p(1)\sqrt[4]{r(0)p(0)}}\sqrt{\lambda}\sin(\sqrt{\lambda}\,c) + O(1) & \\ \qquad\qquad\qquad \text{if } \sin\alpha\sin\beta \ne 0; & \\ \sin\alpha\,\dfrac{\sqrt[4]{r(0)p(0)}}{\sqrt[4]{r(1)p(1)}}\cos(\sqrt{\lambda}\,c) + O\left(\dfrac{1}{\sqrt{\lambda}}\right) & \text{if } \sin\alpha \ne 0,\ \sin\beta = 0; \\ -\sin\beta\,\dfrac{p(0)\sqrt[4]{r(1)p(1)}}{p(1)\sqrt[4]{r(0)p(0)}}\cos(\sqrt{\lambda}\,c) + O\left(\dfrac{1}{\sqrt{\lambda}}\right) & \\ \qquad\qquad\qquad \text{if } \sin\alpha = 0,\ \sin\beta \ne 0; & \\ \dfrac{\sqrt[4]{r(0)p(0)}}{\sqrt[4]{r(1)p(1)}}\,\dfrac{\sin(\sqrt{\lambda}\,c)}{\sqrt{\lambda}} + O\left(\dfrac{1}{\lambda}\right) & \text{if } \sin\alpha = \sin\beta = 0 \end{cases}$$

and

$$c = \int_0^1 \sqrt{\frac{r(s)}{p(s)}}\,ds.$$

From the formulae (3.15) we can calculate asymptotic eigenvalues as follows:

(3.16)

$$\sqrt{\lambda_n} = \begin{cases} \dfrac{n\pi}{c} + O\left(\dfrac{1}{n}\right) & \text{if } \sin\alpha\sin\beta \neq 0 \text{ or } \sin\alpha = \sin\beta = 0; \\ \dfrac{(n+(1/2))\pi}{c} + O\left(\dfrac{1}{n}\right) & \\ \qquad\qquad \text{if } \sin\alpha \neq 0,\ \sin\beta = 0 \text{ or } \sin\alpha = 0,\ \sin\beta \neq 0. \end{cases}$$

By the maximum-minimum principle, see [**3**], we have

$$\lambda_n(Q) = \lambda_n(\widetilde{Q}) \geq \lambda_n(\widetilde{Q}_1),$$

where $\lambda_n(Q)$ is the nth eigenvalue of (1.2) and (1.3), and $\widetilde{Q}_1(t) = s_1(t)E_2$, $s_1(t)$ is the smaller characteristic value of the matrix $\widetilde{Q}(t)$. Since each eigenvalue of (2.2) and (2.3), with $\widetilde{Q}_1(t)$ replacing $\widetilde{Q}(t)$, is of multiplicity two, that is,

$$\begin{array}{cccccccc} \lambda_n(Q): & \lambda_1(Q) & \lambda_2(Q) & \lambda_3(Q) & \lambda_4(Q) & \dots & \lambda_n(Q) & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \lambda_n(\widetilde{Q}_1): & \lambda_1(\widetilde{Q}_1) & \lambda_1(\widetilde{Q}_1) & \lambda_2(\widetilde{Q}_1) & \lambda_2(\widetilde{Q}_1) & \dots & \lambda_{[(n+1)/2]}(\widetilde{Q}_1) & \dots, \end{array}$$

we get

(3.17)

$$\begin{aligned} \lambda_n(Q) &\geq \lambda_{[(n+1)/2]}(\widetilde{Q}_1) \\ &\geq \begin{cases} \dfrac{([(n+1)/2])^2\pi^2}{c^2} + s_1 \\ \qquad\qquad \text{if } \sin\alpha\sin\beta \neq 0 \text{ or } \sin\alpha = \sin\beta = 0; \\ \dfrac{([(n+1)/2]+(1/2))^2\pi^2}{c^2} + s_1 \\ \qquad\qquad \text{if } \sin\alpha \neq 0,\ \sin\beta = 0 \text{ or } \sin\alpha = 0,\ \sin\beta \neq 0, \end{cases} \end{aligned}$$

where $s_1 = \min_{0 \leq t \leq c} s_1(t)$, and $[a]$ denotes the maximal integer not greater than real number a. By (3.17), Theorem 3 can be restated as the following form.

Theorem 4. *Suppose that* $Q(x) = \left(\begin{smallmatrix} p_1(x) & -r_1(x) \\ -r_1(x) & p_2(x) \end{smallmatrix}\right)$, *where* $p_1(x), p_2(x)$ *are continuous functions and* $r_1(x)$ *is a* C^1*-function on* $0 \le x \le 1$, *and*

$$\int_0^1 \frac{r_1(x)}{\sqrt{r(x)p(x)}}\,dx \neq 0.$$

Let m_Q *be the smallest positive integer which satisfies the following condition:*

$$\begin{aligned}
&\frac{([\frac{m_Q+1}{2}])^2\pi^2}{c^2} + s_1 > \Lambda_{*2} && \textit{if } \sin\alpha\sin\beta \neq 0;\\
&\frac{([\frac{m_Q+1}{2}] + \frac{1}{2})^2\pi^2}{c^2} + s_1 > \Lambda_{*2} && \textit{if } \sin\alpha \neq 0, \sin\beta = 0;\\
&\frac{([\frac{m_Q+1}{2}] + \frac{1}{2})^2\pi^2}{c^2} + s_1 > \Lambda_{*1} && \textit{if } \sin\alpha = 0, \sin\beta \neq 0;\\
&\frac{([\frac{m_Q+1}{2}])^2\pi^2}{c^2} + s_1 > \Lambda_{*1} && \textit{if } \sin\alpha = \sin\beta = 0,
\end{aligned}$$

where $s_1 = \min_{0\le t\le c} s_1(t)$, $s_1(t)$ *is the smaller characteristic value of the matrix* $\widetilde{Q}(t)$ *defined by* (2.4), *and* $\Lambda_{*1}, \Lambda_{*2}$ *are the same as Theorem* 3, *then the* n*th eigenvalue of* (1.2) *and* (1.3) *is simple if* $n \ge m_Q$.

By Theorem 4 and the proof of Theorem 2 we also have the following result.

Theorem 5. *Suppose* $q_1(x)$ *and* $q_2(x)$ *are two real-valued* C^1*-functions on* $0 \le x \le 1$, *and*

$$\int_0^1 \frac{q_1(x) - q_2(x)}{\sqrt{r(x)p(x)}}\,dx \neq 0.$$

Let $s_{q_{12}} = \min_{0\le t\le c} s_{q_{12}}(t)$, *where* $s_{q_{12}}(t)$ *is the smaller characteristic value of the matrix* $\widetilde{Q}(t)$ *with* $Q(x) = \operatorname{diag}(q_1(x), q_2(x))$ *in* (2.4). *Let* $n_{q_{12}}$ *be the smallest positive integer which satisfies the following* *condition*:

$$
\begin{aligned}
&\frac{(n_{q_{12}}+1)^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*2,q_{12}} && \textit{if}\ \ \sin\alpha\sin\beta\neq 0;\\
&\frac{(n_{q_{12}}+(3/2))^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*2,q_{12}} && \textit{if}\ \ \sin\alpha\neq 0,\ \sin\beta=0;\\
&\frac{(n_{q_{12}}+(3/2))^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*1,q_{12}} && \textit{if}\ \ \sin\alpha=0,\ \sin\beta\neq 0;\\
&\frac{(n_{q_{12}}+1)^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*1,q_{12}} && \textit{if}\ \ \sin\alpha=\sin\beta=0,
\end{aligned}
\tag{3.18}
$$

where $\Lambda_{*1,q_{12}}, \Lambda_{*2,q_{12}}$ *is the root of the equation* (E_1), (E_2), *replacing* $Q(x)$ *with* $Q_{\pi/4}(x)$, *respectively. Then* $\sigma(q_1)$ *and* $\sigma(q_2)$ *have at most* $n_{q_{12}}$ *elements in common.*

Proof. Similar to Theorem 2, choosing $\theta=\pi/4$, we get

$$
\begin{aligned}
Q_{\pi/4}(x) &= \begin{pmatrix} (1/2)(q_1(x)+q_2(x)) & (1/2)(q_2(x)-q_1(x)) \\ (1/2)(q_2(x)-q_1(x)) & (1/2)(q_1(x)+q_2(x)) \end{pmatrix},\\
r_1(x) &= \frac{1}{2}(q_1(x)-q_2(x)),
\end{aligned}
$$

and $\Lambda_{*1,q_{12}}, \Lambda_{*2,q_{12}}$ is the root of the equation (E_1), (E_2), replacing $Q(x)$ with $Q_{\pi/4}(x)$, respectively. Suppose $\sigma(q_1)$ and $\sigma(q_2)$ have at least $n_{q_{12}}+1$ elements in common. Since the sequence of eigenvalues of (1.2) and (1.3) with $Q(x)=Q_{\pi/4}(x)$ consists of elements of $\sigma(q_1)$ and $\sigma(q_2)$ counting multiplicity, there exists an index n_*, $n_*\geq 2n_{q_{12}}+2$, such that $\lambda_{n_*}(Q_{\pi/4})$ is not simple. But, by (3.17) and (3.18), we have

$$
\begin{aligned}
&\lambda_{n_*}(Q_{\pi/4})\\
&\quad\geq \lambda_{2n_{q_{12}}+2}(Q_{\pi/4})\\
&\quad\geq \begin{cases}
\dfrac{(n_{q_{12}}+1)^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*2,q_{12}} & \text{if } \sin\alpha\sin\beta\neq 0;\\[2mm]
\dfrac{(n_{q_{12}}+(3/2))^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*2,q_{12}} & \text{if } \sin\alpha\neq 0,\ \sin\beta=0;\\[2mm]
\dfrac{(n_{q_{12}}+(3/2))^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*1,q_{12}} & \text{if } \sin\alpha=0,\ \sin\beta\neq 0;\\[2mm]
\dfrac{(n_{q_{12}}+1)^2\pi^2}{c^2}+s_{q_{12}}>\Lambda_{*1,q_{12}} & \text{if } \sin\alpha=\sin\beta=0,
\end{cases}
\end{aligned}
$$

hence

$$\lambda_{n_*}(Q_{\pi/4}) > \begin{cases} \Lambda_{*1,q_{12}} & \text{if } \sin\alpha = 0; \\ \Lambda_{*2,q_{12}} & \text{if } \sin\alpha \neq 0, \end{cases}$$

together with Theorem 3, we see that $\lambda_{n_*}(Q_{\pi/4})$ is simple which contradicts that $\lambda_{n_*}(Q_{\pi/4})$ is not simple. Thus $\sigma(q_1) \cap \sigma(q_2)$ has at most $n_{q_{12}}$ elements in common. We complete the proof of the theorem. $\square$

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