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KERNELS OF ADJOINTS OF COMPOSITION OPERATORS WITH MULTIVALENT SYMBOL VIA A FORMAL ADJOINT

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ABSTRACT. If φ is an analytic map of the unit disk D into itself, the composition operator C_φ on the Hardy space $H^2(D)$ is defined by $C_\varphi(f) = f \circ \varphi$. For a certain class of composition operators with multivalent symbol φ, we give a complete and convenient description of $\operatorname{Ker} C_\varphi^\star$ using intuition from a purely formal adjoint calculation.

1. Introduction. Composition operators can be defined on any Hilbert space of analytic functions. Here we consider composition operators on the classical Hardy Hilbert space of analytic functions on the unit disk D, $H^2(D)$, the set of analytic functions f on D for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, \frac{d\theta}{2\pi} < \infty.$$

If φ is an analytic map of the unit disk D into itself, the composition operator C_φ on $H^2(D)$ is defined by $C_\varphi(f) = f \circ \varphi$ and bounded on $H^2(D)$ [**3**]. These operators have been studied for almost four decades and many properties are known [**3**] but progress in answering several basic questions has been impeded.

Computing the norm, determining the spectrum, and deciding on the hyponormality of C_φ and $C_\varphi^\star$ are all hindered by the absence of a convenient description of the adjoint $C_\varphi^\star$ (except in the case that φ is a linear fractional map [**3**], or an inner function [**3**] and more recently [**6**]). In this note we present an approach around this obstacle whereby a purely formal adjoint provides some intuition in determining $\operatorname{Ker} C_\varphi^\star$ for a general class of multivalent symbols φ, and we expect applications to follow as a result of the utility of the description.

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After dispensing with the necessary preliminary notions in Section 2, we present a successive derivative condition that completely characterizes $\operatorname{Ker} C_\varphi^\star$ in Section 3. In Section 4 we present a formal expression for $C_\varphi^\star$ and show how this formal adjoint provides intuition for discovering the description of $\operatorname{Ker} C_\varphi^\star$. To our knowledge this is the first application of Cowen's method for finding the adjoint (formal or otherwise) to the case where φ has variable multiplicity on the unit disk.

2. Preliminaries. For $w \in D$, evaluation at w is a bounded linear functional so, by the Riesz representation theorem, there is a function K_w in $H^2(D)$ that induces this linear functional: $f(w) = \langle f, K_w\rangle$. The function K_w is called the reproducing kernel function. In the Hardy space $H^2(D)$, the reproducing kernel is

$$K_w(z) = \frac{1}{1-\overline{w}z}$$

and has H^2 norm

$$||K_w|| = \frac{1}{(1-|w|^2)^{1/2}}.$$

A fundamental property of composition operators is that the set $\{K_\alpha : \alpha \in D\}$ is invariant under the action of the adjoint. In fact, for any $f \in H^2(D)$, it is easy to see that

$$\langle f, C_\varphi^\star K_\alpha\rangle = \langle C_\varphi f, K_\alpha\rangle = f(\varphi(\alpha)) = \langle f, K_{\varphi(\alpha)}\rangle$$

so we have $C_\varphi^\star K_\alpha = K_{\varphi(\alpha)}$.

3. A description of Ker $C_\varphi^\star$, $\varphi(z) = ((1-2c)z^2)/(1-2cz)$ for $0 < c < 1/2$. In this section we present a complete and convenient description of $\operatorname{Ker} C_\varphi^\star$ for $\varphi(z) = ((1-2c)z^2)/(1-2cz)$, $0 < c < 1/2$.

Note that for any composition operator $C_\varphi(1) = 1$. So if $f \in \operatorname{Ker} C_\varphi^\star$, then $0 = \langle 1, C_\varphi^\star f\rangle = \langle C_\varphi 1, f\rangle = \langle 1, f\rangle$ so that $f \in zH^2(D)$. Hence, $\operatorname{Ker} C_\varphi^\star$ is necessarily a subspace of zH^2.

Theorem 1. *Let*

$$\varphi(z) = \frac{(1-2c)z^2}{1-2cz}$$

for $0 < c < 1/2$. Suppose f is an element of $H^2(D)$ with constant term equal to zero. Then f is an element of the kernel of $C_\varphi^\star$ if and only if f satisfies the successive derivative condition

$$\left\langle f,\ \frac{K_c^{(2j)}}{(2j)!} - c\,\frac{K_c^{(2j+1)}}{(2j+1)!}\right\rangle = 0 \quad \textit{for} \quad j = 0, 1, \dots .$$

Proof. Suppose f satisfies the successive derivative condition for all nonnegative integers j. Since f is analytic in D, f has a Taylor series expansion in the subdisk $|z-c| < 1-c$ on which

$$\sum_{j=1}^{\infty} a_j (z-c)^j$$

converges absolutely. Since f satisfies the successive derivative condition we have for all nonnegative integers j

$$\begin{aligned} 0 &= \left\langle f,\ \frac{K_c^{(2j)}}{(2j)!} - c\,\frac{K_c^{(2j+1)}}{(2j+1)!}\right\rangle \\ &= \frac{f^{(2j)}(c)}{(2j)!} - c\,\frac{f^{(2j+1)}(c)}{(2j+1)!} \\ &= a_{2j} - c\,a_{2j+1} \end{aligned}$$

and so $a_{2j} = ca_{2j+1}$. Note that for the Taylor series

$$f(z) = \sum_{j=1}^{\infty} a_j (z-c)^j$$

we have

$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty} \big(a_{2j}(z-c)^{2j} + a_{2j+1}(z-c)^{2j+1}\big) \\ &= \sum_{j=0}^{\infty} c\,a_{2j+1}(z-c)^{2j} + a_{2j+1}(z-c)^{2j+1} \\ &= \sum_{j=0}^{\infty} a_{2j+1}(z-c)^{2j}(c+z-c) \\ &= \sum_{j=0}^{\infty} a_{2j+1} z (z-c)^{2j}. \end{aligned}$$

Thus, if f satisfies the successive derivative condition for all nonnegative integers j then f is of the form

$$\sum_{j=0}^{\infty} a_{2j+1} z(z-c)^{2j}.$$

To show that f is an element of the kernel of $C_\varphi^\star$ we'll show that f is in the orthogonal complement of the closure of the range of C_φ. It suffices to show for all nonnegative integers k that $\langle f(z), (\varphi(z))^k \rangle = 0$. By the identity principle for analytic functions applied to $f(z)/z$ on the subdisk $|z-c| < 1-c$, the analytic continuation of $f(z)/z$ to the disk D is an even function of $z-c$ defined and analytic on D. Now,

$$\begin{aligned}
\langle f(z), (\varphi(z))^k \rangle &= \int_0^{2\pi} f(e^{i\theta})\, \overline{\varphi(e^{i\theta})^k}\, \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} f(e^{i\theta}) \left(\frac{(1-2c)e^{-2i\theta}}{1-2c\,e^{-i\theta}} \right)^k \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-2c}{e^{2i\theta}-2c\,e^{i\theta}} \right)^k \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \frac{f(e^{i\theta})(1-2c)^k}{e^{ik\theta}(e^{i\theta}-2c)^k} \frac{d\theta}{2\pi} \\
&= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)(1-2c)^k}{z^{k+1}(z-2c)^k}\, dz \\
&= \frac{1}{2\pi i} \int_{|w+c|=1} \frac{[(f(w+c))/(w+c)](1-2c)^k}{(w+c)^k(w+c-2c)^k}\, dw \\
&= \frac{1}{2\pi i} \int_{|w+c|=1} \frac{[(f(w+c))/(w+c)](1-2c)^k}{(w^2-c^2)^k}\, dw
\end{aligned}$$

by the change of variable $w = z - c$. Recall that

$$\frac{f(w+c)}{w+c} = \sum_{j=0}^{\infty} a_{2j+1}(w+c-c)^{2j} = \sum_{j=0}^{\infty} a_{2j+1}(w)^{2j}$$

so that

$$(1-2c)^k\, \frac{f(w+c)}{w+c}$$

is an even function of w on the disk $|w| < 1 - c$. Let $g(w)$ denote

$$(1-2c)^k \, \frac{f(w+c)}{w+c},$$

and consider the partial fraction expansion for

$$\frac{1}{(w+c)^k(w-c)^k}:$$

$$\frac{a_1}{w+c} + \frac{a_2}{(w+c)^2} + \cdots + \frac{a_k}{(w+c)^k} + \frac{b_1}{w-c} + \frac{b_2}{(w-c)^2} + \cdots + \frac{b_k}{(w-c)^k}.$$

Note that this is an even function of w, since

$$\frac{1}{(w+c)^k(w-c)^k} = \frac{1}{(w^2-c^2)^k},$$

so we must have $a_j = (-1)^j b_j$ for all $j = 1, \dots, k$. By the theory of residues, we have

$$\frac{1}{2\pi i}\int_{|w+c|=1} g(w)\,\frac{a_j}{(w+c)^j}\,dw = \frac{a_j g^{(j-1)}(-c)}{(j-1)!}$$

and

$$\frac{1}{2\pi i}\int_{|w+c|=1} g(w)\,\frac{b_j}{(w-c)^j}\,dw = \frac{b_j g^{(j-1)}(c)}{(j-1)!}.$$

Since $g(w)$ is an even function,

$$g^{(j)}(c) = (-1)^j g^{(j)}(-c)$$

for all nonnegative integers k so that

$$\begin{aligned} a_j g^{(j-1)}(-c) &= (-1)^j\, b_j g^{(j-1)}(-c) \\ &= (-1)^j\, b_j (-1)^{j-1} g^{(j-1)}(c) \\ &= -\, b_j g^{(j-1)}(c). \end{aligned}$$

Thus,

$$\frac{1}{2\pi i}\int_{|w+c|=1}\frac{g(w)}{(w+c)^j(w-c)^j}\,dw=0$$

for all nonnegative k since

$$\int_{|w+c|=1} g(w)\,\frac{a_j}{(w+c)^j}\,dw+\int_{|w+c|=1} g(w)\,\frac{b_j}{(w-c)^j}\,dw$$

is equal to zero for all $j=0,1,\dots,k$.

Conversely, suppose that f is in the kernel of $C_\varphi^\star$. Then f is in the orthogonal complement of the closure of the range of C_φ. Hence, for all nonnegative integers k, we have $\langle f(z),(\varphi(z))^k\rangle=0$. As above, this gives

$$\frac{1}{2\pi i}\int_{|w+c|=1}\frac{g(w)}{(w+c)^k(w-c)^k}\,dw=0$$

for all nonnegative integers k. Recalling the partial fraction expansion involved in the integrand, the theory of residues implies that for each k we have $a_j=(-1)^j b_j$, $j=0,1,\dots,k$. For $k=1$, we have $g(-c)=g(c)$. For $k=2$, the theory of residues yields an expression of four terms in which two terms cancel since $g(c)=g(-c)$ and $a_1=-b_1$. Since $a_2=b_2$ we have $g^{(1)}(-c)=-g^{(1)}(c)$. Continuing in this manner we see that $g^{(j)}(-c)=(-1)^j g^{(j)}(c)$ for all $j=0,1,\dots$. Hence, g is an even function of w. Recalling that

$$g(w)=(1-2c)^k\,\frac{f(w+c)}{w+c}$$

and $w=z-c$, we see that $f(z)/z$ is an even function of $z-c$ or

$$\frac{f(z)}{z}=\sum_{j=0}^{\infty}a_{2j}(z-c)^{2j}$$

so that

$$f(z)=\sum_{j=0}^{\infty}z\,a_{2j}(z-c)^{2j}.$$

Hence,

$$\begin{aligned}
f(z) &= \sum_{j=0}^{\infty} z\, a_{2j}(z-c)^{2j} \\
&= \sum_{j=0}^{\infty} a_{2j}(z-c+c)(z-c)^{2j} \\
&= \sum_{j=0}^{\infty} a_{2j}(z-c)(z-c)^{2j} + \sum_{j=0}^{\infty} c\, a_{2j}(z-c)^{2j} \\
&= \sum_{j=0}^{\infty} a_{2j}(z-c)^{2j+1} + \sum_{j=0}^{\infty} c\, a_{2j}(z-c)^{2j}
\end{aligned}$$

so that $f(z)$ has power series $\sum_{j=0}^{\infty} b_j(z-c)^j$ where $b_{2j+1} = a_{2j}$ and $b_{2j} = c\, a_{2j}$. Thus, for all nonnegative integers j, we have

$$\begin{aligned}
\left\langle f,\ \frac{K_c^{(2j)}}{(2j)!} - c\, \frac{K_c^{(2j+1)}}{(2j+1)!} \right\rangle &= \frac{f^{(2j)}(c)}{(2j)!} - c\, \frac{f^{(2j+1)}(c)}{(2j+1)!} \\
&= b_{2j} - c\, b_{2j+1} \\
&= c\, a_{2j} - c\, a_{2j} \\
&= 0
\end{aligned}$$

so if f is in the kernel of $C_\varphi^\star$, then f satisfies the successive derivative condition for all nonnegative integers j. $\square$

It is easy to construct polynomials of any degree that satisfy the successive derivative condition of the previous theorem. Hence, we record the following immediate corollary.

Corollary 2. *If*

$$\varphi(z) = \frac{(1-2c)z^2}{1-2c\,z}$$

for $0 < c < 1/2$, *then* $\mathit{Ker}\, C_\varphi^\star$ *is infinite dimensional.*

4. A formal expression for $C_\varphi^\star$. We present the formal adjoint for the composition operator C_φ where

$$\varphi(z) = \frac{(1-2c)z^2}{1-2c\,z} \quad \text{for} \quad 0 < c < 1/2.$$

Let M_F denote the formal multiplication operator on $H^2(D)$ given by $M_F(g(z)) = F(z)g(z)$ for $g \in H^2(D)$ and $z \in D$.

Theorem 3. *If*

$$\varphi(z) = \frac{(1-2c)z^2}{1-2c\,z} \quad \text{for} \quad 0 < c < 1/2$$

we have the following formal expression for the adjoint of C_φ

$$C_\varphi^\star = \frac{1}{2} \sum_{\sqrt{c^2+(1-2c)z}} M_{F(\sqrt{c^2+(1-2c)z})}\, C_{\sigma(\sqrt{c^2+(1-2c)z})}$$

where $F(z) = (-c+z)/z$ *and* $\sigma(z) = c+z$ *and where the sum is understood to be taken over the branches of the square root function.*

Proof. Following the technique of Cowen used to find the adjoint of a composition operator with linear fractional symbol [**2**], we compute the action of the adjoint on the reproducing kernel functions:

$$\begin{aligned} C_\varphi^\star(K_w(z)) &= K_{\varphi(w)}(z) \\ &= \frac{1}{1-\overline{\varphi(w)}z} \\ &= \frac{1}{1-(\overline{w}^2(1-2c)z)/(1-2c\,\overline{w})} \\ &= \frac{1-2c\,\overline{w}}{1-2c\,\overline{w}-\overline{w}^2(1-2c)z} \\ &= \frac{2c\,\overline{w}-1}{(1-2c)z\,\overline{w}^2+2c\,\overline{w}-1}. \end{aligned}$$

Now, the denominator

$$(1-2c)z\,\overline{w}^2+2c\,\overline{w}-1$$

has a factorization $(1-2c)z\,(\overline{w}-w_1)(\overline{w}-w_2)$ where

$$\begin{aligned} w_1, w_2 &= \frac{(-2c)/(1-2c) \pm \sqrt{(4c^2)/((1-2c)^2) + (4z)/(1-2c)}}{2z} \\ &= \frac{-c/(1-2c) \pm 1/(1-2c)\sqrt{c^2+(1-2c)z}}{z} \\ &= \frac{-c \pm \sqrt{c^2+(1-2c)z}}{(1-2c)z} \\ &= \frac{1}{c \pm \sqrt{c^2+(1-2c)z}}. \end{aligned}$$

Hence,

$$\begin{aligned} C_\varphi^\star(K_w(z)) &= K_{\varphi(w)}(z) \\ &= \frac{2c\,\overline{w}-1}{(1-2c)z\,(\overline{w}-w_1)(\overline{w}-w_2)} \end{aligned}$$

with w_1 and w_2 as above. By partial fractions,

$$K_{\varphi(w)} = \frac{1}{(1-2c)z}\left[\frac{A}{\overline{w}-w_1} + \frac{B}{\overline{w}-w_2}\right]$$

where

$$A = \frac{2c\,w_1-1}{w_1-w_2}$$

and

$$B = \frac{1-2c\,w_2}{w_1-w_2}$$

so that

$$\begin{aligned} &C_\varphi^\star(K_w(z)) \\ &\quad = \frac{1}{(1-2c)z}\left[\frac{(2c\,w_1-1)/(w_1-w_2)}{\overline{w}-w_1} + \frac{(1-2c\,w_2)/(w_1-w_2)}{\overline{w}-w_2}\right] \\ &\quad = \frac{1}{(1-2c)z(w_1-w_2)}\left[\frac{2c\,w_1-1}{\overline{w}-w_1} + \frac{1-2c\,w_2}{\overline{w}-w_2}\right] \\ &\quad = \frac{1}{(1-2c)z(w_1-w_2)}\left[\frac{(2c\,w_1-1)/(-w_1)}{1-(\overline{w})/(w_1)} + \frac{(1-2c\,w_2)/(-w_2)}{1-(\overline{w})/(w_2)}\right]. \end{aligned}$$

Now,

$$\begin{aligned} w_1 - w_2 &= \frac{1}{c+\sqrt{c^2+(1-2c)z}} - \frac{1}{c-\sqrt{c^2+(1-2c)z}} \\ &= \frac{2\sqrt{c^2+(1-2c)z}}{(1-2c)z}, \end{aligned}$$

$$\begin{aligned} \frac{1-2c\,w_1}{w_1} &= \frac{(2c)/(c+\sqrt{c^2+(1-2c)z}-1)}{-1/(c+\sqrt{c^2+(1-2c)z}\,)} \\ &= -\,c+\sqrt{c^2+(1-2c)z} \end{aligned}$$

and

$$\begin{aligned} \frac{2cw_2-1}{w_2} &= \frac{1-(2c)/(c-\sqrt{c^2+(1-2c)z}\,)}{-1/(c-\sqrt{c^2+(1-2c)z}\,)} \\ &= c+\sqrt{c^2+(1-2c)z}. \end{aligned}$$

Hence,

$$\begin{aligned} &K_{\varphi(w)}(z) \\ &= \frac{1}{(1-2c)z\left((2\sqrt{c^2+(1-2c)z}\,)/((1-2c)z)\right)} \\ &\quad\times\left[\frac{-c+\sqrt{c^2+(1-2c)z}}{1-\overline{w}(c+\sqrt{c^2+(1-2c)z}\,)} + \frac{c+\sqrt{c^2+(1-2c)z}}{1-\overline{w}(c-\sqrt{c^2+(1-2c)z}\,)}\right] \\ &= \frac{1}{2}\left[\frac{(-c+\sqrt{c^2+(1-2c)z}\,)/(\sqrt{c^2+(1-2c)z}\,)}{1-\overline{w}(c+\sqrt{c^2+(1-2c)z}\,)}\right. \\ &\qquad\qquad\left. + \frac{(c+\sqrt{c^2+(1-2c)z}\,)/(-\sqrt{c^2+(1-2c)z}\,)}{1-\overline{w}(c-\sqrt{c^2+(1-2c)z}\,)}\right]. \end{aligned}$$

Thus, we may interpret $C_\varphi^\star$ as the following formal sum:

$$C_\varphi^\star = \frac{1}{2}\sum_{\sqrt{c^2+(1-2c)z}} M_{F(\sqrt{c^2+(1-2c)z})}\, C_{\sigma(\sqrt{c^2+(1-2c)z})}$$

where $F(z) = (-c+z)/z$ and $\sigma(z) = c+z$ and where the sum is understood to be taken over the branches of the square root function. $\square$

In the following, we will use this formal expression for $C_\varphi^\star$ to supply the intuition for the results of Section 3. Note that, for functions of the form $\sum_{j=0}^\infty a_j(z-c)^j$, $0<c<1/2$, the action of $C_\varphi^\star$ is particularly simple. The following calculation shows that

$$C_\varphi^\star(z-c)^{2j} = (c^2+(1-2c)z)^j$$

and

$$C_\varphi^\star(z-c)^{2j+1} = -\,c(c^2+(1-2c)z)^j.$$

Since, formally, we have

$$\begin{aligned}
&2C_\varphi^\star(z-c)^{2j}\\
&= \sum_{\sqrt{c^2+(1-2c)z}} M_{F(\sqrt{c^2+(1-2c)z})}\, C_{\sigma(\sqrt{c^2+(1-2c)z})}\,(z-c)^{2j}\\
&= \frac{-c+\sqrt{c^2+(1-2c)z}}{\sqrt{c^2+(1-2c)z}}\,(c+\sqrt{c^2+(1-2c)z}-c)^{2j}\\
&\quad + \frac{c+\sqrt{c^2+(1-2c)z}}{\sqrt{c^2+(1-2c)z}}\,(c-\sqrt{c^2+(1-2c)z}-c)^{2j}\\
&= \frac{1}{\sqrt{c^2+(1-2c)z}}\left[\left(-c+\sqrt{c^2+(1-2c)z}\right)\left(\sqrt{c^2+(1-2c)z}\right)^{2j}\right]\\
&\quad + \frac{1}{\sqrt{c^2+(1-2c)z}}\left[\left(c+\sqrt{c^2+(1-2c)z}\right)\left(\sqrt{c^2+(1-2c)z}\right)^{2j}\right]\\
&= \frac{1}{\sqrt{c^2+(1-2c)z}}\,\left(c^2+(1-2c)z\right)^j\\
&\quad \cdot\left(-c+\sqrt{c^2+(1-2c)z}+c+\sqrt{c^2+(1-2c)z}\right)\\
&= \frac{1}{\sqrt{c^2+(1-2c)z}}\left(c^2+(1-2c)z\right)^j\left(2\sqrt{c^2+(1-2c)z}\right)\\
&= 2\left(c^2+(1-2c)z\right)^j
\end{aligned}$$

and, again formally,

$$\begin{aligned}
&2C_\varphi^\star(z-c)^{2j+1}\\
&= \sum_{\sqrt{c^2+(1-2c)z}} M_{F(\sqrt{c^2+(1-2c)z})}\, C_{\sigma(\sqrt{c^2+(1-2c)z})}\, (z-c)^{2j+1}\\
&= \frac{-c+\sqrt{c^2+(1-2c)z}}{\sqrt{c^2+(1-2c)z}} \left(c+\sqrt{c^2+(1-2c)z}-c\right)^{2j+1}\\
&\quad + \frac{c+\sqrt{c^2+(1-2c)z}}{\sqrt{c^2+(1-2c)z}} \left(c-\sqrt{c^2+(1-2c)z}-c\right)^{2j+1}\\
&= \frac{-c+\sqrt{c^2+(1-2c)z}}{\sqrt{c^2+(1-2c)z}} \left(\sqrt{c^2+(1-2c)z}^{2j+1}\right)\\
&\quad - \frac{c+\sqrt{c^2+(1-2c)z}}{\sqrt{c^2+(1-2c)z}} \left(\sqrt{c^2+(1-2c)z}\right)^{2j+1}\\
&= \left(-c+\sqrt{c^2+(1-2c)z}\right)\left(c^2+(1-2c)z\right)^j\\
&\quad - \left(c+\sqrt{c^2+(1-2c)z}\right)\left(c^2+(1-2c)z\right)^j\\
&= \left(-c+\sqrt{c^2+(1-2c)z}-c-\sqrt{c^2+(1-2c)z}\right)\left(c^2+(1-2c)z\right)^j\\
&= -2c(c^2+(1-2c)z)^j.
\end{aligned}$$

Hence we have, at least formally,

$$\begin{aligned}
C_\varphi^\star \sum_{j=0}^\infty a_j(z-c)^j &= C_\varphi^\star \sum_{j=0}^\infty a_{2j}(z-c)^{2j} + C_\varphi^\star \sum_{j=0}^\infty a_{2j+1}(z-c)^{2j+1}\\
&= \sum_{j=0}^\infty a_{2j}(c^2+(1-2c)z)^j\\
&\quad + \sum_{j=0}^\infty a_{2j+1}(-c)(c^2+(1-2c)z)^j\\
&= \sum_{j=0}^\infty (a_{2j}-ca_{2j+1})(c^2+(1-2c)z)^j.
\end{aligned}$$

From this it is easy to identify which $zH^2(D)$ functions ought to be in $\operatorname{Ker} C_\varphi^\star$. Since

$$\begin{aligned}0 &= \left\langle f,\ \frac{K_c^{(2j)}}{(2j)!} - c\,\frac{K_c^{(2j+j)}}{(2j+1)!}\right\rangle \\ &= \frac{f^{(2j)}(c)}{(2j)!} - c\,\frac{f^{(2j+1)}(c)}{(2j+1)!} \\ &= a_{2j} - c\,a_{2j+1},\end{aligned}$$

we can see that functions of the form $\sum_{j=0}^{\infty} a_j(z-c)^j$ are in $\operatorname{Ker} C_\varphi^\star$ if they satisfy the successive derivative condition

$$0 = \left\langle f,\ \frac{K_c^{(2j)}}{(2j)!} - c\,\frac{K_c^{(2j+1)}}{(2j+1)!}\right\rangle$$

for all nonnegative integers j.

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