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THE REAL GENUS OF GROUPS OF ODD ORDER

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ABSTRACT. Let G be a finite group. The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts. Here we consider groups of odd order acting on bordered surfaces. First we show that if G is a group of odd order, then the real genus $\rho(G)$ is even. We also obtain a stronger result for p-groups. Let p be an odd prime, and let G be a p-group with $\rho(G) \geq 2$; then the real genus $\rho(G) \equiv p+1 \bmod 2p$. We also examine "large" automorphism groups of odd order. If the odd order group G acts on a bordered Klein surface of genus $g \geq 2$, then $|G| \leq 3(g-1)$. If G acts with the largest possible order $3(g-1)$, then we call G an *O^*-group*. In general, a quotient Q of an O^*-group G is again an O^*-group, and a surface X on which G acts is a full covering of a surface of lower genus on which Q acts. Thus, it is natural to consider the notion of an *O^*-simple* group, that is, an O^*-group with no O^*-quotient. We classify the O^*-simple groups.

1. Introduction. Let G be a finite group. The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts. A *real genus action* of G is an action of G on a bordered surface of (algebraic) genus $\rho(G)$. There are now several results about the real genus parameter. The groups with real genus $\rho \leq 8$ have been classified [**7, 12, 13, 16**], and genus formulas have been obtained for several families of groups [**13–16**]. In particular, McCullough determined the real genus of each finite abelian group [**21**].

Here we consider groups of odd order acting on bordered Klein surfaces. Our main result is the following.

Theorem 1. *If G is a group of odd order, then the real genus $\rho(G)$ is even.*

We also obtain a stronger result for p-groups.

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Theorem 2. *Let p be an odd prime, and let G be a p-group with $\rho(G) \geq 2$. Then the real genus*

$$\rho(G) \equiv p + 1 \bmod 2p.$$

Next we examine "large" automorphism groups of odd order. If the odd order group G acts on a bordered Klein surface of genus $g \geq 2$, then $|G| \leq 3(g-1)$. If G acts with the largest possible order $3(g-1)$, then we call G an *O^*-group*. We start by presenting several examples of infinite families of O^*-groups.

In general, a quotient Q of an O^*-group G is again an O^*-group, and a surface X on which G acts is a full covering of a surface of lower genus on which Q acts. Thus, it is natural to consider the notion of an O^*-*simple* group, that is, an O^*-group with no O^*-quotient. We classify these O^*-simple groups; in addition to the abelian group $Z_3 \times Z_3$, there are two infinite families of groups. In one family, each group has order $3p$, where p is a prime such that $3 \mid p-1$. In the other, each group has order $3q^2$, where q is a prime such that $3 \mid q+1$.

Finally, we complete the determination of the real genus of all groups with odd order less than 100.

We use the standard representation of a group G as a quotient of a non-Euclidean crystallographic group Γ by a bordered surface group K; then G acts on the Klein surface U/K, where U is the open upper half-plane.

2. Preliminaries. We shall assume that all surfaces are compact. A bordered surface X can carry a dianalytic structure and be considered a Klein surface or a nonsingular real algebraic curve (with real points). Thus the surface X has an algebraic genus g. The algebraic genus is the rank of the fundamental group of X, and this number appears naturally in bounds for the order of its automorphism group. The real genus of a group is defined in terms of the algebraic genus.

Associated with the NEC group Γ is its *signature*, which has the form

$$(1) \qquad (p; \pm; [\lambda_1, \ldots, \lambda_r]; \ \{(\nu_{11}, \ldots, \nu_{1\mathrm{s}_1}), \ldots, (\nu_{\mathrm{k}1}, \ldots, \nu_{\mathrm{ks}_\mathrm{k}})\}).$$

The quotient space $X = U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and nonorientable if the minus sign is used. The integers $\lambda_1, \ldots, \lambda_\mathrm{r}$, called the *ordinary periods*, are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The integers $\nu_{\mathrm{i}1}, \ldots, \nu_{is_i}$, called the *link periods*, are the ramification indices in fibers above points on the ith boundary component of X.

Associated with the signature (1) is a presentation for the NEC group Γ, although the form of the presentation depends upon whether the plus or minus sign is present. These presentations are in the monograph [**1**], for instance.

Let Γ be an NEC group with signature (1), and assume $k \geq 1$ so that the quotient space U/Γ is a bordered surface. The non-Euclidean area $\mu(\Gamma)$ of a fundamental region for Γ is given by [**24**, p. 235]:

$$\frac{\mu(\Gamma)}{2\pi} = \gamma - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{\lambda_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{\nu_{ij}}\right), \tag{2}$$

where γ is the algebraic genus of the quotient space U/Γ. If Λ is a subgroup of finite index in Γ, then

$$[\Gamma : \Lambda] = \frac{\mu(\Lambda)}{\mu(\Gamma)}. \tag{3}$$

An NEC group K is called a *bordered surface group* if the quotient map from U to the quotient space U/K is unramified and further, U/K has a nonempty boundary. Bordered surface groups contain reflections but no other elements of finite order.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$. Then X can represented as U/K, where K is a bordered surface group with $\mu(K) = 2\pi(g-1)$. Let G be a group of dianalytic automorphisms of the Klein surface X. Then there are an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$. The group $G \cong \Gamma/K$, so that from (3) the algebraic genus g of the bordered surface X on which G acts is given by

$$g = 1 + |G| \cdot \frac{\mu(\Gamma)}{2\pi}. \tag{4}$$

Thus (2) and (4) give the relationship between the genera g and γ of X and U/Γ, respectively. This relationship is sometimes given as the

Riemann-Hurwitz formula for the quotient mapping $X \to X/G = U/\Gamma$; see [**6, 9**], for example.

The following is basic and quite easy to prove.

Proposition 1. *Each period divides* $|G|$*. If* $|G|$ *is odd, then, further, all period cycles of* Γ *are empty.*

The general upper bound $12(g-1)$ [**9**] for the size of an automorphism group of a surface of genus $g \geq 2$ can be improved considerably for groups of odd order.

Theorem 3. *Suppose the odd order group* G *acts on a surface* X *of genus* $g \geq 2$*. Then* $|G| \leq 3(g-1)$*. The bound* $3(g-1)$ *is attained if and only if* $G \cong \Delta/K$*, where* Δ *is an* NEC *group* Δ *with signature*

$$(0; +; [3,3]; \{()\}) \tag{5}$$

and K *is a bordered surface group. Further, the surface* $X = U/K$ *on which* G *acts is orientable, and the action of* G *on* X *is orientation-preserving.*

Proof. If G is not cyclic, then the bound was obtained in [**16**, Theorem 2]. Assume that G is cyclic. Then, just applying the old bound for the order of a cyclic group of automorphisms, we have $|G| \leq 2g+2$ if g is even and X is orientable; otherwise, $|G| \leq 2g$ [**10**, Theorem 1]. Immediately, $|G| \leq 3(g-1)$ in all cases except $g = 2$, X is orientable, and $|G| = 5$. But it is not hard to see that there is no action of Z_5 on an orientable surface of genus 2. Hence the bound $3(g-1)$ also applies to odd order cyclic groups.

Now represent X as U/K, where K is a bordered surface group. Then the bound $3(g-1)$ is attained if and only if G is a quotient Δ/K, where $\mu(\Delta)/2\pi = 1/3$. Using Proposition 1, it is easy to see that Δ must have signature (5). Here see [**1**, Theorem 4.4.7, p. 130] and [**2**].

Further, the surface $X = U/K$ on which G acts is orientable, since the index of K in Δ is odd [**1**, p. 39]. Finally, since $|G|$ is odd, G has no subgroup of index 2 and thus the action of G on X cannot be orientation-reversing. $\square$

This bound for groups of odd order agrees with the corresponding bound for 3-groups. There are infinite families of 3-groups for which this bound is attained. See [**1**, pp. 130, 131], [**2**, Section 5] and [**13**, Section 4]. We shall see examples of odd order groups that are not 3-groups in Section 4.

The group Δ has presentation

$$x^3 = y^3 = c^2 = [e, c] = xye = 1.$$

Since G has odd order, $c \in K$, and e is clearly redundant. Hence G is generated by two elements of order 3.

We call a noncyclic group of odd order that is generated by two elements of order 3 an *O^*-group*. These groups will be of special interest here. We record the following.

Proposition 2. *A finite group G of odd order is an O^*-group if and only if G is not cyclic and acts as a group of $3(g-1)$ automorphisms of a surface of genus $g \geq 2$.*

We are excluding Z_3 from the O^*-groups, even though Z_3 clearly acts on a bordered surface of genus 2 (a sphere with three holes or a torus with one hole). But these are not genus actions of Z_3, since Z_3 acts on the disk and $\rho(Z_3) = 0$. Further, no other group of real genus 0 or 1 except Z_3 is generated by elements of order 3 [**13**]. Thus, an O^*-group has a genus action of a surface of genus $g \geq 2$, and we have the following.

Corollary 1. *If G is an O^*-group, then $\rho(G) = 1 + |G|/3$ and G has even genus.*

3. Reductions. Let G be a finite group of odd order with $\rho(G) \geq 2$. Then there are an NEC group Γ with signature (1) and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = \text{kernel}\,\phi$ is a bordered surface group. By Proposition 1, all period cycles in the signature of Γ are empty, and we may also assume that the ordinary periods are in increasing order, that is, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$. The group G acts on $X = U/K$, a surface of genus g, where g is given by (4). We want to minimize g or, equivalently, $\mu(\Gamma)$. A series of "reductions" shows that if G has odd order, we only need to consider NEC groups with a single type of signature. A similar approach was successfully employed in [**18**].

Lemma 1. *Suppose* Γ *has signature*

$$(p;+;[\lambda_1,\dots,\lambda_r];\{()^k\})$$

with $p>0$ *and* $k>0$. *Then there exist an NEC group* Γ' *with signature*

$$(p-1;+;[\nu_1,\nu_2,\lambda_1,\dots,\lambda_r];\{()^k\}) \tag{6}$$

and a homomorphism $\alpha:\Gamma'\to G$ *onto* G *such that kernel* α *is a bordered surface group and* $\mu(\Gamma')<\mu(\Gamma)$.

Proof. Let $\nu_1=o(a_p)$, $\nu_2=o(b_p)$, and let Γ' be an NEC group with signature (6). Most generators of Γ' correspond in a general way to generators of Γ; the two elliptic generators y_1,y_2 are "new." Write $z=(b_pa_p)^{-1}$.

We define a homomorphism $\alpha:\Gamma'\to G$ by defining $\alpha(y_1)=\phi(a_p)$, $\alpha(y_2)=\phi(b_p)$, $\alpha(x_i')=\phi(zx_iz^{-1})$ and $\alpha(e_1')=\phi(ze_1)$; otherwise, each generator in Γ' is mapped to the image of its corresponding generator in Γ, e.g., $\alpha(a_i)=\phi(a_i)$ for $i<p$. Then α is clearly surjective, and the defining relations for Γ' are satisfied in G. Thus α is a homomorphism of Γ' onto G. Further, kernel α is a bordered surface group. Finally, using (2) we calculate

$$\frac{\mu(\Gamma')}{2\pi}=\frac{\mu(\Gamma)}{2\pi}-\frac{1}{\nu_1}-\frac{1}{\nu_2}<\frac{\mu(\Gamma)}{2\pi}. \qquad \square$$

The proofs of the following three lemmas are quite similar to the proof of Lemma 1. Since these proofs present no special difficulties, we omit them.

Lemma 2. *Suppose* Γ *has signature*

$$(p;-;[\lambda_1,\dots,\lambda_r];\{()^k\})$$

with $p > 1$ and $k > 0$. Then there exist an NEC *group* Γ' *with signature*

$$(p-1; +; [\nu, \lambda_1, \dots, \lambda_r]; \{()^k\}) \tag{7}$$

and a homomorphism $\beta : \Gamma' \to G$ onto G such that kernel β is a bordered surface group and $\mu(\Gamma') < \mu(\Gamma)$.

Lemma 3. *Suppose Γ has signature*

$$(1; -; [\lambda_1, \dots, \lambda_r]; \{()^k\})$$

with $k > 0$. Then there exist an NEC group Γ' with signature

$$(0; +; [\lambda_1, \dots, \lambda_r]; \{()^{k+1}\}) \tag{8}$$

and a homomorphism $\alpha : \Gamma' \to G$ onto G such that kernel α is a bordered surface group and $\mu(\Gamma') = \mu(\Gamma)$.

Lemma 4. *Suppose Γ has signature*

$$(0; +; [\lambda_1, \dots, \lambda_r]; \{()^k\}),$$

with $k > 1$. Then there exist an NEC *group* Γ' *with signature*

$$(0; +; [\nu, \lambda_1, \dots, \lambda_r]; \{()^{k-1}\}) \tag{9}$$

and a homomorphism $\alpha : \Gamma' \to G$ onto G such that kernel α *is a bordered surface group and $\mu(\Gamma') < \mu(\Gamma)$.*

Therefore, in order to minimize $\mu(\Gamma)$, we only need to consider one type of signature. We record this result in the following.

Proposition 3. *Let G be a finite group of odd order. Let S_G be the set of all* NEC *groups Γ such that G is a quotient group of Γ with the kernel a bordered surface group. If Γ is an* NEC *group in S_G with minimal non-Euclidean area, then the signature of Γ has the form*

$$(0; +; [\lambda_1, \dots, \lambda_r]; \{()\}). \tag{10}$$

Proof. By Proposition 1, all period cycles are empty, and thus we may assume the signature of Γ has the form

$$(p; \pm; [\lambda_1, \dots, \lambda_r]; \{()^k\}),$$

with $k > 0$.

If the minus sign occurs, then $p \geq 1$. Apply Lemma 2 (if $p > 1$), Lemma 3, and Lemma 4 (at least once) to obtain a group Δ in $\mathrm{S_G}$ with signature (10) such that $\mu(\Delta) < \mu(\Gamma)$.

If the plus sign occurs, use Lemma 1 (if $p > 0$) and then Lemma 4 (if $k > 1$). $\square$

4. The main results. It is now easy to establish Theorems 1 and 2. We begin with an easy, but interesting, result about the real genus action of a group of odd order.

Theorem 4. *Let G be a group of odd order with real genus action on the bordered Klein surface X of genus $\rho(G) \geq 2$. Then the quotient space X/G is the disc D, that is, X/G has algebraic genus* 0.

Proof. Represent X as U/K, where K is a bordered surface group. Then by Proposition 2 there is an NEC group Γ with signature (10) and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$. Then $X/G = U/\Gamma$ is the disc D, the only bordered surface with algebraic genus 0. $\square$

This result is interesting, in part, because the corresponding results for two related parameters, the symmetric genus and the strong symmetric genus, do not hold, that is, the quotient space under a genus action need not have genus zero [**20**, Proposition 4]. In connection with our approach in Section 3, we note that the reductions of the lemmas are not possible without connecting generators. With Proposition 3 the proof of Theorem 1 is now easy.

Proof of Theorem 1. First, no group of real genus 1 has odd order [**13**, Theorem 4], and 0 is even, of course. Consequently, we only need to consider groups with genus 2 or more.

Let X be a bordered Klein surface of genus $g = \rho(G) \geq 2$ such that there is a genus action of G on X. Represent X as U/K, where K is a bordered surface group. Then by Proposition 3 there is an NEC group Γ with signature (10) and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$.

Then from (2) we have

$$\frac{\mu(\Gamma)}{2\pi} = -1 + \sum_{i=1}^{r} \left(1 - \frac{1}{\lambda_i}\right).$$

Now by (4)

$$\mathrm{g} = 1 - |G| + \sum_{i=1}^{r} \left(|G| - \frac{|G|}{\lambda_i}\right).$$

Each period divides $|G|$, and each term $|G|/\lambda_i$ is odd. Now each term inside the summation sign is even, and g is even. $\square$

The same approach yields Theorem 2, a stronger result for finite p-groups with positive real genus.

Proof of Theorem 2. Write $|G| = p^n$ and continue the notation of the proof of Theorem 1. The periods are powers of p; denote the exponents $m_1, \ldots, m_r$, that is, $\lambda_i = p^{m_i}$. Each $m_i < n$ since G is not cyclic. Now (2) and (4) give

$$\mathrm{g} = 1 + p^n \left[-1 + \sum_{i=1}^{r} \left(1 - \frac{1}{p^{m_i}}\right)\right],$$

and then

$$\mathrm{g} = 1 - p^n + \sum_{i=1}^{r} p^{n-m_i}(\mathrm{p}^{m_i} - 1).$$

Each term inside the summation sign is even (since p is odd) and divisible by p (since $n - m_i$ is positive). Hence the sum is congruent to $0 \bmod 2p$. Since p is odd, $p^n \equiv p \bmod 2p$ and $-p^n \equiv -p \equiv p \bmod 2p$. Therefore $g \equiv p + 1 \bmod 2p$. $\square$

It is possible to give a shorter proof of Theorem 2, using Theorem 1 together with the complex double of a bordered surface and an important result of Kulkarni [**8**, p. 199].

5. Examples–O^*-groups. Here we present some examples of groups of odd order that act as large groups of automorphisms. Each group has order $3p$, $9p$ or $3p^2$, for some odd prime p. These groups will also be needed when we consider O^*-simple groups.

First let p be an odd prime such that 3 divides $p-1$. Then there is a single nonabelian group of order $3p$. This group has presentation

$$X^p = Y^3 = 1, \quad Y^{-1}XY = X^r, \tag{11}$$

where $r^3 \equiv 1 \bmod p$ and $r \not\equiv 1 \bmod p$. We denote this group G_{3p}; it is a semi-direct product $Z_p \times_\phi Z_3$. This group is generated by two elements (Y and XY) of order 3 and has real genus

$$\rho(G_{3p}) = 1 + p.$$

The real genus of this family of O^*-groups was calculated in [**15**, Theorem 4].

It is not hard to see that each direct product $Z_3 \times G_{3p}$ is also an O^*-group.

Proposition 4. *Let p be an odd prime such that 3 divides $p-1$. Then $Z_3 \times G_{3p}$ is an O^*-group and*

$$\rho(Z_3 \times G_{3p}) = 1 + 3p.$$

Proof. Let G_{3p} have presentation (11), and let A be a generator for Z_3. Then $(1, Y)$ and (A, XY) are elements of order 3 that generate $Z_3 \times G_{3p}$. Thus $Z_3 \times G_{3p}$ is an O^*-group, and the genus formula holds.

Let $G = Z_p \times Z_p$ have presentation

$$X^p = Y^p = 1, \quad XY = YX. \tag{12}$$

Now let p be an odd prime such that 3 divides $p+1$. Then there is an automorphism of G of order 3 defined by

$$X \longrightarrow Y, \quad Y \longrightarrow X^{-1}Y^{-1}.$$

Thus there is a nonabelian group of order $3p^2$ with generators X, Y and A and relations (12) together with

$$A^3 = 1, \quad A^{-1}XA = Y, \quad A^{-1}YA = X^{-1}Y^{-1}. \tag{13}$$

We denote this group H_{3p^2}. It is a semi-direct product $(Z_p)^2 \times_\phi Z_3$ and is the unique nonabelian group of order $3p^2$ in case 3 divides $p+1$ [**3**, p. 80]. This is another family of O^*-groups.

Proposition 5. *Let p be an odd prime such that 3 divides $p+1$. Then H_{3p^2} is an O^*-group and*

$$\rho(H_{3p^2}) = 1 + p^2.$$

Proof. Let $G = H_{3p^2}$ have presentation (12) and (13). Then it is not hard to check that AX and AY are elements of order 3 that generate G. Thus H_{3p^2} is an O^*-group, and $3[\rho(H_{3p^2})-1] = |H_{3p^2}| = 3p^2$. ◻

Now let p be an odd prime such that 3 divides $p-1$. Then there are four nonabelian groups of order $3p^2$ [**3**, Section 59]. These are $Z_3 \times G_{3p}$, a semi-direct product $Z_{p^2} \times_\theta Z_3$, and two semi-direct products $(Z_p)^2 \times_\phi Z_3$; two of these four are O^*-groups.

First we consider the semi-direct product $Z_{p^2} \times_\theta Z_3$. The cyclic group of order p^2 has an automorphism of order 3, and there is a nonabelian group of order $3p^2$ with presentation

$$X^{p^2} = B^3 = 1, \quad B^{-1}XB = X^r, \tag{14}$$

where $r^3 \equiv 1 \bmod p^2$ and $r \not\equiv 1 \bmod p^2$. We denote this group J_{3p^2}.

Proposition 6. *Let p be an odd prime such that 3 divides $p-1$. Then J_{3p^2} is an O^*-group and*

$$\rho(J_{3p^2}) = 1 + p^2.$$

Proof. Let J_{3p^2} have presentation (14). Then BX and BX^p are elements of order 3 (in fact, all elements of the form BX^i have order 3). Further, BX and BX^p generate J_{3p^2}. $\square$

Let $G = Z_p \times Z_p$ have presentation (12), where p is an odd prime such that 3 divides $p-1$. Then there is an automorphism of G of order 3 defined by

$$X \longrightarrow X^{\alpha}, \quad Y \longrightarrow Y^{\alpha^2},$$

where $\alpha^3 \equiv 1 \bmod p$ and $\alpha \not\equiv 1 \bmod p$. Thus there is a nonabelian group of order $3p^2$ with generators X, Y and A and relations (12) together with

$$A^3 = 1, \quad A^{-1}XA = X^{\alpha}, \quad A^{-1}YA = Y^{\alpha^2}. \tag{15}$$

We denote this group K_{3p^2}; it is a semi-direct product $(Z_p)^2 \times_{\phi} Z_3$. Again, these are O^*-groups.

Proposition 7. *Let p be an odd prime such that 3 divides $p-1$. Then K_{3p^2} is an O^*-group and*

$$\rho(K_{3p^2}) = 1 + p^2.$$

Proof. Let K_{3p^2} have presentation (12) and (15). Then it is not hard to check that AX and AY are elements of order 3 that generate G. $\square$

6. Full covers. There are standard techniques (using the theory of covering spaces and the fundamental group) that show that there are infinitely many extensions of abelian groups by each O^*-group that are also O^*-groups. Our basic reference here is [**6**, Section 4].

Let X and X' be bordered Klein surfaces, and let $\phi : X \to X'$ be an unramified normal covering without folding of the surface X' (so that X is a covering space in the usual topological sense and the covering transformations act transitively on fibers). If every automorphism of X' lifts to an automorphism of X, then ϕ is called a *full* covering. In this case, let $G' = A(X')$, and let N be the group of covering transformations of X. Then let G be the group of automorphisms of X generated by N and lifts of elements of G'. We have the exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G' \longrightarrow 1.$$

Of course, in general G is only a subgroup of $A(X)$.

However, if the group G' is maximal in some sense, than it may be that G is as well, and even, perhaps, $G = A(X)$. If G' is an M*-group, then G is too and hence $G = A(X)$ [**6**, Theorem 5]. For groups of odd order, we have the following.

Theorem 5. *Let G' be an O^*-group with genus action on the bordered surface X', and let $\phi : X \to X'$ be a full covering of X'. If the degree of the covering ϕ is odd, then there is an O^*-group G with genus action on the surface X.*

Proof. Let g and g' denote the genera of X and X', respectively, and let r be the degree of ϕ. Since ϕ is unramified,

$$g - 1 = r(g' - 1).$$

Since G' is an O^*-group, $|G'| = 3(g' - 1)$. Now G is a group of odd order $|G| = r \cdot |G'| = 3(g - 1)$ acting on X, that is, G is an O^*-group. $\square$

The constructions of [**6**, Section 4] may now be applied to show the existence of infinite families of O^*-groups. For example, the abelian group $\mathrm{Z}_3 \times \mathrm{Z}_3$ has genus 4 and acts on a torus with three holes. Applying the first construction [**6**, Theorem 8] yields, for each odd positive integer n, a surface with

$$g_n = 3n^4 + 1, \quad p_n = 3n^3(n - 1)/2 + 1, \quad k_n = 3n^3.$$

The application of this construction to a general O^*-group yields the following.

Proposition 8. *If G is an O^*-group of order $3(g - 1)$ and n is an odd positive integer, then G has an O^*-extension G_n of the form*

$$1 \longrightarrow (Z_n)^g \longrightarrow G_n \longrightarrow G \longrightarrow 1.$$

The construction could even be applied to Z_3 acting on a surface of genus 2 (either a torus with one hole or a sphere with three holes) to produce O^*-groups, although the action of Z_3 would not be a genus action.

7. O^*-simple groups. Now we consider quotients of O^*-groups and surfaces with "large" odd order groups of automorphisms. First we show a quotient of an O^*-group is an O^*-group.

Theorem 6. *Let G be an O^*-group with genus action on the bordered surface X of genus $g \geq 2$. Let N be a normal subgroup of G of index $r > 3$. Set $G' = G/N$, $X' = X/N$, let $\phi : X \to X'$ be the quotient map, and let g' be the genus of X'. Then*

(1) *G' is an O^*-group with genus action on X',*

(2) *$g' \geq 2$, and*

(3) *ϕ is a full covering.*

Proof. Since G is generated by two elements of order 3, so is its quotient group G'. Further G' is not cyclic since $r > 3$. Thus G' is an O^*-group with $\rho(G') \geq 2$.

The group $G' = G/N$ acts on the surface $X' = X/N$, and hence $g' \geq \rho(G') \geq 2$ immediately.

Since G' has odd order, $|G'| \leq 3(g'-1)$. Then applying the Riemann-Hurwitz formula to the mapping ϕ yields $g - 1 \geq |N| \cdot (g' - 1)$, with equality if and only if ϕ is unramified. Now

$$|G'| = |G/N| = 3(g-1)/|N| \geq 3(g'-1).$$

Hence $|G'| = 3(g'-1)$, the action of G' on X' is a genus action, and ϕ is unramified. Also, there can be no folding in a covering ϕ of odd degree. Thus ϕ is a full covering. □

This result suggests the following definition. An O^*-group is called O^*-*simple* if it has no proper O^*-quotient group, or equivalently, if it has no nontrivial normal subgroup of index larger than 3. If G is an O^*-group with genus action on the surface X, then G has a quotient group

G' that is O^*-simple with a genus action on a surface X'. Further, X is a full covering of X'.

It is not too hard to classify the O^*-simple groups. It is necessary to consider groups of order $3p^2$ in case 3 divides $p-1$.

Lemma 5. *Let p be an odd prime such that 3 divides $p-1$. Then there are no O^*-simple groups of order $3p^2$.*

Proof. There are four nonabelian groups of this order [**3**, Section 59], $Z_p \times G_{3p}$, J_{3p^2}, K_{3p^2} and the semi-direct product $(Z_p)^2 \times_\phi Z_3$ with presentation (12) plus

$$A^3 = 1, \quad A^{-1}XA = X^\beta, \quad A^{-1}YA = Y^\beta,$$

where $\beta^3 \equiv 1 \bmod p$ and $\beta \not\equiv 1 \bmod p$.

Obviously, $Z_p \times G_{3p}$ is not generated by elements of order 3. The O^*-group J_{3p^2} has a normal subgroup of order p (generated by X^p) and thus is not O^*-simple. The O^*-group K_{3p^2} has two normal subgroups of order p (one generated by X, the other by Y) and is not O^*-simple either. The other semi-direct product $(Z_p)^2 \times_\phi Z_3$ is determined by the action

$$X \longrightarrow X^\beta, \quad Y \longrightarrow Y^\beta,$$

on $(Z_p)^2$. This is a power automorphism of $(Z_p)^2$, and it is not hard to see that the semi-direct product has rank 3 and consequently cannot be an O^*-group. $\square$

Theorem 7. *Let G be an O^*-simple group. Then G is isomorphic to one of the following.*

(1) $Z_3 \times Z_3$.

(2) G_{3p} *for some prime p such that $3 \mid p-1$.*

(3) H_{3q^2} *for some prime q such that $3 \mid q+1$.*

Proof. A major result about finite groups is that all groups of odd order are solvable [**4**].

First $Z_3 \times Z_3$ is clearly the smallest O^*-simple group. Any 3-group with larger order has an O^*-quotient of order 9. Thus $Z_3 \times Z_3$ is the only O^*-simple 3-group.

Assume that G is not a 3-group. Since G is solvable, the commutator subgroup $G' \neq G$. Since G is O^*-simple, G does not have a normal subgroup of index larger than 3 and the only possibility is that $G/G' \cong Z_3$. Further, G' must be a minimal normal subgroup. Otherwise, there would be a minimal normal subgroup $M \subset G'$ with $[G : M] > 3$, and G would not be O^*-simple, by Theorem 6. But every minimal normal subgroup of a solvable group is an elementary abelian group [**23**, p. 117]. Thus G' is isomorphic to $(Z_p)^m$ for some odd prime $p > 3$ and some positive integer m.

The O^*-group G is generated by two elements of order 3. Assume $G = \langle a, b\rangle$, where $o(a) = o(b) = 3$. Let $\mu : G \to G/G'$ be the natural quotient mapping. Then a and b are not in the commutator subgroup G'. In $G/G' \cong Z_3$, either $\mu(a) = \mu(b)$ or $\mu(a) = \mu(b^2)$, that is, either ab^{-1} or ab is in G'. We may assume that $a = nb$ for some element n in the elementary abelian p-group G' (since a and b^{-1} are alternate generators for G). Now $G = \langle b, n\rangle$, where $o(n) = p$.

The elements bnb^{-1} and b^2nb^{-2} are elements of order p and thus in the normal Sylow p-subgroup G'. Let $M = \langle n, bnb^{-1}, b^2nb^{-2}\rangle$. Then clearly $M \subset G'$ and M is normal in $G = \langle b, n\rangle$. Thus $M = G' \cong (Z_p)^m$, since G' is a minimal normal subgroup of G. Now $m \leq 3$.

Suppose $bnb^{-1} \in \langle n\rangle$. Then $M = \langle n\rangle$ and $m = 1$. Now G is a non-abelian group of order $3p$, and thus $G \cong G_{3p}$ for some prime p such that $3 \mid p - 1$.

Suppose $bnb^{-1} \notin \langle n\rangle$. Then n and bnb^{-1} generate an abelian group of order p^2 and $m \geq 2$. First assume $m = 2$. Then G is a non-abelian group of order $3p^2$, and by Lemma 5, 3 does not divide $p - 1$. Hence $3 \mid p + 1$ and $G \cong H_{3p^2}$, since this is the unique non-abelian group of this order.

The only remaining case cannot occur. To see this, assume that $bnb^{-1} \notin \langle n\rangle$ and $m = 3$; then $b^2nb^{-2} \notin \langle n, bnb^{-1}\rangle$. Now let $x = n(bnb^{-1})(b^2nb^{-2}) = (nb)^3$. Then $x \neq 1$, x is in the abelian group M, and $nx = xn$. Also,

$$bxb^{-1} = b[n(bnb^{-1})(b^2nb^{-2})]b^{-1} = (bnb^{-1})(b^2nb^{-2})\, n = x,$$

since M is abelian. Thus the subgroup $J = \langle x \rangle$ is normal in $G = \langle b, n \rangle$, and $|J| = p$. Since M is a minimal normal subgroup of G, this is an obvious contradiction. □

For the companion ideas about M^*-groups and a classification of the solvable M^*-simple groups, see [**6**].

8. Groups of odd order less than 100**.** Here we finish the calculation of the real genus of the non-abelian groups of odd order in this range. The real genus of each abelian group has been determined by McCullough [**21**]. Also, the real genus of each group with order less than 32 has been determined [**16**], and all groups with order 81 were considered in [**17**]. There is not much left to do; most of the groups were considered in Section 5.

The table gives $\rho(G)$ for each nonabelian group G with $32 < |G| < 100$, $|G| \neq 81$.

TABLE 1. Groups of odd order less than 100.

Group	ρ	Reference	Group	ρ	Reference
$G_{3\cdot 13}$	14	[**15**, Theorem 4]	$Z_7 \times_\phi Z_9$	48	
$G_{5\cdot 11}$	34	[**15**, Theorem 4]	$H_{3\cdot 5^2}$	26	Proposition 5
$G_{3\cdot 19}$	20	[**15**, Theorem 4]	$G_{3\cdot 31}$	32	[**15**, Theorem 4]
$Z_3 \times G_{3\cdot 7}$	22	Proposition 4			

The only group that has not already been considered is one of the two non-abelian groups of order 63, the semi-direct product $G = Z_7 \times_\phi Z_9$ with presentation

$$X^7 = Y^9 = 1, \quad Y^{-1}XY = X^2.$$

The center $Z(G) = \langle Y^3 \rangle$ contains the only elements of order 3, and G/Z is isomorphic to the non-abelian group of order 21. Also, the Sylow 7-subgroup is normal. Thus, any two element generating set for G must contain at least one element with order larger than 7. The group G is clearly generated by two elements of orders 7, 9, and $\rho(G) = 48$.

9. Comments and open problems. For each integer $g \geq 2$, define $\nu(g)$ to be the number of groups with real genus g. Of course, $\nu(g)$ is a finite number for each g. We know that $\nu(2) = 0$ [**13**], $\nu(3) = 2$ [**13**], $\nu(4) = 4$ [**12**] and $\nu(5) = 9$ [**16**]. Also, $\nu(6) = 4$ and $\nu(7)$ and $\nu(8)$ are known [**7**]. Further, $\nu(g)$ is positive for all odd $g \geq 3$ [**13**, Theorem 9], and, in fact, the function ν is unbounded.

Mockiewicz has recently obtained the interesting result that there is no group of real genus 12, that is, $\nu(12) = 0$ [**22**]. Thus there is a value besides $g = 2$ for which $\nu(g) = 0$. It may be that this is only a small genus phenomenon, however. In any case, general questions remain. Is $\nu(g) = 0$ for any even integer $g > 12$? Is $\nu(g) = 0$ for infinitely many values of g?

These questions are especially interesting because the corresponding question for the strong symmetric genus, a related parameter, has recently been settled. If n is a non-negative integer, then there is at least one group of strong symmetric genus n [**19**, Theorem 1].

There are similar questions about groups of odd order. For each integer $g \geq 2$, define $\tau(g)$ to be the number of groups of *odd* order that have real genus g. Then $\tau(g) = 0$ for all odd $g \geq 3$, by Theorem 1. Further, using the bound $3(g-1)$ together with Table 1 and known results from [**16, 17, 21**], it is easy to determine $\tau(g)$ for all $g \leq 34$ (and classify all groups of odd order in the range). We find that $\tau(g) = 0$ for $g = 2, 6, 12, 18, 24$ and 30; otherwise $\tau(g)$ is positive for g in this range. Here also see [**7, 22**]. It might be interesting to consider $\tau(g)$, for $g = p + 1$, where p is an odd prime such that 3 divides $p + 1$.

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