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ON ABSOLUTE SUMMABILITY FACTORS

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ABSTRACT. The purpose of this paper is to determine the conditions for which $\sum a_n \lambda_\nu$ is summable $|T|_s$ whenever $\sum a_n$ is summable $|\overline{N}, p_n|_k$ where T is a lower triangular matrix with positive entries and row sums one. As special cases we obtain inclusion theorems for pairs of weighted mean matrices.

In [**5**], Sarigöl obtained necessary and sufficient conditions for $|N, p_n|_k \Rightarrow |N, q_n|_s$ for the case $1 \le k \le s$.

The concept of absolute summability of order k was defined by Flett [**3**] as follows. Let $\sum a_n$ be a given infinite series with partial sums s_n, and let σ_n^α denote the nth Cesaro means of order α, $\alpha > -1$, of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \ge 1$, $\alpha > -1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta \sigma_{n-1}^\alpha|^k < \infty, \tag{1}$$

where, for any sequence $\{b_n\}$, $\Delta b_n = b_n - b_{n+1}$.

In defining absolute summability of order k for weighted mean methods, Bor [**1**] and others used the definition

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta u_{n-1}|^k < \infty, \tag{2}$$

where

$$u_n := \sum_{\nu=0}^{n} p_\nu s_\nu.$$

In using (2) as the definition, it was apparently assumed that the n in (1) represented the reciprocal of the nth main diagonal term of $(C, 1)$.

But this interpretation cannot be correct. For, if it were, then the Cesaro methods (C,α) for $\alpha \neq 1$ would have to satisfy the condition

$$\sum_{n=1}^{\infty}(n^{\alpha})^{k-1}|\Delta_{n-1}^{\alpha}|^{k} < \infty.$$

However, Fleet [**3**] stays with n for all values of $\alpha > -1$.

Let T denote a lower triangular matrix with nonzero entries and row sums 1. Define

$$\bar{t}_{n\nu} = \sum_{i=\nu}^{n} t_{\nu i}, \quad n,\nu = 0,1,\ldots$$

and

$$\hat{t}_{n\nu} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, \quad n = 1,2,\ldots .$$

It is the purpose of this paper to prove the following generalization of the necessary part of the theorem in [**5**], using definition (1).

Theorem 1. *Let $1 < k \leq s < \infty$. Suppose that $\{p_n\}$ is a positive sequence such that $P_n \to \infty$ as $n \to \infty$ and*

$$\sum_{n=\nu+1}^{\infty} n^{k-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^{k} = O\left(\frac{1}{P_\nu}\right)^{k}. \tag{3}$$

If $\sum a_n\lambda_\nu$ is summable $|T|_s$ whenever $\sum a_n$ is summable $|\overline{N},p_n|_k$, then

(i) $t_{\nu\nu}\lambda_v = O\left(\left(\frac{p_\nu}{P_\nu}\right)\nu^{1/s-1/k}\right)$

(ii) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_v)|^{s} = O\left(\left(\frac{p_\nu}{P_\nu}\right)^{s}\nu^{s-s/k}\right).$

Proof. Let $\{t_n\}$ denote the sequence of $(\overline{N},p_n)$ means of the series $\sum a_n$. Then

$$\begin{aligned} t_n &= \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu, \\ X_n = t_n - t_{n-1} &= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu; \quad P_{-1} = 0 \end{aligned} \tag{4}$$

and

$$T_n = \sum_{\nu=0}^{n} \sum_{i=\nu}^{n} t_{n\nu} \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \bar{t}_{n\nu} \lambda_\nu a_\nu$$

and

$$Y_n = T_n - T_{n-1} = \sum_{\nu=0}^{n} (\bar{t}_{n\nu} - \bar{t}_{n-1,\nu}) \lambda_\nu a_\nu \tag{5}$$

since $\hat{t}_{n0} = 0$.

We are given that

$$\sum_{n=1}^{\infty} n^{s-1} |Y_n|^s < \infty \tag{6}$$

whenever

$$\sum_{n=1}^{\infty} n^{k-1} |X_n|^k < \infty. \tag{7}$$

Now the space of sequences $\{a_n\}$ satisfying (7) is a Banach space if normed by

$$\|X\| = \left(|X_0|^k + \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \right)^{1/k}. \tag{8}$$

We also consider the space of those sequences $\{Y_n\}$ that satisfy (6).

This is also a BK-space with respect to the norm

$$\|Y\| = \left(|Y_0|^s + \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s \right)^{1/s}. \tag{9}$$

Observe that (5) transforms the space of sequences satisfying (7) into the space of sequences satisfying (6). Applying the Banach-Steinhaus theorem, there exists a constant $K > 0$ such that

$$\|Y\| \le K \|X\|. \tag{10}$$

Applying (4) and (5) to $a_\nu = e_\nu - e_{\nu+1}$, where e_ν is the νth coordinate vector, we have

$$X_n = \begin{cases} 0, & \text{if } n < \nu, \\ \dfrac{p_\nu}{P_\nu}, & \text{if } n = \nu, \\ \dfrac{-p_\nu p_n}{P_n P_{n-1}}, & \text{if } n > \nu; \end{cases}$$

and

$$Y_n = \begin{cases} 0, & \text{if } n < \nu, \\ \hat{t}_{n\nu}\lambda_\nu, & \text{if } n = \nu, \\ \Delta_\nu(\hat{t}_{nv}\lambda_\nu), & \text{if } n > \nu. \end{cases}$$

By (8) and (9) it follows that

$$\|X\| = \left\{ \nu^{k-1}\Big(\frac{p_\nu}{P_\nu}\Big)^k + \sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{p_\nu p_n}{P_n P_{n-1}}\Big)^k \right\}^{1/k}$$

and

$$\|Y\| = \left\{ \nu^{s-1}|t_{\nu\nu}\lambda_\nu|^s + \sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)|^2 \right\}^{1/s},$$

recalling that $\hat{t}_{\nu\nu} = \bar{t}_{\nu\nu} = t_{\nu\nu}$.

Using (10) and (3),

$$\begin{aligned} \nu^{s-1}|t_{\nu\nu}\lambda_\nu|^s &+ \sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)|^s \\ &\le K^s\bigg(\nu^{k-1}\Big(\frac{p_\nu}{P_\nu}\Big)^k + \sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{p_\nu p_n}{P_n P_{n-1}}\Big)^k\bigg)^{s/k} \\ &\le K^s\bigg(\nu^{k-1}\Big(\frac{p_\nu}{P_\nu}\Big)^k + \Big(\frac{p_\nu}{P_\nu}\Big)^k\bigg)^{s/k} \\ &= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^k \nu^{k-1}\bigg)^{s/k}. \end{aligned}$$

The above inequality will be true if and only if each term on the left-hand side is $O((p_\nu/P_\nu)^k \nu^{k-1})^{s/k}$. Taking the first term

$$\begin{aligned}
\nu^{s-1}|t_{\nu\nu}\lambda_\nu|^s &= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^k \nu^{k-1}\bigg)^{s/k} \\
|t_{\nu\nu}\lambda_\nu|^s &= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^s \nu^{1-s/k}\bigg) \\
|t_{\nu\nu}\lambda_\nu| &= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^s \nu^{1-s/k}\bigg)^{1/s} \\
&= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big) \nu^{1/s-1/k}\bigg),
\end{aligned}$$

which verifies that (i) is necessary.

Using the second term we have

$$\begin{aligned}
\sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)|^s &= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^k \nu^{k-1}\bigg)^{s/k} \\
&= O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^s \nu^{s-s/k}\bigg),
\end{aligned}$$

which is condition (ii).

Applications.

Corollary 1. *Suppose that $\{p_n\}, \{q_n\}$ are positive sequences with $\{p_n\}$ satisfying $P_n \to \infty$ and condition (3). If $\sum a_n\lambda_n$ is summable $|\overline{N}, q_n|_s$, whenever $\sum a_n$ is summable $|\overline{N}, p_n|_k$, then*

(i) $\lambda_\nu = O\Big(\dfrac{p_\nu Q_\nu}{q_\nu P_\nu}\Big)(\nu^{1/s-1/k})$.

(ii) $|\Delta_\nu(Q_{\nu-1}\lambda_v)|^s\bigg(\displaystyle\sum_{n=\nu+1}^{\infty} n^{s-1}\Big(\frac{q_n}{Q_nQ_{n-1}}\Big)^s\bigg) = O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^s \nu^{s-s/k}\bigg)$.

Proof. Apply the theorem with $T = (t_{n\nu})$ a weighted mean matrix $(\overline{N}, q_n)$. It is easy to see that

$$\hat{t}_{n\nu} = -\frac{q_nQ_{\nu-1}}{Q_nQ_{n-1}}$$

and
$$\Delta_\nu(\hat{t}_{n\nu}\lambda_v) = \hat{t}_{n\nu} - \hat{t}_{n,\nu+1} = -\frac{q_n}{Q_n Q_{n-1}}\Delta(Q_{\nu-1}\lambda_\nu).$$

Corollary 2. *Let $\{p_n\}$ be a positive sequence satisfying $P_n \to \infty$ and* (3)*. If $\sum a_n\lambda_n$ is summable, $|T|_k$ whenever $\sum a_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$, then*

(i) $t_{\nu\nu}\lambda_\nu = O\Big(\dfrac{p_\nu}{P_\nu}\Big)$

(ii) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{t}_{n\nu}\lambda_\nu)|^k = O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^k \nu^{k-1}\bigg).$

To prove Corollary 2, simply set $s = k$ in Theorem 1.

Corollary 3. *Suppose that $\{p_n\}, \{q_n\}$ are positive sequences with $\{p_n\}$ satisfying $P_n \to \infty$ and condition* (3)*. If $\sum a_n\lambda_n$ is summable $|\overline{N}, q_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$, then*

(i) $\lambda_\nu = O\Big(\dfrac{p_\nu Q_\nu}{q_\nu P_\nu}\Big)$

(ii) $\displaystyle|\Delta_\nu(Q_{\nu-1}\lambda_v)|^k \sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k = O\bigg(\Big(\frac{p_\nu}{P_\nu}\Big)^k \nu^{k-1}\bigg).$

To prove Corollary 3, simply set $s = k$ in Corollary 1.

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