ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 2003

ON CONTINUOUS SOLUTIONS OF A FUNCTIONAL EQUATION OF ITERATIVE TYPE

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ABSTRACT. Properties of continuous solutions of the functional equation $\sum_{i=1}^{n} \lambda_i f^{2i-1}(x) = F(x)$ are discussed. Under some conditions we prove the existence, uniqueness and stability of the continuous solutions of the equation.

1. Introduction. The iterative equation

$$f^n(x) = F(x), \tag{1.1}$$

is an important form of functional equations, where $f : I = [a, b] \to I$ is an unknown function, f^n denotes the n-th iterate of f. Abel [**1**], Bödewadt [**2**], Dubbey [**4**], Fort [**6**], Kuczma [**7, 8**] and others established the existence of solutions for equation (1.1). It is well known that equation (1.1) has a continuous solution for any n if F is a strictly increasing continuous function and equation (1.1) has no continuous solutions for even n if F is a strictly decreasing continuous function. Recently, a few elegant results for equation

$$\sum_{i=1}^{n} \lambda_i f^i(x) = F(x) \tag{1.2}$$

have been obtained in [**3**] and [**9–12**]. In particular, Zhang [**10,11**] discussed the existence, uniqueness and stability of continuous solutions of equation (1.2), where F is a strictly increasing continuous function in $[a, b]$ and has fixed points a, b.

1991 AMS *Mathematics Subject Classification.* 39B12, 39B20.
Key words and phrases. Functional equation, Ascoli-Arzela lemma, continuous solution, existence, uniqueness, stability, Schauder's fixed point theorem, Edelstein's fixed point theorem.
The second author was supported by Korea Research Foundation grant (KRF-2001-015-DP0025).
Received by the editors on October 31, 2000, and in revised form on August 23, 2001.

The purpose of this paper is to study the properties of continuous solutions of the functional equation

$$\sum_{i=1}^{n} \lambda_i f^{2i-1}(x) = F(x), \tag{1.3}$$

where F is a strictly decreasing continuous function in $[a, b]$, $n \geq 2$ and

$$\lambda_1 \in (0,1), \lambda_2, \lambda_3, \dots, \lambda_n \geq 0, \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1. \tag{1.4}$$

Under some conditions we prove the existence, uniqueness and stability of continuous solutions of equation (1.3).

Throughout this paper, let $R = (-\infty, \infty)$, $I = [a, b] \subseteq R$ and N denote the set of all positive integers. i_I stands for the identity mapping on I. For $X, Y \subseteq R$, $C^0(X, Y)$ denotes the set of all continuous functions from X into Y. Obviously, $(C^0(I, R), \|\cdot\|_0)$ is a Banach space, where $\|f\|_0 = \max\{|f(x)| : x \in I\}$ for any $f \in C^0(I, R)$. Given $r, s > 0$, let

$$A(r, s) = \{F : F \in C^0(I, I), F(a) = b, F(b) = a \text{ and (1.5) is satisfied}\},$$
$$B(I, s) = \{t : t \in C^0(I, I), t(a) = b, t(b) = a \text{ and (l.6) is satisfied}\},$$

$$r(y - x) \leq F(x) - F(y) \leq \lambda_1 s(y - x) \tag{1.5}$$

for all $x, y \in I$ with $x < y$, and

$$0 \leq t(x) - t(y) \leq s(y - x) \tag{1.6}$$

for all $x, y \in I$ with $x < y$.

2. The existence and uniqueness of continuous solutions. Our main results are as follows:

Theorem 2.1. *Suppose that F is in $A(r, s)$ and* (1.4) *holds. Then equation* (1.3) *has a solution in $B(I, s)$.*

Proof. It is clear that $\phi \neq B(I,s) \subseteq C^0(I,I) \subseteq C^0(I,R)$. Set

$$h_g(x) = \sum_{i=1}^{n} \lambda_i g^{2i-2}(x) \tag{2.1}$$

for any $g \in B(I,s)$ and $x \in I$, where $g^0 = i_I$. Note that $g^{2i-2}(a) = a, g^{2i-2}(b) = b$, and

$$\begin{aligned} 0 &\le g^{2i}(y) - g^{2i}(x) \le s\big(g^{2i-1}(x) - g^{2i-1}(y)\big) \\ &\le s^2\big(g^{2i-2}(y) - g^{2i-2}(x)\big) \le \cdots \le s^{2i}(y-x) \end{aligned} \tag{2.2}$$

for any $g \in B(I,s)$, $x, y \in I$ with $x < y$, and $i \in N$. Using (2.1) and (2.2), we conclude easily that $h_g(a) = a, h_g(b) = b, h_g \in C^0(I,I)$ and for all $x, y \in I$ with $x < y$,

$$\begin{aligned} 0 < \lambda_1(y-x) &\le h_g(y) - h_g(x) \\ &= \sum_{i=1}^{n} \lambda_i\big(g^{2i-2}(y) - g^{2i-2}(x)\big) \\ &\le \sum_{i=1}^{n} \lambda_i s^{2i-2}(y-x) \le m(y-x), \end{aligned} \tag{2.3}$$

where $m = \max\{s^{2i-2} : 1 \le i \le n\}$. Thus (2.3) ensures that

$$\frac{1}{m}\,(y-x) \le h_g^{-1}(y) - h_g^{-1}(x) \le \frac{1}{\lambda_1}\,(y-x) \tag{2.4}$$

for all $x, y \in I$ with $x < y$. Define a mapping $D : B(I,s) \to B(I,s)$ by

$$D\big(g(x)\big) = h_g^{-1}F(x) \tag{2.5}$$

for any $g \in B(I,s)$ and $x \in I$, where $F \in A(r,s)$. Then $D(g(a)) = h_g^{-1}F(a) = h_g^{-1}b = b, D(g(b)) = a$, and from (2.4) and (2.5) we have

$$\begin{aligned} 0 &< \frac{1}{m}\,r(y-x) \le \frac{1}{m}\big(F(x) - F(x)\big) \\ &\le D\big(g(x)\big) - D\big(g(y)\big) = h_g^{-1}F(x) - h_g^{-1}F(y) \\ &\le \frac{1}{\lambda_1}\,\big(F(x) - F(y)\big) \le s(y-x) \end{aligned} \tag{2.6}$$

for all $x, y \in I$ with $x < y$. That is, $D(B(I,s)) \subseteq B(I,s)$. We claim that D is continuous in $B(I,s)$. For any $g, t \in B(I,s)$, let $u = D(g), v = D(t)$ and $k \in N$. By virtue of (1.6), we have

$$
\begin{aligned}
\|u^k - v^k\|_0 &\le \max\big\{|uu^{k-1}(x) - uv^{k-1}(x)| : x \in I\big\} \\
&\quad + \max\big\{|uv^{k-1}(x) - vv^{k-1}(x)| : x \in I\big\} \\
&\le s\|u^{k-1} - v^{k-1}\|_0 + \|u - v\|_0 \\
&\le s\big(s\|u^{k-2} - v^{k-2}\|_0 + \|u - v\|_0\big) + \|u - v\|_0 \\
&\le s^2\|u^{k-2} - v^{k-2}\|_0 + s\|u - v\|_0 + \|u - v\|_0 \\
&\le \cdots \\
&\le \sum_{i=1}^{k} s^{i-1}\|u - v\|_0.
\end{aligned} \tag{2.7}
$$

In view of(1.5), we have

$$
\frac{1}{\lambda_1 s}(y - x) \le F^{-1}x - F^{-1}y \le \frac{1}{r}(y - x) \tag{2.8}
$$

for all $x, y \in I$ with $x < y$. From (2.5), (2.6), (2.1), (2.7) and (2.8), we conclude that

(2.9)

$$
\begin{aligned}
\|D(g) - D(t)\|_0 &= \|u - v\|_0 = \|u - uu^{-1}v\|_0 \\
&\le s\|i_I - u^{-1}v\|_0 = s \max\big\{|i_I(x) - u^{-1}v(x)| : x \in I\big\} \\
&= s \max\big\{|v^{-1}(y) - u^{-1}(y)| : y = v(x) \text{ and } x \in I\big\} \\
&= s\|u^{-1} - v^{-1}\|_0 = s\|F^{-1}h_g - F^{-1}h_t\|_0 \\
&\le \frac{s}{r}\,\|h_g - h_t\|_0 \le \frac{s}{r}\,\sum_{i=2}^{n} \lambda_i \|g^{2i-2} - t^{2i-2}\|_0 \\
&\le \frac{s}{r}\,\sum_{i=2}^{n} \left(\lambda_i \sum_{k=1}^{2i-2} s^{k-1}\right)\|g - t\|_0.
\end{aligned}
$$

That is, D is continuous in $B(I,s)$. We now assert that $B(I,s)$ is a compact convex subset of $C^0(I,R)$. Given $p, q \in B(I,s)$ put $w = cp + (1-c)q$ for $c \in [0,1]$. Then $w(a) = b, w(b) = a$, $w \in C^0(I,I)$ and

$$
\begin{aligned}
0 \le w(x) - w(y) &= c\big(p(x) - p(y)\big) + (1-c)\big(q(x) - q(y)\big) \\
&\le cs(y - x) + (1-c)s(y - x) = s(y - x)
\end{aligned} \tag{2.10}
$$

for any $x, y \in I$ with $x < y$. Hence $w \in B(I, s)$. That is, $B(I, s)$ is convex. It is easy to verify that $B(I, s)$ is closed from the definition of $B(I, s)$. Let $\{t_n\}_{n \in N}$ be a sequence in $B(I, s)$. Since $\|t_n\|_0 \leq \max\{|a|, |b|\}$ for all $n \in N$, so $\{t_n\}_{n \in N}$ is uniformly bounded on I. Note that for any $\varepsilon > 0$, there exists $\delta = \varepsilon/s > 0$ such that

$$|t_n(x) - t_n(y)| \leq s|x - y| < \varepsilon$$

for all $n \in N$ and $x, y \in I$ with $|x - y| < \delta$. Therefore, $\{t_n\}_{n \in N}$ is equicontinuous in I. It follows from the Ascoli-Arzela lemma that there is a uniformly convergent subsequence of $\{t_n\}_{n \in N}$. Thus $B(I, s)$ is sequentially compact. Consequently, $B(I, s)$ is a compact subset of the Banach space $C^0(I, R)$. It follows from Schauder's fixed point theorem that there exists $f \in B(I, s)$ satisfying

$$f(x) = D\big(f(x)\big) = h_f^{-1} F(x)$$

for all $x \in I$. That is,

$$\sum_{i=1}^{n} \lambda_i \, f^{2i-1}(x) = h_f f(x) = F(x)$$

for all $x \in I$. This completes the proof. $\square$

We now give sufficient conditions for the uniqueness of the continuous solution of equation (1.3).

Theorem 2.2. *Suppose that F is in $A(r, s)$ and* (1.4) *holds. Assume that the following condition*

$$(1 - \lambda_1) \sum_{k=1}^{2n-2} s^k - \sum_{i=2}^{n-1} \left(\sum_{k=2}^{i} \lambda_k \right) \big(s^{2i-1} + s^{2i}\big) < r \tag{2.11}$$

is satisfied, where $n \in N - \{1, 2\}$. Then equation (1.3) *has a unique solution in $B(I, s)$.*

Proof. Let D be as in the proof of Theorem 2.1 and $n \in N - \{1, 2\}$. It follows from (2.9) and (2.11) that

$$\begin{aligned}
&\|D(g) - D(t)\|_0 \\
&\le \frac{s}{r} \sum_{i=2}^{n} \left(\lambda_i \sum_{k=1}^{2i-2} s^{k-1} \right) \|g - t\|_0 \\
&= \frac{s}{r} \left[\left(\sum_{i=2}^{n} \lambda_i \right) \sum_{k=1}^{2n-2} s^{k-1} + \sum_{i=2}^{n-1} \left(\sum_{k=2}^{i} \lambda_k \right) \left(\sum_{k=1}^{2i-2} s^{k-1} - \sum_{k=1}^{2i} s^{k-1} \right) \right] \|g - t\|_0 \\
&= \frac{1}{r} \left[(1 - \lambda_1) \sum_{k=1}^{2n-2} s^k - \sum_{i=2}^{n-1} \left(\sum_{k=2}^{i} \lambda_k \right) (s^{2i-1} + s^{2i}) \right] \|g - t\|_0 \\
&< \|g - t\|_0,
\end{aligned}$$

for all distinct $g, t \in B(I, s)$. That is, D is contractive from a compact metric space $B(I, s)$ into itself. Theorem 1 of Edelstein [**5**] ensures that D has a unique fixed point in $B(I, s)$. Therefore, equation (1.3) has a unique solution in $B(I, s)$. This completes the proof. □

Theorem 2.3. *Assume that F is in $A(r, s)$ and $(l.4)$ holds. If $n = 2$ and*

$$(1 - \lambda_1) s (1 + s) < r, \tag{2.12}$$

then equation (1.3) *has a unique solution in $B(I, s)$.*

Proof. Let D be as in the proof of Theorem 2.1 and $n = 2$. It follows from (2.9) and (2.12) that

$$\|D(g) - D(t)\|_0 \le \frac{s}{r} \lambda_2 (1 + s) \|g - t\|_0 < \|g - t\|_0$$

for all distinct $g, t \in B(I, s)$. The rest of the proof is identical with the proof of Theorem 3.2. This completes the proof. □

From Theorem 2.2 , we have

Corollary 2.1. *Assume that F is in $A(r,s)$ and (1.4) holds. If the following condition*

$$(1-\lambda_1)\sum_{k=1}^{2n-2} s^k < r, \quad n \in N - \{1,2\} \tag{2.13}$$

is satisfied, then equation (1.3) has a unique solution in $B(I,s)$.

Next we consider the existence and uniqueness of the solution of the functional equation

$$-c_0(b+a-x) + \sum_{k=1}^{n} c_k\, f^{2k-1}(x) = 0, \quad x \in I, \tag{2.14}$$

where $n \in N - \{1\}$ and the c_k's are given real numbers satisfying the condition

(2.15)

$$\sum_{k=1}^{n} c_k = c_0,\ 1 < \frac{c_0}{c_1} < 2^{1/(2n-2)},\ \frac{c_k}{c_0} \geq 0 \quad \text{for all } k \in \{2,3,\dots,n\}.$$

Theorem 2.4. *The solution of equation (2.14) which is continuous and decreasing in I, and maps a,b into b,a, respectively, is only $b+a-x$.*

Proof. Obviously $b+a-x$ belongs to $B(I,s)$ for all $s \geq 1$ and it is also a solution of equation (2.14). Set $\lambda_k = c_k/c_0$ for each $k \in \{2,3,\dots,n\}$, $F(x) = b+a-x$ for all $x \in I$ and $r = 1 = \lambda_1 s$. Then $F(x) \in B(I,s)$, (1.4) holds and equation (2.14) is changed into the form

$$\sum_{k=1}^{n} \lambda_k\, f^{2k-1}(x) = F(x), \quad x \in I. \tag{2.16}$$

Suppose that there exists a decreasing and continuous solution $f(x)$ of equation (2.16) with $f(a) = b$ and $f(b) = a$. For any $x, y \in I$ with $x < y$, by (2.16) we have

$$\begin{aligned} y - x &= (b+a-x) - (b+a-y) \\ &= \sum_{k=1}^{n} \lambda_k \big(f^{2k-1}(x) - f^{2k-1}(y)\big) \\ &\geq \lambda_1 \big(f(x) - f(y)\big) \geq 0, \end{aligned}$$

which implies that

$$0 \le f(x) - f(y) \le \frac{1}{\lambda_1}(y-x) = s(y-x).$$

That is, $f \in B(I, s)$. We now consider two cases.

Case 1. Suppose that $n \in N - \{1, 2\}$. It follows from (2.15) that

$$\begin{aligned}(1-\lambda_1)\sum_{k=1}^{2n-2} s^k &= \left(1 - \frac{1}{s}\right)\sum_{k=1}^{2n-2} s^k = s^{2n-2} - 1 \\ &= \left(\frac{c_0}{c_1}\right)^{2n-2} - 1 < 1 = r.\end{aligned}$$

Case 2. Suppose that $n = 2$. Then (2.15) implies that

$$(1-\lambda_1)s(1+s) = \left(1 - \frac{1}{s}\right)s(1+s) = s^2 - 1 = r.$$

Thus Corollary 2.1 and Theorem 2.2 ensure that equation (2.16) for each $n \in N - \{1\}$ has a unique solution in $B(I, s)$. Therefore, the continuous and decreasing solution of equation (2.14) which maps a, b into b, a, respectively, is only $f(x) = b + a - x$. This completes the proof. □

3. The stability of continuous solution.

Lemma 3.1. *Let f and g be bijections from I into itself. Suppose that there is a positive constant c satisfying the following condition*

$$|f(x) - f(y)| \le c|x - y| \tag{3.1}$$

for all $x, y \in I$. Then

$$\|f - g\|_0 \le c\|f^{-1} - g^{-1}\|_0.$$

Proof. Note that $ff^{-1} = i_I$ and $g^{-1}(I) = I$. It follows from (3.1) that

$$\begin{aligned}\|f-g\|_0 &= \|f - ff^{-1}g\|_0 \le c\|i_I - f^{-1}g\|_0 \\ &= c\max\big\{|i_I(x) - f^{-1}g(x)| : x \in I\big\} \\ &= c\max\big\{|g^{-1}(y) - f^{-1}(y)| : y \in I\big\} \\ &= c\|f^{-1} - g^{-1}\|_0.\end{aligned}$$

This completes the proof. □

Now we establish the conditions to guarantee the stability of the continuous solution of equation (1.3)

Theorem 3.1. *Assume that* (1.4) *and* (2.11) *hold. Then the solution of equation* (1.3) *in* $B(I,s)$ *is continuously dependent on the given function* $F \in A(r,s)$.

Proof. Let F and G be arbitrary elements in $A(r,s)$. Theorem 2.2 ensures that there exist f and g in $B(I,s)$ such that

$$f(x) = h_f^{-1}F(x) \quad \text{and} \quad g(x) = h_g^{-1}G(x) \tag{3.3}$$

for each $x \in I$. In view of (2.8), we conclude that

$$\max\big\{\big|F^{-1}(x) - F^{-1}(y)\big|, \big|G^{-1}(x) - G^{-1}(y)\big|\big\} \le \frac{1}{r}\,|x-y| \tag{3.4}$$

for all $x, y \in I$. Lemma 3.1 and (3.4) yield that

$$\|F^{-1} - G^{-1}\|_0 \le \frac{1}{r}\,\|F-G\|_0. \tag{3.5}$$

Using Lemma 3.1 and (2.6), we have

$$\|f-g\|_0 \le s\|f^{-1} - g^{-1}\|_0. \tag{3.6}$$

It follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned}\|f-g\|_0 &\le s\|f^{-1} - g^{-1}\|_0 = s\|F^{-1}h_f - G^{-1}h_g\|_0 \\ &\le s\|F^{-1}h_f - F^{-1}h_g\|_0 + s\|F^{-1}h_g - G^{-1}h_g\|_0 \\ &\le \frac{s}{r}\,\|h_f - h_g\|_0 + \frac{s}{r}\|F-G\|_0.\end{aligned} \tag{3.7}$$

Thus (2.5), (2.7) and (3.3) ensure that

$$
\begin{aligned}
(3.8)\quad & \|h_f - h_g\|_0 \\
&\le \sum_{i=2}^{n} \lambda_i \|f^{2i-2} - g^{2i-2}\|_0 \\
&\le \sum_{i=2}^{n} \left(\lambda_i \sum_{k=1}^{2i-2} s^{k-1}\right) \|f-g\|_0 \\
&= \left[(1-\lambda_1) \sum_{k=1}^{2n-2} s^{k-1} - \sum_{i=2}^{n-1} \left(\sum_{k=2}^{i} \lambda_k \right) \left(s^{2i-2} - s^{2i-1}\right) \right] \|f-g\|_0.
\end{aligned}
$$

It follows from (3.7) and (3.8) that

$$
\begin{aligned}
& \|f-g\|_0 \\
&\le \frac{s}{r} \left[(1-\lambda_1) \sum_{k=1}^{2n-2} s^{k-1} - \sum_{i=2}^{n-1} \left(\sum_{k=2}^{i} \lambda_k \right) \left(s^{2i-2} + s^{2i-1}\right) \right] \|f-g\|_0 \\
&\quad + \frac{s}{r} \|F-G\|_0,
\end{aligned}
$$

which implies that

$$\|f-g\|_0 \le \frac{s}{rm} \|F-G\|_0,$$

where

$$m = 1 - \frac{1}{r} \left[(1-\lambda_i) \sum_{k=1}^{2n-2} s^k - \sum_{i=2}^{n-1} \left(\sum_{k=2}^{i} \lambda_k \right) \left(s^{2i-1} + s^{2i}\right) \right] > 0.$$

That is,the solution of equation (1.3) in $B(I,s)$ is continuously dependent on the given function in $A(r,s)$. This completes the proof. □

Theorem 3.2. *Assume that* (1.4) *and* (2.12) *hold for* $n = 2$. *Then the solution of equation* (1.3) *in* $B(I,s)$ *is continuously dependent on the given function* $F \in A(r,s)$.

Proof. Let F and G be arbitrary elements in $A(r,s)$. Theorem 2.3 ensures that there exist f and g in $B(I,s)$ satisfying (3.3) As in the proof of Theorem 3.1, we infer that

$$\begin{aligned}\|f-g\|_0 &\le \frac{s}{r}\,\|h_f-h_g\|_0+\frac{s}{r}\,\|F-G\|_0\\ &\le \frac{s}{r}\,\lambda_2(1+s)\|f-g\|_0+\frac{s}{r}\,\|F-G\|_0,\end{aligned}$$

which implies that

$$\|f-g\|_0 \le \frac{s}{rq}\,\|F-G\|_0,$$

where

$$q=1-\frac{s}{r}(1+s)\lambda_2>0.$$

That is, the solution of equation (1.3) in $B(I,s)$ is continuously dependent on the given function in $A(r,s)$. This completes the proof. $\square$

4. Examples. In this section we give two examples in support of our results.

Example 4.1. Let $r\in(0,1)$ be fixed and $I=[0,1]$. Define $F:I\to I$ by

$$F(x)=1+(2r-3)x+3(1-r)x^2+(r-1)x^3 \quad \text{for all } x\in I.$$

Then $F(0)=1$, $F(1)=0$, and for $x,y\in I$

$$\begin{aligned}F(x)-F(y) &= (2r-3)(x-y)+3(1-r)(x^2-y^2)+(r-1)(x^3-y^3)\\ &= (x-y)\left[2r-3+3(1-r)(x+y)+(c-1)(x^2+xy+y^2)\right]\\ &= (y-x)\left\{r+(1-r)[3-3(x+y)+x^2+xy+y^2]\right\}\\ &= (y-x)\left\{r+(1-r)[(1-x)^2+(1-x)(1-y)+(1-y)^2]\right\},\end{aligned}$$

which implies that

$$r(y-x)\le F(x)-F(y)\le(3-2r)(y-x)$$

for all $x,y\in I$ with $x<y$. Therefore $F\in A\left(r,(3-2r)/(\lambda_1)\right)$. Theorem 2.1 ensures that equation (1.3) has a solution in $B\left(I,(3-2r)/(\lambda_1)\right)$. Now suppose that $n=2$. Note that

$$(1-\lambda_1)\,\frac{3-2r}{\lambda_1}\left(1+\frac{3-2r}{\lambda_1}\right)<r$$

is equivalent to

$$(3-r)\lambda_1^2 - 2(r-1)(3-2r)\lambda_1 - (3-2r)^2 > 0.$$

Thus for

$$\lambda_1 \in \left(\frac{(3-2r)(r-1+\sqrt{4-3r+r^2}\,)}{3-r}, 1 \right),$$

by Theorem 2.3 we conclude that equation (1.3) with $n = 2$ has exactly one solution in $B\,(I, (3-2r)/(\lambda_1))$.

Example 4.2. Let $I = [0, c]$, $F(x) = t(e^{c-x} - 1)$ for all $x \in I$, and $c = t(e^c - 1) > 0$, where $t \in (0, 1)$ is fixed. Then

$$F(c) = 0, \quad F(0) = t(e^c - 1) = c.$$

Given $x, y \in I$ with $x < y$, by the mean value theorem there exists $\xi \in (x, y)$ such that

$$t(y-x) \leq F(x) - F(y) = -te^{c-\xi}(x-y) \leq te^c(y-x).$$

Since

$$\lim_{\lambda \to 1^-} \lambda = 1 > 1 - \frac{1}{e^c(1+te^c)} = \lim_{\lambda \to 1^-} \left[1 - \frac{t}{((te^c)/\lambda)\,[1 + ((te^c)/\lambda)]} \right],$$

so there exists $p \in (0, 1)$ satisfying for all $\lambda \in [p, 1)$,

$$\lambda > 1 - \frac{t}{((te^c)/\lambda)\,[1 + ((te^c)/\lambda)]}.$$

Thus F belongs to $A\,[t, ((te^c)/\lambda_1)]$ for any $\lambda_1 \in [p, 1)$. It follows from Theorem 3.2 that the following equation

$$\lambda_1 f(x) + (1-\lambda_1) f^3(x) = F(x), \quad \lambda_1 \in [p, 1)$$

has a unique solution in $B\,[I, ((te^c)/\lambda_1)]$ and it is continuously dependent on

$$F(x) = t(e^{c-x} - 1).$$

Acknowledgment. This work was supported by Korea Research Foundation Grant (KRF-2001-015-DP0025).

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