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RESTRICTIONS OF ESSENTIALLY NORMAL OPERATORS

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Let $\mathcal{H}$ be a complex, infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{C}$ denote the ideal of all compact operators in $\mathcal{L}(\mathcal{H})$, and let π denote the natural quotient map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}$. For T in $\mathcal{L}(\mathcal{H})$, let $\tilde{T} = \pi(T)$. Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *essentially normal* if $\tilde{T}$ is normal, or, equivalently, if the self-commutator $T^*T - TT^*$ is compact. Let T be an operator in $\mathcal{L}(\mathcal{H})$ that is unitarily equivalent to the bilateral shift of infinite multiplicity. There exists an invariant subspace $\mathcal{M}$ for T such that $T|\mathcal{M}$ is unitarily equivalent to the unilateral shift of infinite multiplicity. Note that T is essentially normal (it's normal), but $T|\mathcal{M}$ is not essentially normal. Thus the restriction of an essentially normal operator to an invariant subspace is not necessarily essentially normal.

Recall that an operator S in $\mathcal{L}(\mathcal{H})$ is said to be subnormal if it has a normal extension. Bunce and Deddens proved in [**2**] that an operator S in $\mathcal{L}(\mathcal{H})$ is *subnormal* if and only if, for each $B_0, B_1, \ldots, B_n$ in $C^*(S)$, the C^*-algebra generated by S and $1_{\mathcal{H}}$, (or equivalently in $\mathcal{L}(\mathcal{H})$),

$$(1) \qquad \sum_{k=0}^{n} \sum_{j=0}^{n} B_j^* S^{*k} S^j B_k \geq 0_{\mathcal{H}}.$$

(See also [**4**]). This characterization of a subnormal operator is completely algebraic, and Bunce has used it to define a subnormal element of an abstract C^*-algebra [**1**]. Accordingly, we shall say that an element S of the Calkin algebra is *subnormal* if, for each $B_0, B_1, \ldots, B_n$ in $\mathcal{L}(\mathcal{H})/\mathcal{C}$, (1) holds. An operator S in $\mathcal{L}(\mathcal{H})$ is said to be *essentially subnormal* if $\tilde{S}$ is a subnormal element of the Calkin algebra. Observe that each essentially normal operator is essentially subnormal. We noticed above that the restriction of an essentially normal operator to an invariant subspace is not necessarily essentially normal. However, it follows readily that the restriction of an essentially normal operator to an

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invariant subspace is essentially subnormal. These observations suggest the following question:

Question A. Does each essentially subnormal operator in $\mathcal{L}(\mathcal{H})$ have an essentially normal extension?

Let S be a subnormal operator in $\mathcal{L}(\mathcal{H})$. Halmos showed in [**5**] that the minimal normal extension of S is unitarily equivalent to the operator

$$\begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix} \tag{2}$$

in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. The operator T is called the dual of S. (See [**3**] for a discussion of the dual of a subnormal operator.) Since the matrix in (2) is normal, a calculation shows that $S^*S - SS^* = XX^*$, $T^*T - TT^* = X^*X$, and $XT = S^*X$. We shall say that an element $\tilde{T}$ of the Calkin algebra $\mathcal{A}$ is an *algebraic dual* of a subnormal element $\tilde{S}$ of $\mathcal{A}$ if there exists $\tilde{X}$ in $\mathcal{A}$ such that

$$\tilde{S}^*\tilde{S} - \tilde{S}\tilde{S}^* = \tilde{X}\tilde{X}^*, \quad \tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^* = \tilde{X}^*\tilde{X}, \quad \text{and } \tilde{X}\tilde{T} = \tilde{S}^*\tilde{X}. \tag{3}$$

In [**6**], it was shown that two algebraic duals of a subnormal element of a C^*-algebra need not be unitarily equivalent. Also in [**6**], an abstract C^*-algebra is defined to be *dual closed* if it contains an algebraic dual of each of its subnormal elements. The C^*-algebra $\mathcal{L}(\mathcal{H})$ is obviously dual closed. The following question appears in [**6**].

Question B. Is the Calkin algebra dual closed?

The purpose of this note is to show that Questions A and B are equivalent.

We shall begin with the following theorem.

Theorem 1. *Suppose that $S \in \mathcal{L}(\mathcal{H})$ and S has an essentially normal extension. Then S has an essentially normal extension in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.*

Theorem 1 follows easily from the following two lemmas.

LEMMA 2. *Suppose that $S \in \mathcal{L}(\mathcal{H})$ and S has an essentially normal extension in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H}_0)$, where $\mathcal{H}_0$ is a Hilbert space and $\dim(\mathcal{H}_0) \leq \dim(\mathcal{H})$. Then S has an essentially normal extension in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.*

PROOF. Let N be an essentially normal extension of S in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H}_0)$. Then N is unitarily equivalent to

$$\begin{bmatrix} S & A \\ 0 & B \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}_0$. Let $M = N \oplus 0_{\mathcal{H}}$ on $(\mathcal{H} \oplus \mathcal{H}_0) \oplus \mathcal{H}$. Then M is essentially normal and is unitarily equivalent to

$$\begin{bmatrix} S & A & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}_0 \oplus \mathcal{H}$. Since $\dim(\mathcal{H}_0 \oplus \mathcal{H}) = \dim(\mathcal{H})$, then M is unitarily equivalent to an operator of the form

$$\begin{bmatrix} S & A_0 \\ 0 & B_0 \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$. Hence S has an essentially normal extension on $\mathcal{H} \oplus \mathcal{H}$. $\square$

LEMMA 3. *Suppose that $S \in \mathcal{L}(\mathcal{H})$ and S has an essentially normal extension on $\mathcal{H} \oplus \mathcal{H}_0$, where $\mathcal{H}_0$ is a Hilbert space and $\dim(\mathcal{H}) \leq \dim(\mathcal{H}_0)$. Then S has an essentially normal extension on $\mathcal{H} \oplus \mathcal{H}$.*

PROOF. Let N be an essentially normal extension of S in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H}_0)$. Now N is unitarily equivalent to the operator

$$\begin{bmatrix} S & A \\ 0 & B \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}_0$. Let

$$\Lambda = \{B^{*m_k} B^{nk} \dots B^{*m_1} B^{n_1} : m_i, n_i \in \mathbf{Z}^+ \cup \{0\}, i = 1, \dots, k, k \in \mathbf{Z}^+\}.$$

and let $\mathcal{M} = \overline{\text{span}}\{Rx : R \in \Lambda \text{ and } x \in \ker(A)^{\perp}\}$. Since Λ is countable, $\dim(\mathcal{M}) \leq \max\{\chi_0, \dim(\ker(A)^{\perp}, \}$. But, since range $(A) \subseteq \mathcal{H}, \dim(\ker(A)^{\perp}) = \dim(\text{range}\,(A)^{-}) \leq \dim(\mathcal{H})$. Hence $\dim(\mathcal{M}) \leq \dim(\mathcal{H})$. Also, since $\mathcal{M}$ reduces B, the operator N is unitarily equivalent to

$$\begin{bmatrix} S & A_0 & 0 \\ 0 & B_0 & 0 \\ 0 & 0 & B_1 \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{M} \oplus (\mathcal{H}_0 \ominus \mathcal{M})$. It follows that the operator

$$\begin{bmatrix} S & A_0 \\ 0 & B_0 \end{bmatrix}$$

is an essentially normal extension of S on $\mathcal{H} \oplus \mathcal{M}$. Since $\dim(\mathcal{M}) \leq \dim(\mathcal{H})$, Lemma 1 implies that S has an essentially normal extension on $\mathcal{H} \oplus \mathcal{H}$. □

We shall use the following notation and terminology. Let $\mathcal{A}$ be a C^*-algebra. Recall that an element W of $\mathcal{A}$ is called an isometry if $W^*W = 1$. A pair of isometries W_1 and W_2 in $\mathcal{A}$ is said to be *complementary* if $W_1W_1^* + W_2W_2^* = 1$. Let W_1 and W_2 be a pair of complementary isometries in $\mathcal{L}(\mathcal{H})$. Define a linear map $\Gamma : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ by

$$\Gamma(T) = \begin{bmatrix} W_1^*TW_1 & W_1^*TW_2 \\ W_2^*TW_1 & W_2^*TW_2 \end{bmatrix}.$$

Let $U : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be defined by $Ux = W_1^*x \oplus W_2^*x$. Note that $W_2^*W_1 = W_1^*W_2 = 0_{\mathcal{H}}$. Thus $U^*(x \oplus y) = W_1x + W_2y, U^*U = 1_{\mathcal{H}}, UU^* = 1_{\mathcal{H}\oplus\mathcal{H}}$, and $\Gamma(T) = UTU^*$ for each T in $\mathcal{L}(\mathcal{H})$. Hence Γ is implemented by a Hilbert space isomorphism, and thus is a $*$-algebra isomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. Furthermore, $\Gamma(T)$ is unitarily equivalent to T for each T in $\mathcal{L}(\mathcal{H})$.

Let $\mathcal{A} = \mathcal{L}(\mathcal{H})/\mathcal{C}$. There exists a Hilbert space $\mathcal{K}$ and a $*$-algebra isomorphism $\delta : \mathcal{A} \to \mathcal{L}(\mathcal{K})$. Let $\phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ be the $*$-algebra homomorphism defined by $\phi = \delta o \pi$, where $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{A}$ is the Calkin map. For T in $\mathcal{L}(\mathcal{H})$, let $\hat{T} = \phi(T)$.

Now define a map $\Phi : \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \to \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ by

$$\Phi\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}.$$

It is routine to verify that Φ is a $*$-algebra homomorphism.

Note that $\hat{W}_1$ and $\hat{W}_2$ is a pair of complementary isometries in $\mathcal{L}(\mathcal{K})$. Thus we define $\hat{\Gamma} : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ by

$$\hat{\Gamma}(T) = \begin{bmatrix} \hat{W}_1^* T \hat{W}_1 & \hat{W}_1^* T \hat{W}_2^* \\ \hat{W}_2^* T \hat{W}_1 & \hat{W}_2^* T \hat{W}_2 \end{bmatrix}$$

for each T in $\mathcal{L}(\mathcal{K})$. As above, $\hat{\Gamma}$ is a $*$-algebra isomorphism of $\mathcal{L}(\mathcal{K})$ onto $\mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ that is implemented by a Hilbert space isomorphism. Thus $\hat{\Gamma}(T)$ is unitarily equivalent to T for each T in $\mathcal{L}(\mathcal{K})$. □

The proof of the following lemma is self-evident.

Lemma 4. *The following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{H}) & \xrightarrow{\;\Gamma\;} & \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathcal{L}(\mathcal{K}) & \xrightarrow[\hat{\Gamma}]{} & \mathcal{L}(\mathcal{K} \oplus \mathcal{K}). \end{array}$$

Observe that an operator S in $\mathcal{L}(\mathcal{H})$ is essentially subnormal if and only if $\hat{S}$ is subnormal.

The following theorem shows that Questions A and B are equivalent.

Theorem 5. *Suppose that $S \in \mathcal{L}(\mathcal{H})$. Then S has an essentially normal extension if and only if S is essentially subnormal and the Calkin algebra contains an algebraic dual of S.*

Proof. Suppose that N, T, and X belong to $\mathcal{L}(\mathcal{H})$ and

$$\Gamma(N) = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}.$$

Then using Theorem 1 and Lemma 4, we can show that N is essentially normal if and only if

$$\hat{\Gamma}(\hat{N}) = \begin{bmatrix} \hat{S} & \hat{X} \\ 0 & \hat{T}^* \end{bmatrix}$$

is normal in $\mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ if and only if $\tilde{S}, \tilde{T}$, and $\tilde{X}$ satisfy (3). □

Note that a compact perturbation of a subnormal operator is essentially subnormal. Let V be the unilateral shift of multiplicity one, and let $S = V^*$. Then S is essentially subnormal (it's essentially normal). But since the Fredholm index of S equals one and the Fredholm index of a Fredholm subnormal operator is less than or equal to zero, S is not equal to a subnormal operator plus a compact operator. Thus an essentially subnormal operator need not be equal to a subnormal operator plus a compact.

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