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ON ALMOST COMMUTING HERMITIAN OPERATORS

STANISLAW J. SZAREK

ABSTRACT. It is an old problem in operator theory whether a pair of norm one compact Hermitian operators with "small" (in norm) commutator can be "well" approximated by a commuting pair of Hermitian operators. We show that, for operators of rank not exceeding n, such approximants exist provided $||[A,B]||/n^{1/2}$ is small. This improves a result of Pearcy and Shields and sheds some new light on the original question and its relationship to a few related ones.

The following is an old question in the "local" operator theory (cf. [**8**]): If two norm one compact Hermitian operators have small commutators, are they close to a commuting pair? More precisely,

(1) *Given $\varepsilon > 0$, does there exist $\delta > 0$ such that, whenever A, B are norm one, compact Hermitian operators on a Hilbert space with $||[A,B]|| \leq \delta$, then one can find* (*compact Hermitian operators*) *A_1, B_1 satisfying $||A_1 - A|| \leq \varepsilon, ||B_1 - B|| \leq \varepsilon$ and $[A_1, B_1] = 0$?*

We are going to refer to (1) as the Main Problem. An equivalent version follows: *If T is a norm one compact operator with "small" selfcommutator $[T^*, T]$, is T "close" to a normal operator?* This one is clearly related to the work of Brown, Douglas, Filmore [**3**] on essentially normal operators. By approximation, questions of the above type reduce to the case of operators acting on finite dimensional spaces (i.e., to matrices) with dimension-free dependence of δ on ε.

Two positive results in the direction of the Main Problem are certainly worth mentioning. First, it was proved by Pearcy and Shields [**7**] that, if just *one* of the operators is assumed to be Hermitian and they

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act on an n-dimensional space, then (1) holds with non-dimension-free dependence of δ on ε, namely, $\delta = \varepsilon^2/(n-1)$. Secondly, Davidson [**5**] proved the following absorption theorem: If A, B have small commutators, then there exists a commuting pair C, D of Hermitian operators such that $A \oplus C$ and $B \oplus D$ are close to a commuting pair (moreover, C and D can be chosen to act on a space of the same dimension as that of A and B). This was later used to deduce a quantitative version of the theorem of Brown-Douglas-Filmore by Berg and Davidson [**1**].

On the other hand, a number of analogues of the Main Problem were shown to be false: the one, obtained by replacing everywhere the word "*Hermitian*" by "*unitary*," by Voiculescu [**9**] and the one for "*arbitrary*" operators by Choi [**4**]. In both cases a sequence of pairs of rank n operators is constructed, for which the norm of the commutator does not exceed C/n, while the distance from a commuting pair is at least c, where $C, c > 0$ are numerical constants.

The purpose of this paper is to give an (asymptotic) improvement of the result of Pearcy and Shields in the case when *both* A and B are Hermitian, which shows that the situation in the Hermitian setting is completely different than in the "unitary" or "arbitrary" cases. A counterexample to the Main Problem, if it exists, must be of different nature than those from [**4**] or [**9**]. Our main result is

THEOREM. *There exists a positive constant* $\mathbf{c}$ *such that, whenever* n *is a positive integer,* $\varepsilon > 0$ *and* A, B *are two Hermitian Hilbert space operators of rank at most* n *satisfying* $||[A,B]|| \le \delta \equiv c\varepsilon^{13/2} n^{-1/2}$, $||A|| \le 1, ||B|| \le 1$, *then there exist commuting Hermitian operators* A_1, B_1 *such that* $||A_1 - A|| \le \varepsilon$ *and* $||B_1 - B|| \le \varepsilon$.

We have an immediate

COROLLARY. *If* n, ε, δ *are as in the Theorem and* T *is a rank* n *Hilbert space operator such that* $||T|| \le 1$ *and* $||[T^*, T]|| \le \delta$, *then there exists a normal operator* N *with* $||N - T|| \le \varepsilon$.

PROOF. Apply the Theorem with $A = \operatorname{Re} T$ and $B = \operatorname{Im} T$. $\square$

REMARKS. (1) The dependence of δ on ε is certainly not optimal, one can easily reduce the exponent $13/2$ at the cost of making the argument more complicated. The best possible exponent is clearly 2 (as in [**7**]) and in that case the hypothesis that A, B are of norm ≤ 1 wouldn't be necessary. We did not work hard on optimizing the exponent; our main point is that $\varepsilon = o(1)$ if $\delta = o(n^{-1/2})$. On the other hand, the exponent "$-1/2$" in $n^{-1/2}$ does not seem to be easily improvable without introducing new ideas into argument.

(2) Our argument is very strongly based on [**5**], where a deep analysis of the Main Problem was presented and where most of the ideas used here appear explicitly or implicitly. Our contribution consists mainly of the explicit condition on m, s and ε in the Proposition below, the argument leading to it and observing that it may be useful.

(3) The noncompact version of the Main Problem is also false due to an index obstruction (see Berg and Olsen [**2**]). See also [**4**] for other relevant references.

For the proof of the Theorem we will need the following technical result, inspired by question $(\mathcal{Q}')$ from [**5**] (see also Theorem 5.2 there).

PROPOSITION. *There exists $c > 0$ such that, whenever $\varepsilon \in (0,1)$, m, s are positive integers with $m < c\varepsilon^9 s$, $\mathcal{H} = \oplus_{0 \leq j \leq s} \mathcal{H}_j$ (an orthogonal decomposition) with $\min_{0 \leq j \leq s} \dim \mathcal{H}_j \leq m$ for all j, and B (a selfadjoint operator on $\mathcal{H}$, $0 \leq B \leq 1$) is block tridiagonal with respect to $\mathcal{H}_j$'s (i.e., $B\mathcal{H}_i \perp \mathcal{H}_j$ if $|i-j| > 1$), then there exists an orthogonal projection P with $\mathcal{H}_0 \subset \operatorname{ran}(P), \mathcal{H}_s \subset \ker(P)$ and $||BP - PB|| < \varepsilon$.*

PROOF OF THE THEOREM. (Assuming the Proposition). Without serious loss of generality we can assume that $0 \leq A, B \leq 1$ (just replace A, B by $(I + A)/2, (I + B)/2$ respectively and modify the constant c accordingly). We sketch briefly the procedure from [**5**], which reduces a "Theorem-like" statement to a "Proposition-like" statement. For completeness, we provide more of the details at the end of our argument.

Given positive integer s (to be specified later), partition the interval $[0, 1]$ into subintervals $I_0, I_1, I_2, \ldots$, each (except possibly for the last one) of length $\rho \equiv \varepsilon/s$. Let $A = \oplus_{j\geq 0} A_j$ be the corresponding spectral decomposition, i.e., if A acts on $\mathcal{H}$, then $\mathcal{H} = \oplus_{j\geq 0}\mathcal{H}_j$, each A_j acts on $\mathcal{H}_j$ and the spectrum $\sigma(A_j) \subset I_j$. Let (B_{ij}) be the matrix of B (with operator entries) with respect to the decomposition $\mathcal{H} = \oplus\mathcal{H}_j$ and let B' be its block-triangular part (i.e., $B' = (B'_{ij})$ with $B'_{ij} = B_{ij}$ if $|i-j| \leq 1$ and $B'_{ij} = 0$ otherwise). Then

$$||B - B'|| \leq 16\delta\rho^{-1} \tag{2}$$

If, moreover, for every integer $q \geq 0$, there is a projection L with $\oplus_{0\leq j\leq q}\mathcal{H}_j \subset \operatorname{ran}(L), \oplus_{j\geq q+s}\mathcal{H}_j \subset \ker(L)$ and $[L, B'] \leq \varepsilon/5$, then there exist A_1, B_1 with

$$||A_1 - A|| \leq \varepsilon, \quad ||B_1 - B'|| \leq 4\varepsilon/5 \ \text{ and } \ [A_1, B_1] = 0. \tag{3}$$

Consequently, if additionally

$$\delta \leq \varepsilon^2/80s, \tag{4}$$

then also $||B_1 - B|| \leq 4\varepsilon/5 + 16\delta\rho^{-1} \leq \varepsilon$, and so (A_1, B_1) is the required pair of commuting approximants.

Let us now show that the above procedure, together with our Proposition, proves the Theorem. Indeed, if we set, for given q, $m \equiv \min_{0\leq j\leq s} \dim \mathcal{H}_{j+q}$, then certainly $m \leq n/s$, and so the Proposition would yield the required projection L provided that $n/s \leq c_1\varepsilon^9 s$ or $s \geq (c_1^{-1}\varepsilon^{-9}n)^{1/2}$ (where $c_1 = 5^9 c$; we must change the constant due to the fact that we are replacing ε by $\varepsilon/5$ in the assertion). Combining this with (4) concludes the proof of the Theorem.

Let us recall now briefly how (2) and (3) were achieved in [**5**]. For (2), take a function $f \in L_1(\mathbf{R})$ such that $||f||_1 < 2\rho^{-1}$ and that its Fourier transform $\hat{f}$ satisfies $\hat{f}(x) = x^{-1}$ for $|x| > \rho$. Next let $B'' = B - B'$ and define

$$\mathcal{Q} = \int_{\mathbf{R}} \exp(-itA)[A, B'']\exp(itA)f(t)dt.$$

It is easy to check that $||\mathcal{Q}|| \leq 2\rho^{-1}||[A, B'']|| \leq 8\delta\rho^{-1}$. On the other hand, an examination of the scalar products $\langle [A, \mathcal{Q}]v, u\rangle$ and

$\langle [A, B'']v, u\rangle$, where u and v are (any) eigenvectors of A and B respectively, shows that $[A, \mathcal{Q}] = [A, B'']$. Since $\mathcal{Q} - B''$ is block-diagonal with respect to $\oplus\mathcal{H}_j$, $||B''|| \le 2||\mathcal{Q}||$, and so (2) follows.

For (3), divide $\oplus\mathcal{H}_j$ into "superblocks" corresponding to spectral intervals $J_0, J_1, J_2, \ldots$ of A of length ε and let $L_0, L_1, L_2, \ldots$ be the projections corresponding to these "superblocks," whose existence is guaranteed by the Proposition. Next let $A_1 = \sum_i \mu_i(L_{i+1} - L_i)$, where μ_i is the center of J_i, and $B_1 = \sum_i (L_{i+1} - L_i)B'(L_{i+1} - L_i)$; (3) is then easily checked. □

It remains to prove the Proposition. We are going to need a few auxiliary results. We start with an elementary Lemma:

LEMMA 1. *If μ is a finite measure on $[0,1]$ and $\kappa, \eta > 0$, then there exists a collection $\{I_1, \ldots, I_r\}$ of subintervals of $[0,1]$ such that*

(a) $r \le 2\kappa^{-1}$,

(b) $|I_j| \le \kappa$ *for all* j,

(c) $\mathrm{dist}(I_i, I_j) \ge \eta$ *if* $i \ne j$, *and*

(d) $\mu(\sim \cup_{j \le r} I_j) \le 4\eta\kappa^{-1}\mu([0,1])$.

PROOF. (Sketch) Divide $[0,1]$ into subintervals of length $\le \kappa/2$. Remove from each of them (except for the extreme ones) a subinterval of length η (we may clearly assume that $\eta < \kappa$), which has the smallest μ-mass. The complement of the removed intervals constitutes the required collection. □

The next lemma is a variant of Lemma 2.2 from [**5**] (cf. also [**9**]).

LEMMA 2. *If E, F' are (orthogonal) projections with $E \le G$, $||E(I - F')|| \le \alpha$ and $||F'(I - G)|| \le \beta$, then there exists F such that $E \le F \le G$ and $||F' - F|| \le (\alpha^2 + \beta^2)^{1/2}$.*

PROOF. Set $F = E + G(E^\perp \cap F')$, where we identify a projection with its range (note that the two summands are orthogonal). □

The following is very well-known (see, e.g., [**6**]).

LEMMA 3. *There exists a universal positive constant M such that, if f is a continuous function on $[0,1]$ with modulus of continuity $\omega(\cdot)$ and $n \in \mathbf{N}$, then there exists a polynomial p of degree $\leq n$ such that $||f-p||_\infty \leq M\omega(1/n)$. Moreover, the map $A_n : f \to g$ may be chosen to be a positive linear operator.*

PROOF OF PROPOSITION. Observe first that, without serious loss of generality, we can replace the hypothesis "$\min_j \dim \mathcal{H}_j \leq m$" by

$$\max_j \dim \mathcal{H}_j \leq m \tag{5}$$

(one may have to change the constant c, though). Indeed, if, e.g., $\dim \mathcal{H}_i \leq m$, then we can construct another orthogonal decomposition $\mathcal{H} = \oplus_{0\leq j\leq s}\mathcal{H}'_j$ with $\mathcal{H}_0 \subset \mathcal{H}'_0$, $\mathcal{H}_s \subset \mathcal{H}'_s$ and $\dim \mathcal{H}'_j \leq m$, for $j = 1, 2, \ldots, s-1$, with respect to which B is also block tridiagonal: set $\mathcal{H}'_i = \mathcal{H}_i, \mathcal{H}'_{i-1}$ (respectively, $\mathcal{H}'_{i+1}$) to be the orthogonal projection of $B\mathcal{H}'_i$ onto $\oplus_{0\leq j\leq -1}\mathcal{H}_j$ (respectively $\oplus_{0\leq j\leq i+1}\mathcal{H}_j$), etc. Then apply the Proposition to the "middle section" of B (acting on $\mathcal{H}' = \oplus_{1\leq j\leq s-1}\mathcal{H}'_j$), to obtain a projection P' on $\mathcal{H}'$, and finally set $P = I_{\mathcal{H}'_0} \oplus P' \oplus \mathbf{O}_{\mathcal{H}'_s}$.

As was observed in [**5**], under the additional assumption (5), the Proposition is equivalent to the following statement (cf. question $(\mathcal{Q}'')$, end of §3 there; for $f \in L_\infty(0,1)$, we denote by M_f the operator of multiplication by f acting on $L_2(0,1)$):

(6) Let ε, m, s be as in the Proposition and let $\mathcal{K}$ be an m-dimensional subspace of $L_2(0,1)$. Then there exists a projection P such that, $\mathcal{K} \subset \mathrm{ran}(P) \subset \mathrm{span}(M_x^j\mathcal{K} : 0 \leq j \leq s-1)$ and $||M_xP - PM_x|| < \varepsilon$.

Indeed, to see that (6) implies the Proposition (which is what we need), it is enough to use the spectral theorem and approximation, identifying M_x with B and $\mathcal{K}$ with $\mathcal{H}_0$. We want to remark here that if we insist only on the first of the inclusions in the condition above, then P can always be chosen with rank$P \leq m/\varepsilon$; this can be thought of as a weak (cf. [**5**], comments at the end of §3) kind of uniform quasidiagonality of self-adjoint operators.

Now denote $\mathcal{F} = \mathrm{ran}(P)$ and observe that

$$||M_x P - PM_x|| < \varepsilon \quad \Leftrightarrow \quad (f \in \mathcal{F}, ||f|| \leq 1 \Rightarrow \mathrm{dist}(M_x f, \mathcal{F}) < \varepsilon).$$

Let $\varphi_1, \dots, \varphi_m$ be an orthonormal basis of $\mathcal{K}$ and set $\Phi = (\sum_{j \leq m} |\varphi_j|^2)^{1/2}$. Then (i) $||\Phi|| = m^{1/2}$; (ii) Φ does not depend on the choice of (φ_j); (iii) $f \in \mathcal{K}, ||f|| \leq 1 \Rightarrow |f| \leq \Phi$. Apply Lemma 1 with $\kappa = \varepsilon, \eta = 2^{-16}\varepsilon^6/m$ and $d\mu = \Phi^2 d\lambda$, where λ denotes the Lebesgue measure, obtaining $I_1, \dots, I_r, r \leq 2\varepsilon^{-1}$. Let $E(\cdot)$ be the spectral measure of M_x (i.e., for $I \subset [0,1], E(I) = M_{\chi_I}$). Given $j \in \{1, 2, \dots, r\}$, consider the operator $A_j = E(I_j)|_{\mathcal{K}}$ and its polar decomposition $A_j = U_j \mathcal{Q}_j$. Denote further, by $E_j(\cdot)$, the spectral measure of $\mathcal{Q}_j (\geq 0)$ and let

$$a = 2^{-4}\varepsilon^{3/2}, \quad \mathcal{K}_j = E_j((a,1])\mathcal{K}, \quad \mathcal{L} = \mathrm{span}(A_j \mathcal{K}_j, j = 1, 2, \dots, r).$$

We now want to show that the projection $P_{\mathcal{L}}$ "nearly" satisfies the assertion of the Proposition and so, by Lemma 2, there is a "nearby" projection which works there. To this end observe that if $f \in \mathcal{K}$ with $||f|| \leq 1$, then $|f| \leq \Phi$ and so, by Lemma 1(d),

$$||f - \sum_{j \leq r} A_j f|| \leq (2\eta m/\kappa)^{1/2} \leq (2^{-15}\varepsilon^5)^{1/2}.$$

Also, for any $j, ||A_j f - A_j E_j((a,1])f|| \leq a$ and so

$$||\sum_j A_j f - \sum_j A_j E_j((a,1])f|| \leq r^{1/2} a \leq 2^{-7/2}\varepsilon.$$

Hence $\mathrm{dist}(f, \mathcal{L}) \leq (2^{-15}\varepsilon^5)^{1/2} + 2^{-7/2}\varepsilon < \varepsilon/8$ or, in other words,

$$||(I - P_{\mathcal{L}})P_{\mathcal{K}}|| \leq \varepsilon/8. \tag{7}$$

Now let $g \in \mathcal{L}$ with $||g|| = 1$. Then $g = \sum_j t_j A_j f_j$ with $f_j \in \mathcal{K}_j$; we can also assume that $||A_j f_j|| = 1$ for all j and $\sum_j |t_j|^2 = 1$. It then follows from the definition of $\mathcal{K}_j$ that $||f_j|| \le a^{-1}$. Observe that if c_j is the center of I_j, then

$$||M_x g - \sum_j c_j t_j A_j f_j|| \le 1/2 \max_j |I_j| \le \varepsilon/2$$

and, consequently,

$$\operatorname{dist}(M_x g, \mathcal{L}) \le \varepsilon/2. \tag{8}$$

Next, for each $j \le r$, let ψ_j be the function which is 1 on $I_j, 0$ on I_i's for $i \ne j$ and affine on each of the intervals in the complement of $S \equiv \cup_{i \le r} I_i$; then $(\psi_j)_{j \le r}$ form a partition of unity on $[0,1]$. Applying Lemma 3 to each of the ψ_j's and setting $\gamma = M/\eta(s-1)$, we get polynomials $(p_j)_{j \le r}$ such that, for all j, $\deg p_j \le s-1, 0 \le p_j(t) \le 1$ for $t \in [0,1]$, $p_j(t) \ge 1-\gamma$ for $t \in I_j, p_j(t) \le \gamma$ for $t \in I_i, i \ne j$ and, additionally, $\sum_j p_j(t) = 1$ for $t \in [0,1]$. It follows that $\sum_j |\chi_{I_j}(t) - p_j(t)| \le 2\gamma$ for $t \in S$. We now want to estimate $||\sum_j t_j A_j f_j - \sum_j t_j p_j f_j|| = ||\sum_j t_j(\chi_{I_j} - p_j) f_j||$. One has, for $t \in S$,

$$|\sum_j t_j(\chi_{I_j}(t) - p_j(t)) f_j(t)| \le 2\gamma \max_j |t_j f_j(t)| \le 2\gamma (\sum_{j \le r} |f_j(t)|^2)^{1/2}$$

and, for $t \notin S$,

$$\begin{aligned} |\sum_j t_j(\chi_{I_j}(t) - p_j(t)) f_j(t)| &= |\sum_j t_j p_j(t) f_j(t)| \\ &\le \max_j |t_j f_j(t)| \le a^{-1}\Phi(t). \end{aligned}$$

Putting these together, we get

$$\begin{aligned} ||\sum_j t_j A_j f_j - \sum_j t_j p_j f_j|| &\le a^{-1}(4\gamma^2 r + 2\eta m/\kappa)^{1/2} \\ &\le 2^4 \varepsilon^{-3/2}(8\gamma^2\varepsilon^{-1} + 2^{-15}\varepsilon^5)^{1/2}. \end{aligned}$$

If now the constant c from the hypothesis is chosen small enough, then our assumptions on s imply $\gamma < 2^{-9}\varepsilon^3$ so the above does not exceed

$\varepsilon/8$. It follows that, setting $\mathcal{G} = \mathrm{span}(M_x^j \mathcal{K} : 0 \leq j \leq s-1)$ (remember $g = \sum_j t_j A_j f_j$),

$$\mathrm{dist}(g, \mathcal{G}) \leq \varepsilon/8$$

and hence

$$||(I - P_{\mathcal{G}})P_{\mathcal{L}}|| \leq \varepsilon/8. \tag{9}$$

To conclude the argument, use (7), (8) and (9) to apply Lemma 2 with $F' = P_{\mathcal{L}}, E = P_{\mathcal{K}}$ and $G = P_{\mathcal{G}}$; this gives a projection F. It is then easily checked that $P = F$ satisfies the assertion of the Proposition. $\square$

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Department of Mathematics, Case Western Reserve University, Cleveland, OH 44106