ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 2, Spring 1990

DUALITY TYPE RESULTS AND ERGODIC ACTIONS OF SIMPLE LIE GROUPS ON OPERATOR ALGEBRAS

C. PELIGRAD

1. Introduction. Let G be a locally compact second countable group. We shall consider actions of G on a C^*-algebra A or on a von Neuman algebra M. By an action we shall mean a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ (respectively $\mathrm{Aut}(M)$ of G into the group of all $*$-automorphisms of A (respectively M) such that the mapping $g \to \alpha_g(a)$ (respectively $g \to \alpha_g(m)$) is continuous for the topology of G and norm-topology of A (respectively ultraweak topology of M) for all $a \in A$ (respectively all $m \in M$).

In these cases we shall call the triple (A, G, α) (respectively (M, G, α) a C^*-dynamical system (respectively W^*-dynamical system). A W^*-dynamical system is called ergodic if the fixed point algebra $M^\alpha = \{m \in M | \alpha \in M | \alpha_g(m) = m, \text{ for all } g \in G\}$ is trivial, i.e., $M^\alpha = C \cdot 1$. A C^*-dynamical system (A, G, α) with the property that A^α is trivial will be called weakly ergodic. In the C^*-case, even in the classical, abelian case, further notions such as minimality and topological transitivity are to be considered. In order to give the noncommutative formulation of the above two notions, some notations are required.

A C^*-subalgebra $B \subset A$ is called a hereditary subalgebra of A if its positive part B_+ is a hereditary subcone of A_+ (i.e., for $x \in A_+, y \in B_+$ and $x \leq y$ it follows that $x \in B_+$).

If (A, G, α) is a C^*-dynamical system, we denote by $\mathcal{H}^\alpha(A)$ the collection of all nonzero globally α-invariant hereditary subalgebras of A. Following [**6**, Definition 2.1] the action α is called topologically transitive if, for every B_1, $B_2 \in \mathcal{H}^\alpha(A)$ it follows that $B_1 \cdot B_2 \neq 0$.

With the generic name of ergodicity for all the notions formulated above, we shall investigate the following

PROBLEM. Let (A, G, α) be an ergodic C^*-dynamical system (respectively (M, G, α), an ergodic W^*-dynamical system). If $H \subset G$ is a closed subgroup, when is H ergodic on A (respectively M)?

Supported in part by NSF Grant No. DMS-8502353 and by a Taft Travel Grant.

In the case in which one is dealing with transitive actions there is a useful duality principle formulated independently by G.W. Mackey and C.C. Moore. This principle asserts that if H and K are closed subgroups of G then H is ergodic on G/K if and only if K is ergodic on G/H [**7**, Proposition 6]. Using this approach, Moore has shown that $\mathrm{SL}_n(Z)$ acts ergodically on $\mathbf{R}^n$ [**6**, Corollary to Theorem 6].

R.J. Zimmer [**9**, Theorem 4.2] has extended the Mackey-Moore duality principle to the more general case of actions of groups on measure spaces. Assume now that $G = \prod_{i=1}^n G_i$ where G_i is a connected, simple, noncompact Lie group with finite center. If (M, G, α) is an ergodic W^*-dynamical system we will call the action α irreducible [**10**] if, for every noncentral normal subgroup $N \subset G, N$ is still ergodic on M.

In [**7**] C.C. Moore has proved the following:

Moore's Theorem: *Let $G = \prod_{i=1}^n G_i$ be a finite product, where each G_i is a connected, simple, noncompact Lie group with finite center. Suppose that S is an irreducible ergodic G-space with finite invariant measure. If $H \subset G$ is a closed noncompact subgroup, then H is ergodic on S.*

In this paper (Theorem 3.3 below) we will extend Moore's Theorem to the general noncommutative case.

We also extend Zimmer's duality result stated above to the case of topologically transitive C^*-dynamical systems (Theorem 2.1 below). Using this noncommutative duality principle we construct an example (Example 3.4) of an ergodic action of $\mathrm{SL}_2(\mathbf{R})$ on a noncommutative von Neumann algebra with a faithful normal invariant state which constitutes an example for our Theorem 3.3.

2. The duality principle. Let G be a locally compact group and $H \subset G$ a closed subgroup. Let (B, H, β) be a C^*-dynamical system. We define the induced C^*-dynamical system $(\mathrm{ind}_{H\uparrow G}B, G, \alpha)$ as follows:

We denote by $\mathrm{ind}_{H\uparrow G}B$ the set of all $f \in C(G, B)$ with the following two properties:

(1) $f(xh) = \beta_h^{-1}(f(x))$ for all $h \in H, x \in G$,

(2) $x \to ||f(x)||$ is in $\mathrm{C}_0(G/H)$ (this latter being the algebra of continuous functions vanishing at infinity on G/H).

Obviously $\mathrm{ind}_{H\uparrow G}B$ is a C^*-subalgebra of $\mathrm{C}(G, B)$. Let α be the action of G on $\mathrm{ind}_{H\uparrow G}B$ by left translations. We shall call the system

$(\mathrm{ind}_{H\uparrow G}B, G, \alpha)$ the system induced by (B, H, β). The corresponding construction for W^*-dynamical systems is the following:

Let (N, H, β) be a W^*-dynamical system. We denote by $\mathrm{ind}_{H\uparrow G}N$ the set of all elements $f \in L^\infty(G, N)$ with the property $f(xh) = \beta_h^{-1}(f(x))$ for all $x \in G, h \in H$. Then $\mathrm{ind}_{H\uparrow G}N$ is a W^*-algebra. Let α be the action of G on $\mathrm{ind}_{H\uparrow G}N$ by left translations. Then $(\mathrm{ind}_{H\uparrow G}N, G, \alpha)$ will be called the system induced by (N, H, β).

We can now state our duality principle for topologically transitive actions.

THEOREM 2.1. *Let G be a locally compact group and $H \subset G$ a closed subgroup. If (B, H, β) is a C^*-dynamical system, then the following conditions are equivalent*:

(i) *H acts topologically transitively on B*;

(ii) *The induced action α of G on* $\mathrm{ind}_{H\uparrow G}B$ *is topologically transitive.*

PROOF. (i) $\Rightarrow$ (ii). Let $\tilde{B}_1, \tilde{B}_2 \in \mathcal{H}^\alpha(\mathrm{ind}_{H\uparrow G}B)$ with $\tilde{B}_1 \cdot \tilde{B}_2 = (0)$. For $i \in \{1, 2\}$ denote $S_i = \{g \in G \mid (\exists)\ f \in \tilde{B}_i \text{ with } f(g) \neq 0\}$. Since both $\tilde{B}_1 \neq 0$ and $\tilde{B}_2 \neq 0$ we have that both S_1 and S_2 are nonempty. We shall show that S_i are translation invariant and thus $S_i = G, i = 1, 2$.

Indeed let $g \in S_i$ and $g_0 \in G$. Since $g \in S_i$, there is a $\tilde{f} \in \tilde{B}_i$ with $\tilde{f}(g) \neq 0$. Then since $\tilde{B}_i$ is globally α-invariant, we have that $\alpha_{g_0}^{-1}\tilde{f} \in \tilde{B}_i$ and $(\alpha_{g_0}^{-1}\tilde{f})(g_0^{-1}g) = \tilde{f}(g) \neq 0$. Therefore $g_0^{-1}g \in S_i$ and S_i is translation invariant. It follows that $S_i = G, i = 1, 2$. In particular $e \in S_i$ (where e is the neutral element of G). Denote $B_i = \{\tilde{f}(e) \mid \tilde{f} \in \tilde{B}_i\}$. Then $B_i, i = 1, 2$, are hereditary C^*-subalgebras of B which are globally H-invariant and $B_i \cdot B_2 = (0)$.

(ii) $\Rightarrow$ (i). Let $B_1, B_2 \in \mathcal{H}^\beta(B)$ with $B_1 \cdot B_2 = (0)$.

Let $\tilde{B}_i = \{f \in \mathrm{ind}_{H\uparrow G}B \mid f(g) \in B_i \text{ for all } g \in G\}$. Obviously $\tilde{B}_i$ is a hereditary C^*-subalgebra of $\mathrm{ind}_{H\uparrow G}B$ which is globally α-invariant, and $\tilde{B}_i \cdot \tilde{B}_2 = (0)$. In order to prove the implication (ii) $\Rightarrow$ (i) it remains only to show that $\tilde{B}_i \neq (0),\ i = 1, 2$.

For every $f \in \mathrm{C}_c(G, B_i)$ (here $\mathrm{C}_c(G, B_i)$ stands for continuous functions with compact support from G to B_i), let us consider the element

$f^\beta \in \mathrm{C}(G, B_i)$ with

$$f^\beta(g) = \int_H \beta_h(f(gh))dh$$

(where dh is the left Haar measure on H). It is immediate that $f^\beta(gh) = \alpha_h^{-1} f^\beta(g)$ for every $h \in H$ and $g \in G$.

Since $B_i \neq (0)$, one can show by using Tietze's extension Theorem that there is $f \in \mathrm{C}_c(G, B_i)$ such that $f^\beta \neq (0)$.

Let now $\phi \in \mathrm{C}_c(G/H)$ be such that $h(g) = \phi(\dot{g}) f^\beta(g) \neq 0$ for some $g \in G$.

Then $h \in \tilde{B}_i$ and $h \neq 0$ which completes the proof. □

2.2. Let (A, G, α) be a C^*-dynamical system and $H \subset G$ a closed subgroup. If we denote $\beta = \alpha|_H$ the restriction of the action α to H, then we can form the induced system $(\mathrm{ind}\,_{H\uparrow G} A, G, \tau)$. Via the map $(TF)(\dot{g}) = \alpha_g(f(g))$, this system is covariantly isomorphic with the system $(\mathrm{C}_0(G/H)\overline{\otimes} A, G, \lambda \otimes \alpha)$ where λ is the action of G on G/H by left translations. Therefore, from Theorem 2.1, we have

COROLLARY. *Adopt the above notations. Then the following conditions are equivalent.*

(i) $\alpha|_H$ *is topologically transitive on* A.

(ii) *The action* $\lambda \otimes \alpha$ *of* G *on* $\mathrm{C}_0(G/H)\overline{\otimes} A$ *is topologically transitive.*

2.3. Let (N, H, β) be a W^*-dynamical system and $(\mathrm{ind}\,_{H\uparrow G} N, G, \alpha)$ its induced system.

COROLLARY. *The following conditions are equivalent*:

(i) β *is an ergodic action of* H *on* N.

(ii) α *is an ergodic action of* G *on* $\mathrm{ind}\,_{H\uparrow G} N$.

PROOF. Completely similar with that of Theorem 2.1. □

3. Ergodic actions of semisimple Lie groups: A noncommutative version of a theorem of C.C. Moore. In this section we shall discuss a generalization of Moore's Theorem (stated in the Introduction).

The proof given in [**10**] of the original Theorem of Moore is based on a deep result of Howe and Moore on unitary representations of the group G[**4**].

The basic idea in using group representations for problems concerning ergodicity is the following: If G acts on the measure space (S, μ) where μ is a finite G-invariant then there is a natural representation of G on $\mathrm{L}^2(S, \mu)$ associated with the action of G on S. Then G acts ergodically on (S, μ) if and only if the corresponding representation of G on $\mathrm{L}^2(S, \mu)$ contains the identity representation exactly once.

We will see below (Lemma 3.2) that this fact has a noncommutative version that will enable us to prove the noncommutative version of Moore's Theorem.

Let (M, G, α) be a W^*-dynamical system. Assume that there is a normal, faithful G-invariant state ϕ on M. Let $(H^\phi, \pi^\phi, U^\phi, \xi^\phi)$ be the GNS-system associated with (M, G, α), where H^ϕ is the Hilbert space, π^ϕ the representation of M on H^ϕ, U^ϕ the unitary representation of G on H^ϕ coming from the action α.

Let E^ϕ denote the orthogonal projection on the subspace of H^ϕ consisting of vectors which are left fixed by U_G^ϕ and let 1^ϕ denote the identity operator on H^ϕ.

We collect some well-known facts in

THEOREM 3.1. (i) *E^ϕ is in the strong closure of the convex hull, $\mathrm{co}\,(U_G^\phi)$ of U_G^ϕ.*

(ii) *There exists a unique normal, G-invariant conditional expectation $E : \pi^\phi(M) \to \pi^\phi(M) \cap U_G^{\phi'}$ (where $U_G^{\phi'}$ is the commutant of U_G^ϕ) such that $E(M)E^\phi = E^\phi M E^\phi$ for all $m \in \pi^\phi(M)$.*

PROOF. (i). This is a special case of the Alaoglu-Birkhoff mean ergodic Theorem (see for instance [**1**]).

(ii). This is the Kovács and Szücs Theorem [**5, 1**]. □

The following Lemma is a Von Neumann algebra version of some results in [**3, 1**].

LEMMA 3.2. *Adopt the above notations. The following conditions are equivalent*:

(i) *The action α is ergodic.*

(ii) *The representation U^ϕ has no invariant vectors in $H^\phi \Theta C\xi^\phi$.*

PROOF. (i) $\Rightarrow$ (ii). By Theorem 3.1(ii) there is a unique normal G-invariant conditional expectation $E : \pi^\phi(M) \to \pi^\phi(M) \cap U_G^{\phi'}$ such that $E(M)E^\phi = E^\phi M E^\phi$ for every $m \in \pi^\phi(M)^G$. If $E(\pi^\phi(M)) = C1^\phi$ then it is easy to see that E^ϕ has rank one.

(i) $\Rightarrow$ (i). Assume $E^\phi = C\xi^\phi$. Since ϕ is faithful it follows that ξ^ϕ is separating for $\pi^\phi(M)$ and therefore cyclic for $\pi^\phi(M)'$. It is then easy to see that the family of operators $\pi^\phi(M)' \cup E^\phi$ is irreducible on H^ϕ.

On the other hand, by Theorem 3.1(i), $E^\phi \in (U_G^\phi)''$. Therefore, $\pi^\phi(M)' \cup E^\phi \subset (\pi^\phi(M)' \cup U_G^\phi)''$ and since the left-hand side of the inclusion is irreducible, it follows that $(\pi^\phi(M)' \cup U_G^\phi)''$ is irreducible. Hence the commutant $(\pi^\phi(M)' \cup U_G^\phi)' = \pi^\phi(M) \cap U_G^{\phi'}$ reduces to the scalar multiple of the identity.

But this means that G acts ergodically on $\pi^\phi(M)$. Since π^ϕ is faithful this means that the original action α is ergodic.

We can now state and prove the noncommutative version of Moore's Theorem:

THEOREM 3.3. *Let $G = \prod_{i=1}^n G_i$ be a finite product, where each G_i is a connected noncompact simple Lie group with finite center. Let (M, G, α) be an irreducible, ergodic W^*-dynamical system, with a faithful normal invariant state ϕ. If $H \subset G$ is a closed noncompact subgroup then the restriction $\alpha|_H$ is still ergodic.*

PROOF. Let $(H^\phi, \pi^\phi, U^\phi, \xi^\phi$ be the covariant GNS representation of the system (M, G, α, ϕ).

Since α is irreducible, it follows that $\alpha|_{G_i}$ is ergodic for every $i = 1, 2, \dots, n$.

By Lemma 3.2, $U^{\phi}_{G_i}$ has no invariant vectors in $H^{\phi}\Theta C\xi^{\phi}$ for any $i = 1, 2, \dots, n$.

Using a deep result of Howe and Moore [**4**, Theorem 5.2] it follows that all the matrix coefficients of $U^{\phi}_{G_i}, i = 1, 2, \dots, n$, vanish at infinity. Therefore all the matrix coefficients of U^{ϕ}_{G} vanish at infinity. It then follows that if $H \subset G$ is a closed noncompact subgroup, then U^{ϕ}_{H} has no invariant vectors in $H^{\phi}\Theta C\xi^{\phi}$. Applying Lemma 3.2 again it follows that the restriction $\alpha|_H$ is ergodic.

EXAMPLE 3.4. We will give an example of an ergodic action of $\mathrm{SL}_2(\mathbf{R})$ on a von Neumann algebra with a faithful, normal invariant state.

In [**11**] Watani has indicated an action of $\mathrm{SL}_2(\mathbf{Z})$ on the irrational rotation C^*-algebra A_θ. This action extends to an action of $\mathrm{SL}_2(\mathbf{Z})$ on the irrational rotation von Neumann algebra M_θ. As observed in [**11**, Theorem 1] some subgroups of $\mathrm{SL}_2(\mathbf{Z})$ and thus $\mathrm{SL}_2(\mathbf{Z})$ itself acts ergodically on M_θ.

By the duality principle, Corollary 2.3, $\mathrm{SL}_2(\mathbf{R})$ acts ergodically on $\mathrm{ind}_{\Gamma\uparrow G} M_\theta$ where we denoted $G = \mathrm{SL}_2(\mathbf{R})$ and $\Gamma = \mathrm{SL}_2(\mathbf{Z})$. To show that $\mathrm{ind}_{\Gamma\uparrow G} M_\theta$ has a normal, faithful G-invariant state, notice that if Θ is the canonical trace on M_θ then $f \to \int_{G/\Gamma} \Theta(f(g))d\dot{g}$ is a faithful normal G-invariant state on $\mathrm{ind}\, M_\theta$ (Here $d\dot{g}$ is the finite G-invariant measure on G/Γ).

REFERENCES

1. O. Bratteli and D.W. Robinson, *Quantum statistical mechanics and operator algebras,* Springer Verlag, 1979.

2. S. Doplicher, D. Kastler and D.W. Robinson, *Covariance algebras in field theory and statistical mechanics,* Comm. Math. Phys. **3** (1966), 1–28.

3. S. Doplicher, D. Kastler and E. Stormer, *Invariant states and asymptotic abelianess,* J. Funct. Anal. **3** (1969), 419–434.

4. R.E. Howe and C.C. Moore, *Asymptotic properties of unitary representations,* J. Funct. Anal. **32** (1979), 72–96.

5. I. Kovács and J. Szücs, *Ergodic type theorems in von Neumann algebras,* Acta Sci. Math. **27** (1966), 233–248.

6. R. Longo and C. Peligrad, *Noncommutative topological dynamics and compact actions on C^*-algebras,* J. Funct. Anal. **58** (1984), 157–174.

7. C.C. Moore, *Ergodicity of flows on homogeneous spaces,* Amer. J. Math. **88** (1966), 154–178.

8. G.K. Pedersen, C^**-algebras and their automorphism groups,* Academic Press, New York, 1979.

9. R.J. Zimmer, *Orbit spaces of unitary representation, ergodic theory and simple Lie groups,* Ann. of Math. **106** (1977), 573–588.

10. R.J. Zimmer, *Ergodic theory and semisimple groups,* Birkhäuser Verlag, 1984.

11. Y. Watani, *Toral automorphisms on irrational rotation algebras,* Math. Japon. **26** (1981), 479–484.

Department of Mathematics, University of Cincinnati, Cincinnati, OH 45221-0025