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HYPONORMAL AND QUASINORMAL WEIGHTED COMPOSITION OPERATORS ON ℓ^2

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ABSTRACT. Characterizations of hyponormal and quasinormal weighted composition operators on the Hilbert space of complex valued functions on the integers are given in this article. Similar results are also presented for the adjoints of weighted composition operators.

Let H be a Hilbert space of complex-valued functions defined on a set X. A *weighted composition operator* on H is usually defined by $Tf = u(f \circ g)$ for all f in H where $u : X \to \mathbf{C}$ is a *weight function* and $g : X \to X$ is a *composition function*. An operator A on H is *hyponormal* if $A^*A - AA^* \geq 0$, is *seminormal* if one of A or A^* is hyponormal, and is *quasinormal* if $AA^*A = A^*AA$. In the very general setting where H is a sigma-finite L^2 space, measure theoretic characterizations of hyponormal weighted composition operators have been obtained by A. Lambert [**3**] and measure theoretic characterizations of hyponormal, seminormal, and quasinormal unweighted composition operators have been obtained by D. Harrington and R. Whitley [**2**].

In this article we will restrict ourselves to the Hilbert space ℓ^2 of complex-valued functions on the integers ($\mathbf{Z}$ = integers, $\mathbf{C}$ = complex numbers). We will also generalize the definition of a weighted composition operator: for y a subset of $\mathbf{Z}$, $g : y \to \mathbf{Z}$ and $u : \mathbf{Z} \to \mathbf{C} \setminus \{0\}$, define the weighted composition operator T_{ug} by

$$T_{ug} f(n) = \begin{cases} u(n) f \circ g(n) & \text{for } n \text{ in } y, \\ 0, & \text{for } n \text{ not in } y. \end{cases}$$

Characterizations of when T_{ug} and T_{ug}^* are hyponormal, seminormal and quasinormal are presented here. The characterization of T_{ug} hyponormal is a more concrete example of the characterization given in Lambert [**3**]. Here we have slightly generalized the definition of T_{ug} to encompass more operators on ℓ^2 and we have given an independent

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proof relying on the properties of T_{ug} on ℓ^2. It is hoped that this presentation of T_{ug} hyponormal will offer the reader some new insight.

Certain definitions and notations concerning a function $g : y \to \mathbf{Z}$ are needed before we can proceed. Let $g : y \to \mathbf{Z}$ and n be in $\mathbf{Z}$. Then $g^{(i)} = g \circ g \circ \cdots \circ g$ (g composed with itself i times), $g^{-1}(n) = \{k \text{ in } y : g(k) = n\}$ and *the orbit of* g *containing* n is the set

$$\{k \text{ in } Y: \text{ for some } i \geq 0 \text{ either } g^{(i)}(k) = n \text{ or } g^{(i)}(n) = k\}.$$

The space ℓ^2 can also be denoted by $L^2(\mathbf{Z}, m)$, where m is counting measure on $\mathbf{Z}$. In what follows we will sometimes work in a weighted space $L^2(\mathbf{Z}, \beta)$ where β is not counting measure. To denote norms and inner products in $L^2(\mathbf{Z}, \beta)$, when β is not counting measure, we will use a subscript β. In Carlson [**1**] it was shown that if β has sigma-algebra the power set of $\mathbf{Z}$, then a weighted composition operator T_{ug} acting on $L^2(\mathbf{Z}, \beta)$, is unitarily equivalent to a weighted composition operator T_{vg} acting on ℓ^2. Using the unitary operator constructed for showing T_{ug} on $L^2(\mathbf{Z}, \beta)$ is unitarily equivalent to T_{vg} on ℓ^2, the results given here can easily be given for any $L^2(\mathbf{Z}, \beta)$ instead of ℓ^2.

Let Y be a subset of $\mathbf{Z}, g : Y \to \mathbf{Z}$, and $u : \mathbf{Z} \to \mathbf{C} \setminus \{0\}$, also let $T = T_{ug}$. We will need formulas for $T^*T, TT^*, TT^*T, T^*TT, T^*TT^*$, and TT^*T^*. Let n be in $\mathbf{Z}$

$$(1) \qquad T^*Te_n = \begin{cases} \sum_{k \in g^{-1}(n)} |u(k)|^2 e_n & \text{if } n \text{ is in Image}\, g, \\ 0 & \text{otherwise} \end{cases}$$

$$(2) \qquad TT^*e_n = \begin{cases} \sum_{k \in g^{-1}(g(n))} \overline{u(n)} u(k) e_k, & \text{if } n \text{ is in } Y, \\ 0, & \text{otherwise,} \end{cases}$$

$$(3) \quad TT^*Te_n = \begin{cases} \sum_{j \in g^{-1}(n)} \sum_{k \in g^{-1}(n)} |u(j)|^2 u(k) e_k, & \text{if } n \text{ is in Image}\, g, \\ 0, & \text{otherwise,} \end{cases}$$

$$(4) \quad T^*TTe_n = \begin{cases} \sum_{k \in g^{-1}(n)} \sum_{j \in g^{-1}(k)} |u(j)|^2 u(k) e_k, & \text{if } n \text{ is in Image}\, g, \\ 0, & \text{otherwise,} \end{cases}$$

$$(5)\qquad T^*TT^*e_n = \begin{cases} \overline{u(n)} \displaystyle\sum_{k\in g^{-1}(g(n))} |u(k)|^2 e_{g(n)}, & \text{if } n \text{ is in } Y, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(6)\quad TT^*T^*e_n = \begin{cases} \overline{u(n)}\,\overline{u(g(n))} \displaystyle\sum_{k\in g^{-1}(g^{(2)}(n))} u(k)e_k, & \text{if } g(n) \text{ is in } Y, \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 1. *Let Y be a subset of $\mathbf{Z}$, $g : Y \to \mathbf{Z}$, u be a nonzero weight function, and $T = T_{ug}$ be a bounded operator. For each n in* Image g, *let*

$$v(n) = \Big(\sum_{m\in g^{-1}(n)} |u(m)|^2\Big)^{-1}.$$

The operator T is hyponormal if and only if Y is a subset of Image g *and*

$$\sum_{k\in g^{-1}(n)} |u(k)|^2 v(k) \le 1 \qquad \textit{for all } n \textit{ in } \text{Image}\, g.$$

PROOF. Some of the computations that follow involve changing norms and inner products in ℓ^2 to norms and inner products in $L^2(\mathbf{Z}, |u|^2g^{-1})$. Let $f = \sum_{n\in\mathbf{Z}} f_n e_n$ be in ℓ^2,

$$(7)\qquad \begin{aligned} \langle T^*Tf, f\rangle &= \Big\langle \sum_{n\in \text{Image}\, g}\ \sum_{k\in g^{-1}(n)} |u(k)|^2 f_n e_n, \sum_{n\in\mathbf{Z}} f_n e_n \Big\rangle \\ &= \sum_{n\in \text{Image}\, g}\ \sum_{k\in g^{-1}(n)} |f_n|^2 |u(k)|^2 = ||f||^2_{|u|^2g^{-1}} \end{aligned}$$

and

$$
\begin{aligned}
\langle TT^*f, f\rangle &= \Big\langle \sum_{n\in Y}\sum_{k\in g^{-1}(g(n))} f_n\overline{u(n)}u(k)e_k, \sum_{n\in \mathbf{Z}} f_n e_n\Big\rangle\\
&= \sum_{n\in Y}\sum_{k\in g^{-1}(g(n))} f_n\overline{u(n)}\,\overline{f_k}u(k)\\
&= \sum_{j\in \text{Image}\, g}\sum_{k\in g^{-1}(j)}\sum_{t\in g^{-1}(j)} f_t\overline{u(t)}\,\overline{f_k}u(k),\\
&\qquad\qquad (\text{where } j = g(n) \text{ for } n\in Y)\\
&= \sum_{j\in \text{Image}\, g}\Big|\Big\langle f\Big|_{g^{-1}(j)}, u\Big\rangle\Big|^2\\
&= \sum_{j\in \text{Image}\, g^{(2)}}\Big|\Big\langle f\Big|_{g^{-1}(j)}, h\Big\rangle_{|u|^2g^{-1}}\Big|^2\\
&\quad + \sum_{j\in \text{Image}\, g\setminus \text{Image}\, g^{(2)}}\Big|\Big\langle f\Big|_{g^{-1}(j)}, u\Big\rangle\Big|^2,
\end{aligned}
\tag{8}
$$

where $h = \frac{du}{d(|u|^2g^{-1})}$ (Randon-Nikodym derivative).

Let T be hyponormal. For each f in ℓ^2, $\langle T^*Tf, f\rangle \geq \langle TT^*f, f\rangle$. For n in $Y\setminus\text{Image}\, g$, $\langle T^*Te_n, e_n\rangle = 0$ and $\langle TT^*e_n, e_n\rangle = |u(n)|^2$. Thus, Y is a subset of $\text{Image}\, g$ so that $\text{Image}\, g = \text{Image}\, g^{(2)}$. Now (8) becomes

$$
\langle TT^*f, f\rangle = \sum_{j\in \text{Image}\, g}\Big|\Big\langle f\Big|_{g^{-1}(j)}, h\Big\rangle_{|u|^2g^{-1}}\Big|^2.
$$

Let j be in $\text{Image}\, g$ and let N be the subspace of ℓ^2 generated by $\{e_k\}_{k\in g^{-1}(j)}$. For all f in N, $\langle T^*Tf, f\rangle \geq \langle TT^*f, f\rangle$. Using (7) and (8) we see that

$$
||f||^2_{|u|^2g^{-1}} = \langle T^*Tf, f\rangle \geq \langle TT^*f, f\rangle = |\langle f, h\rangle_{|u|^2g^{-1}}|^2
$$

for all f in N. Thus, inner product with $h|_{g^{-1}(j)}$ in $L^2(\mathbf{Z}, |u|^2g^{-1})$ corresponds to a linear functional with bound less than or equal to one. Hence,

$$
\sum_{k\in g^{-1}(j)} |u(k)|^2v(k) = ||h|_{g^{-1}(j)}||^2_{|u|^2g^{-1}} \leq 1.
$$

Conversely, suppose that Y is a subset of Image g and

$$\sum_{k\in g^{-1}(n)} |u(k)|^2 v(k) \leq 1$$

for all n in Image g. Then $||h|_{g^{-1}(n)}||^2_{|u|^2 g^{-1}} \leq 1$ for all n in Image g where $h = \frac{du}{d(|u|^2 g^{-1})}$. Therefore, for all f in ℓ^2,

$$\begin{aligned}\langle TT^* f, f\rangle &= \sum_{n\in \text{Image } g} |\langle f|_{g^{-1}(n)}, h\rangle_{|u|^2 g^{-1}}|^2 \\ &\leq \sum_{n\in \text{Image } g} ||f|_{g^{-1}(n)}||^2_{|u|^2 g^{-1}} \\ &= ||f||^2_{|u|^2 g^{-1}} = \int_{\mathbf{Z}} |f|^2 d|u|^2 g^{-1} = \int_{\mathbf{Z}} [u(f\circ g)][\overline{u(f\circ g)}]dm \\ &= \langle Tf, Tf\rangle = \langle T^*Tf, f\rangle,\end{aligned}$$

where m is counting measure on $\mathbf{Z}$. Thus, T is hyponormal. □

PROPOSITION 2. *Let Y be a subset of $\mathbf{Z}$, $g : Y \to \mathbf{Z}$, u be a nonzero weight function, and $T = T_{ug}$ be a bounded operator. The operator T^* is hyponormal if and only if Y contains* Image g, $g^{-1}(n)$ *contains exactly one element for each n in* Image $g^{(2)}$, *and*

$$\sum_{j\in g^{-1}(k)} |u(j)|^2 \leq |u(k)|^2 \leq |u(g(k))|^2$$

for all k in Image g.

PROOF. Let T^* be a hyponormal operator. The formulas (7) and (8) hold for each f in ℓ^2. For each f in ℓ^2, $\langle TT^*f, f\rangle \geq \langle T^*Tf, f\rangle$ since T^* is hyponormal. Let n be in Image $g\backslash Y$, then $\langle TT^*e_n, e_n\rangle = 0$ and $\langle T^*Te_n, e_n\rangle = \sum_{k\in g^{-1}(n)} |u(k)|^2 \neq 0$. Thus, Image $g\backslash Y$ is empty and Image g is a subset of Y. Let $h = \frac{d(u)}{d(|u|^2 g^{-1})}$. Then, just as in the proof of Proposition 1, $|\langle f, h\rangle_{|u|^2 g^{-1}}|^2 = \langle TT^* f, f\rangle$ if f in ℓ^2 is supported on $g^{-1}(n)$ for some n in Image $g^{(2)}$. Hence, if f in ℓ^2 is supported on $g^{-1}(n)$ for some n in Image $g^{(2)}$ then $|\langle f, h\rangle_{|u|^2 g^{-1}}|^2 \geq ||f||^2_{|u|^2 g^{-1}}$. Therefore, for each n in Image $g^{(2)}$, the subspace N generated by $\{e_k\}_{k\in g^{-1}(n)}$

is one dimensional and $\sum_{k\in g^{-1}(n)}|u(k)|^2(\sum_{j\in g^{-1}(k)}|u(j)|^2)^{-1} \geq 1$. So, $g^{-1}(n)$ contains exactly one element for each n in Image $g^{(2)}$ and $|u(k)|^2 \geq \sum_{m\in g^{-1}(k)}|u(m)|^2$ for each k in Image g.

Conversely, suppose Image g is a subset of $Y, g^{-1}(n)$ contains exactly one element for n in Image $g^{(2)}$, and $\sum_{j\in g^{-1}(k)}|u(j)|^2 \leq |u(k)|^2$ for k in Image g. Let $f = \sum_{n\in \mathbf{Z}} f_n e_n$ be in ℓ^2, then

$$\begin{aligned}\langle T^*Tf, f\rangle &= \sum_{n\in \text{Image } g}|f_n|^2 \sum_{k\in g^{-1}(n)}|u(k)|^2\\ &\leq \sum_{n\in \text{Image } g}|u(n)|^2|f_n|^2\\ &= \sum_{j\in \text{Image } g^{(2)}}\sum_{k\in g^{-1}(j)}\sum_{t\in g^{-1}(j)} f_t\overline{u(t)}\,\overline{f_k}, u(k)\\ &\leq \sum_{j\in \text{Image } g}\sum_{k\in g^{-1}(j)}\sum_{t\in g^{-1}(j)} f_t\overline{u(t)}\,\overline{f_k}u(k)\\ &= \langle TT^*f, f\rangle.\end{aligned}$$

Thus, T^* is hyponormal. □

The seminormal weighted composition operators on ℓ^2 are characterized by combining the results of Proposition 1 and Proposition 2. We will now take up the topics of when T_{ug} is quasinormal and when T_{ug}^* is quasinormal.

PROPOSITION 3. *Let Y be a subset of $\mathbf{Z}$, $g: Y \to \mathbf{Z}$, u be a nonzero weight function and T_{ug} be a bounded operator. The operator $T_{ug} = T$ is quasinormal if and only if ($Y \subset$ Image g) and, for each orbit G of g, there is a constant $K > 0$ such that*

$$\sum_{j\in g^{-1}(k)}|u(j)|^2 = K \quad \text{for all } k \text{ in } (\text{Image } g)\cap G.$$

PROOF. Let T be quasinormal. For any n in $\mathbf{Z}$, the equality $TT^*Te_n = T^*TTe_n$ holds. Thus, by observing (3) and (4) we see that $g^{-1}(n) \cap \operatorname{Image} g = g^{-1}(n)$ for any n in $\operatorname{Image} g$. Therefore, Y is a subset of $\operatorname{Image} g$. Again equating (3) and (4),

$$\sum_{j \in g^{-1}(n)} |u(j)|^2 = \sum_{t \in g^{-1}(g(n))} |u(t)|^2 \neq 0 \quad \text{for } n \text{ in } \operatorname{Image} g.$$

Hence, for each orbit G of g there is a constant $K > 0$ such that

$$\sum_{j \in g^{-1}(n)} |u(j)|^2 = K \text{ for all } n \text{ in } G.$$

Conversely, suppose that Y is a subset of $\operatorname{Image} g$ and, for each orbit G of g, there is a constant K_G such that $\sum_{j \in g^{-1}(n)} |u(j)|^2 = K_G$ for all n in G. Now (3) becomes $TT^*Te_n = \sum_{k \in g^{-1}(n)} K_G u(k) e_k$ if n is in the orbit G and (4) becomes $T^*TTe_n = \sum_{k \in g^{-1}(n)} K_G u(k) e_k$ if n is in the orbit G. Both (3) and (4) are zero if n is not in $\operatorname{Image} g$. Thus, $TT^*T = T^*TT$. □

PROPOSITION 4. *Let Y be a subset of $\mathbf{Z}$, $g : Y \to \mathbf{Z}$, u be a nonzero weight function and $T_{ug} = T$ be a bounded operator. The operator T^* is quasinormal if and only if* $\operatorname{Image} g$ *is a subset of Y, $g^{-1}(n)$ contains exactly one element for each n in* $\operatorname{Image} g^{(2)}$*, and, for each orbit G of g, there is a constant $K > 0$ such that*

$$\sum_{k \in g^{-1}(j)} |u(k)|^2 = |u(j)|^2 = K \quad \textit{for } j \textit{ in } \operatorname{Image} g.$$

PROOF. Let T^* be quasinormal. If n is in $\operatorname{Image} g \backslash Y$ then $TT^*T^*e_n$ is not zero and $T^*TT^*e_n$ is zero. Hence, $\operatorname{Image} g \backslash Y = \varnothing$ and $\operatorname{Image} g$ is a subset of Y. If n is in $\operatorname{Image} g$ then, by equating (5) and (6), $\{k : g(k) = g(n)\} = \{n\}$. So, for j in $\operatorname{Image} g^{(2)}$, $g^{-1}(j)$ contains exactly one element. Let j be in $\operatorname{Image} g$. Then $\overline{u(n)} \sum_{k \in g^{-1}(j)} |u(k)|^2 = \overline{u(n)}\,\overline{u(j)} u(j)$ for each n such that $g(n) = j$. Thus, $\sum_{k \in g^{-1}(j)} |u(k)|^2 = |u(j)|^2$ for each j in $\operatorname{Image} g$. Therefore, for each orbit G of g there is a constant K such that

$$\sum_{k \in g^{-1}(j)} |u(k)|^2 = |u(j)|^2 = K \quad \text{for } j \text{ in } G \cap \operatorname{Image} g.$$

Conversely, suppose Image g is a subset of $Y, g^{-1}(n)$ contains exactly one element for each n in Image $g^{(2)}$, and for each orbit G of g there is a constant $K_G > 0$ such that $|u(j)|^2 = \sum_{k \in g^{-1}(j)} |u(k)|^2 = K_G$ for j in Image g. The formulas (5) and (6) now become

$$T^*TT^*e_n = \begin{cases} \overline{u(n)}, K_G e_g(n) & \text{if } n \text{ is in } Y \cap G, \\ 0, & \text{otherwise} \end{cases}$$

and

$$TT^*T^*e_n = \begin{cases} \overline{u(n)} K_G e_{g(n)}, & \text{if } g(n) \text{ is in } G \cap Y, \\ 0, & \text{otherwise.} \end{cases}$$

Since Image g is a subset of Y, n is in Y if and only if $g(n)$ is in Y. Thus, $TT^*T^* = T^*TT^*$. □

COROLLARY 5. *Let Y be a subset of $\mathbf{Z}, g : Y \to \mathbf{Z}, u$ be a nonzero weight function, and $T = T_{ug}$ be a bounded operator. The operator T on ℓ^2 is normal if and only if*

(i) *g is one-to-one and $Y =$ Image g,*

(ii) *For each orbit G of g there is a constant $K > 0$ such that $|u(n)| = K$ for all n in G.*

PROOF. First, it is clear that (i) and (ii) imply (1) is equal to (2) for each n in $\mathbf{Z}$. Thus, T is normal.

Conversely, let T be normal. The operators T and T^* are both quasinormal. Propositions 3 and 4 imply Y is a subset of Image g and Image g is a subset of Y. Hence, $Y =$ Image g. Let G be an orbit of g. Then, by Proposition 2, there exists a constant $K > 0$ such that $|u(n)| = K$ for all n in G. □

The following example illustrates a quasinormal operator T_{ug}^* that is of the type farthest from being a shift with constant weights (i.e., a normal shift). Let $g : \mathbf{N} \to \mathbf{Z}$ be defined by $g(1) = 3$, and $g(n) = n+1$ for $n \geq 2$ and let $u : \mathbf{Z} \to \mathbf{C}$ be defined by $u(n) = 1$ for $n \geq 3$, $u(1) = u(2) = 1/(1\backslash\sqrt{2})$. The operator T_{ug}^* is quasinormal.

REFERENCES

1. J. Carlson, *Weighted composition operators on* ℓ^2, Dissertation, Purdue University, West Lafayette, Indiana, 1985.

2. D. Harrington, and R. Whitley, *Seminormal composition operators,* J. Operator Theory **2** (1984), 125–135.

3. A. Lambert, *Hyponormal composition operators,* Bull. London Math. Soc. **18** (1986), 395–400.

4. E. Nordgren, *Composition operators on hilbert spaces,* in Hilbert Space Operators, Lecture Notes in Math. **693**, Springer-Verlag, Berlin, 1978.

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