

## INTERPOLATION OF SPECTRUM OF BOUNDED OPERATORS ON LEBESGUE SPACES

BRUCE A. BARNES

ABSTRACT. Let  $\mu$  be a  $\sigma$ -finite positive measure. Assume  $1 \leq p < s < \infty$ . Let  $T$  be a linear operator on  $L^p(\mu) \cap L^s(\mu)$  that has bounded extensions  $T_p$  and  $T_s$  on  $L^p(\mu)$  and  $L^s(\mu)$  respectively. Then  $T$  has a bounded extension  $T_r$  on  $L^r(\mu)$ ,  $p \leq r \leq s$ . The aim of this paper is to study the relationship between the spectral and Fredholm properties of the operator  $T_r$  and those of  $T_p$  and  $T_s$ .

**1. Introduction.** Let  $\mu$  be a fixed positive  $\sigma$ -finite measure, and let  $L^p = L^p(\mu)$  be the usual Lebesgue spaces relative to  $\mu$  for  $1 \leq p \leq \infty$ . Assume  $1 \leq p < s < \infty$ . Suppose  $T$  is a linear operator mapping  $L^p \cap L^s$  into itself such that  $T$  has bounded extensions  $T_p$  on  $L^p$  and  $T_s$  on  $L^s$ . Then the Riesz Convexity Theorem [7; Theorem 11, p. 525] implies that, for  $p < r < s$ ,  $T$  has a bounded extension  $T_r$  on  $L^r$  with

$$\|T_r\| \leq \max\{\|T_p\|, \|T_s\|\}.$$

Let  $\sigma(T)$  denote the spectrum of an operator  $T$ . It is not difficult to find examples where  $\sigma(T_r)$  is different for different  $r \in [p, s]$ ; see for example [6] or [10, pp. 328–329].

One aim of this paper is to deal with the following questions in the situation described above:

- (i) How does  $\sigma(T_r)$  relate to  $\sigma(T_p)$  and  $\sigma(T_s)$ ?
- (ii) If  $T_p$  and  $T_s$  are Fredholm operators, then under what conditions is  $T_r$  a Fredholm operator?
- (iii) How does the Fredholm spectrum and the Weyl spectrum of  $T_r$  relate to the same spectra of  $T_p$  and  $T_s$ ?

Some answers to these questions are given in §4 and §5. (The case where  $s = \infty$  is also included.) Question (i) has been considered by a number of mathematicians; see [2], [8], [9], and [15].

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These questions are more easily answered when  $\mu$  is a finite measure. We give a sample of the type of results obtained in this case. Assuming  $\mu$  is finite, if  $p < r < s$ , then  $\sigma(T_r) \subseteq \sigma(T_p) \cup \sigma(T_s)$ , and for  $u = r, p, s$  and  $v = r, p, s$  every component of  $\sigma(T_u)$  has nonempty intersection with  $\sigma(T_v)$ . Also, if  $T_p$  and  $T_s$  are Fredholm operators with the same index  $k$ , then  $T_r$  is Fredholm with index  $k$ . It follows from this that the Weyl spectrum of  $T_r$ , denoted  $W(T_r)$ , has the property

$$W(T_r) \subseteq W(T_p) \cup W(T_s).$$

Our methods for dealing with these questions involve Banach algebra theory applied to certain algebras of operators. In this regard, the results of the author's paper [3] are used frequently throughout. As a consequence of this method, our results go further than providing answers to questions (i)–(iii). In general we characterize when certain operators are in the underlying Banach algebra of operators. For example, in the finite measure case we prove more than the inclusion  $W(T_r) \subseteq W(T_p) \cup W(T_s)$ . In fact, we show that when  $T_k$  is Fredholm of index zero on  $L^k$  for  $k = p$  and  $s$ , then  $\exists R$  and  $\exists G$  linear maps on  $L^p \cap L^s$  such that  $T = R + G$ , and  $R_k$  is invertible in  $B(L^k)$  and  $G_k$  is of finite rank on  $L^k$  for  $k = p$  and  $s$ . Although perhaps not obvious, this latter result is much stronger than the inclusion result mentioned above.

**2. Notation:  $L_0^\infty$ .** Throughout  $\mu$  is a positive  $\sigma$ -finite measure defined on a  $\sigma$ -algebra of subsets of a set  $\Omega$ . When  $f$  and  $g$  are measurable functions on  $\Omega$  with  $fg \in L^1(\mu)$ , then the notation  $\langle f, g \rangle$  is defined by

$$\langle f, g \rangle = \int_{\Omega} fg d\mu.$$

The space  $L^p = L^p(\mu)$  is the usual Lebesgue space of (equivalence classes of) complex-valued measurable functions on  $\Omega$  with the usual norm  $\|f\|_p$ . Let  $L^{p,s} = L^p \cap L^s$  with norm

$$\|f\|_{p,s} = \max\{\|f\|_p, \|f\|_s\}.$$

Then  $(L^{p,s}, \|\cdot\|_{p,s})$  is a Banach space. As is well-known, when  $1 \leq p < s \leq \infty$  and  $p \leq r \leq s$ , then  $L^{p,s} \subseteq L^r$ . A linear operator  $T : L^{p,s} \rightarrow L^{p,s}$  is  $r$ -continuous when  $T$  is continuous on  $L^{p,s}$  with respect to the  $r$ -norm. In this case, assuming  $r < \infty$ ,  $T$  has a unique extension to a bounded

linear operator  $T_r$  on  $L^r$  (the case  $r = \infty$  is considered later). When  $T : L^{p,s} \rightarrow L^{p,s}$  as above, and  $T$  is bounded as an operator on the Banach space  $L^{p,s}$ , then we often write  $T_{p,s}$  for  $T$  when referring to properties of  $T$  as an operator on  $L^{p,s}$ .

Let  $X$  be a normed linear space. We denote the dual space of  $X$  by  $X'$ . Also,  $B(X)$  denotes the Banach algebra of all bounded linear operators mapping  $X$  into  $X$ . For  $T \in B(X)$ , let  $\|T\|_{\text{op}}$  be the operator norm of  $T$  (often the subscript “op” will be omitted). There is the usual bilinear form  $\langle \cdot, \cdot \rangle$  on  $X \times X'$  given by

$$\langle x, \alpha \rangle = \alpha(x), \quad x \in X, \alpha \in X'.$$

For  $T \in B(X)$ , let  $T' \in B(X')$  be the adjoint operator of  $T$ . Thus,

$$\langle Tx, \alpha \rangle = \langle x, T'\alpha \rangle, \quad x \in X, \alpha \in X'.$$

Assume  $\mathcal{B}$  is a Banach algebra of operators containing the identity operator  $I$ . Let  $\text{Inv}(\mathcal{B})$  be the group of invertible elements in  $\mathcal{B}$ . The spectrum relative to  $\mathcal{B}$  of an element  $T \in \mathcal{B}$  is denoted  $\sigma_{\mathcal{B}}(T)$ . Assume  $\mathcal{F}$  is a designated inessential ideal of  $\mathcal{B}$  [5, p. 42]. We use the notation  $\Phi(\mathcal{B})$  for the set of elements in  $\mathcal{B}$  which are invertible in  $\mathcal{B}$  modulo  $\mathcal{F}$ . The set  $\Phi^0(\mathcal{B})$  are those elements in  $\Phi(\mathcal{B})$  which have general index function zero [5, pp. 38–39]. When  $\mathcal{B} = B(X)$  and  $\mathcal{F}$  is the ideal in  $B(X)$  consisting of finite rank operators, then we use the notation  $\text{Inv}(X)$ ,  $\Phi(X)$ , and  $\Phi^0(X)$  for these same sets. Also, for  $T \in B(X)$ , the spectrum of  $T$  relative to  $B(X)$  is written simply as  $\sigma(T)$ . When  $T \in \Phi(X)$ ,  $\text{ind}(T)$  denotes the usual index of  $T$  on  $X$  (the nullity of  $T$  minus the defect of  $T$ ).

Now we define the Banach space  $L_0^\infty$  and look at some of the properties of this space. Let  $\mathcal{S}$  be the linear space of all simple functions in  $L^1(\mu)$ . Let  $L_0^\infty$  be the closure of  $\mathcal{S}$  in  $L^\infty$ . It follows from the definition that if  $1 \leq p < \infty$ , then the closure of  $L^{p,\infty}$  in  $L^\infty$  is  $L_0^\infty$ . In the case where  $\mu(\Omega) < \infty$ , the space  $L_0^\infty = L^\infty$ , but in general,  $L_0^\infty \subsetneq L^\infty$ .

Assuming  $1 \leq p < \infty$ , when  $T$  is a linear operator on  $L^{p,\infty}$  which is  $\infty$ -continuous, then  $T$  has a unique extension to a bounded linear operator  $T_\infty$  on  $L_0^\infty$ .

Next we note a property of  $L_0^\infty$  which is useful in what follows. The verification of this property is straightforward.

NOTE. For  $g \in L^1(\mu)$ ,

$$(2.1) \quad \|g\|_1 = \sup\{|\langle g, f \rangle| : f \in L_0^\infty, \|f\|_\infty \leq 1\}.$$

Now we prove a result which is used frequently later.

PROPOSITION 2.2. *Assume  $1 < s \leq \infty$ . Assume  $T : L^{1,s} \rightarrow L^{1,s}$  is a linear operator that is both 1-continuous and  $s$ -continuous. Then*

$$T'_1(L_0^\infty) \subseteq L_0^\infty.$$

PROOF. First assume  $s \neq \infty$  and  $t$  is the conjugate exponent of  $s$ . For  $f \in L^{1,s}$  and  $g \in L^{t,\infty}$ ,

$$\|T_s\| \|f\|_s \|g\|_t \geq |\langle Tf, g \rangle| = |\langle f, T'_1(g) \rangle|.$$

Therefore  $f \rightarrow \langle f, T'_1(g) \rangle$  has a unique extension to a continuous linear functional  $\alpha$  on  $L^s$ . There exists a unique  $h \in L^t$  such that

$$\alpha(f) = \langle f, h \rangle, \quad f \in L^s.$$

Thus  $T'_1(g) = h \in L^{t,\infty}$ . Now assume  $g \in L_0^\infty$ . Then  $\exists \{s_n\} \subseteq \mathcal{S}$  such that  $\|s_n - g\|_\infty \rightarrow 0$ . Since  $\mathcal{S} \subseteq L^{t,\infty}$ ,  $\{T'_1(s_n)\} \subseteq L^{t,\infty}$ . Also  $\|T'_1(s_n) - T'_1(g)\|_\infty \rightarrow 0$  and this implies  $T'_1(g) \in L_0^\infty$ .

Now suppose  $s = \infty$ . Fix  $r, 1 < r < \infty$ . Then  $T : L^{1,r} \rightarrow L^{1,r}$  and  $T$  is 1-continuous and  $r$ -continuous on  $L^{1,r}$ . Thus, it follows from the previous case that  $(T_1)'(L_0^\infty) \subseteq L_0^\infty$ .  $\square$

**3. Certain Banach algebras of operators.** Fix  $p$  and  $s, 1 \leq p < s \leq \infty$ . Let  $\mathcal{B}_{p,s}$  be the algebra of all linear operators  $T : L^{p,s} \rightarrow L^{p,s}$  such that  $T$  is both  $p$ -continuous and  $s$ -continuous on  $L^{p,s}$ . Then  $T$  has unique continuous extensions  $T_p \in B(L^p)$  and  $T_s \in B(L^s)$  ( $T_\infty \in B(L_0^\infty)$  when  $s = \infty$ ). The algebra  $\mathcal{B}_{p,s}$  is a Banach algebra in the norm

$$\|T\| = \max\{\|T_p\|, \|T_s\|\}.$$

PROPOSITION 3.1. *Assume  $1 \leq p < s \leq \infty$  and  $p < r < s$ . If  $T \in \mathcal{B}_{p,s}$ , then  $T$  has a unique extension  $T_r \in B(L^r)$ . Furthermore,*

$$\|T_r\| \leq \max\{\|T_p\|, \|T_s\|\}.$$

PROOF. When  $s \neq \infty$ , the proposition follows immediately from the Riesz Convexity Theorem [7; Theorem 11, p.525].

Now assume  $s = \infty$ . Fix  $\Gamma$  a measurable subset of  $\Omega$  such that  $\mu(\Gamma) < \infty$ . Let  $\tilde{\mu}$  be the restriction of  $\mu$  to  $\Gamma$ . When  $f \in L^u(\tilde{\mu}, \Gamma)$ , define  $f_e \in L^u(\mu, \Omega)$  by

$$f_e(\omega) = f(\omega) \quad \text{for } \omega \in \Gamma$$

and

$$f_e(\omega) = 0 \quad \text{for } \omega \notin \Gamma.$$

Define  $T_\Gamma : L^{p,\infty}(\tilde{\mu}) \rightarrow L^{p,\infty}(\mu)$  by  $T_\Gamma(f) = T(f_e)$  when  $f \in L^{p,\infty}(\tilde{\mu})$ . Then  $T_\Gamma$  has continuous extensions  $(T_\Gamma)_p : L^p(\tilde{\mu}) \rightarrow L^p(\mu)$  and  $(T_\Gamma)_\infty : L^\infty(\tilde{\mu}) \rightarrow L^\infty(\mu)$  (note here that  $L^{p,\infty}(\tilde{\mu}) = L^\infty(\tilde{\mu})$  since  $\tilde{\mu}(\Gamma) < \infty$ ). By the Riesz Convexity Theorem  $T_\Gamma$  has a continuous extension  $(T_\Gamma)_r : (L^r(\tilde{\mu}) \rightarrow L^r(\mu))$  with

$$\|(T_\Gamma)_r\| \leq \max\{\|(T_\Gamma)_p\|, \|(T_\Gamma)_\infty\|\} \leq M,$$

where  $M = \max\{\|T_p\|, \|T_\infty\|\}$ . If  $g \in \mathcal{S}$ , then  $g$  vanishes outside of some measurable set  $\Gamma$  with  $\mu(\Gamma) < \infty$ . Thus, when  $g \in \mathcal{S}$  and  $\|g\|_r \leq 1$ , we have  $\|T(g)\|_r = \|(T_\Gamma)_r(g)\|_r \leq M$ . Since  $\mathcal{S}$  is a dense subspace of  $L^r(\mu)$ , this implies  $T$  has a unique extension  $T_r \in B(L^r)$  with  $\|T_r\| \leq M$ .  $\square$

Fix  $1 \leq p < s \leq \infty$ . We shall always denote the conjugate exponent of  $p$  by  $q$  and the conjugate exponent of  $s$  by  $t$ . Thus  $p + q = pq$ , and when  $p = 1$ , then  $q = \infty$ .

If  $f \in L^{p,s}$  and  $g$  is contained in either  $L^q$  or  $L^t$ , then let  $g^* \otimes f$  denote the operator on  $L^{p,s}$  given by

$$(g^* \otimes f)(h) = \langle h, g \rangle f, \quad h \in L^{p,s}.$$

Clearly,  $g^* \otimes f \in \mathcal{B}_{p,s}$  when  $g \in L^{q,t}$ . Now define  $\mathcal{F}_{p,s}$  to be the set of all finite rank linear operators on  $L^{p,s}$  which are both  $p$ -continuous and  $s$ -continuous. Then  $\mathcal{F}_{p,s}$  is an ideal of  $\mathcal{B}_{p,s}$ . When  $s \neq \infty$ ,

$$\mathcal{F}_{p,s} = \text{span}\{g^* \otimes f : f \in L^{p,s}, g \in L^{q,t}\}.$$

When  $s = \infty$ , then  $\mathcal{F}_{p,\infty}$  is the span of operators of the form  $g^* \otimes f$ , where  $f \in L^{p,\infty}$  and  $g \in L^q$  with the property  $h \rightarrow \langle h, g \rangle$  is  $\infty$ -continuous on  $L^{p,\infty}$ . In every case,  $\mathcal{F}_{p,s}$  is the socle of  $\mathcal{B}_{p,s}$  [5, p. 106].

Following [5; Definition F.2.5, p. 31], we set  $\Phi(\mathcal{B}_{p,s})$  to be the set of all  $T \in \mathcal{B}_{p,s}$  such that  $\exists S \in \mathcal{B}_{p,s}$  and  $\exists F, G \in \mathcal{F}_{p,s}$  with

$$TS = I - F \quad \text{and} \quad ST = I - G.$$

Now we introduce another Banach algebra of operators which will prove useful in what follows. Let  $X$  and  $Y$  be Banach spaces and assume there is a nondegenerate bilinear form  $\langle \cdot, \cdot \rangle$  defined on  $X \times Y$  and  $\exists c > 0$  such that

$$|\langle x, y \rangle| \leq c \|x\| \|y\|, \quad x \in X, y \in Y.$$

Let  $\mathcal{A} = \mathcal{A}(X, Y)$  be the algebra of all  $T \in B(X)$  such that  $T$  has an adjoint  $T' \in B(Y)$  relative to the given bilinear form

$$\langle Tx, y \rangle = \langle x, T'y \rangle, \quad x \in X, y \in Y.$$

Then  $\mathcal{A}$  is a Banach algebra with norm

$$\|T\| = \max\{\|T\|_{\text{op}}, \|T'\|_{\text{op}}\}.$$

The algebra  $\mathcal{A}(X, Y)$  is used extensively by K. Jörgens in his book [10] to study linear integral operators. Also, these algebras and Fredholm theory relative to them is the focus of [3].

For  $x \in X, y \in Y$ , let  $y^* \otimes x$  be the operator in  $\mathcal{A}(X, Y)$  defined by

$$(y^* \otimes x)(z) = \langle z, y \rangle x, \quad z \in X.$$

Let  $\mathcal{F} = \mathcal{F}(X, Y)$  be the algebraic span of the set  $\{y^* \otimes x : y \in Y, x \in X\}$ . Then  $\mathcal{F}$  is an inessential ideal of  $\mathcal{A}$ .

**PROPOSITION 3.2.** (1) *Assume  $1 \leq p < s \leq \infty$ , and  $t \leq u \leq q, u \neq 1$ . Then  $\mathcal{B}_{p,s} \subseteq \mathcal{A}(L^{p,s}, L^u)$ .*

(2) *For  $1 < s \leq \infty, \mathcal{B}_{1,s} \subseteq \mathcal{A}(L^{1,s}, L_0^\infty)$ .*

(3) *For  $1 \leq p < \infty, \mathcal{B}_{p,\infty} \subseteq \mathcal{A}(L^{p,\infty}, (L_0^\infty)')$ .*

**PROOF.** Assume  $p, s$ , and  $u$  are as in (1). Let  $r$  be the conjugate index of  $u$ , so  $p \leq r \leq s, r \neq \infty$ . If  $f \in L^{p,s}$ , then  $f \in L^r$ . Let  $(L^{p,s}) \times L^u$  have the natural bilinear form

$$\langle f, g \rangle = \int_{\Omega} fg d\mu, \quad f \in L^{p,s}, g \in L^u.$$

Note that, for  $f \in L^{p,s}$ ,

$$\|f\|_r \leq \max\{\|f\|_p, \|f\|_s\} = \|f\|_{p,s}$$

by [7; Lemma 9, p. 524]. Thus, using Hölder's Inequality we have, for  $f \in L^{p,s}$ ,  $g \in L^u$ ,

$$|\langle f, g \rangle| \leq \|f\|_r \|g\|_u \leq \|f\|_{p,s} \|g\|_u.$$

Now assume  $T \in \mathcal{B}_{p,s}$ . Then  $T \in B(L^{p,s})$  since  $T$  is both  $p$ -continuous and  $s$ -continuous. For  $g \in L^u$ , define  $T'(g) = (T_r)'(g)$ . Then, for  $f \in L^{p,s}$  and  $g \in L^u$ ,  $\langle Tf, g \rangle = \langle f, T'(g) \rangle$ . Therefore  $T \in \mathcal{A}(L^{p,s}, L^u)$ .

Assume  $1 < s \leq \infty$ . Let  $(L^{1,s}) \times L_0^\infty$  have the natural bilinear form  $\langle f, g \rangle$  just as above. Assume  $T \in \mathcal{B}_{1,s}$ . Then  $T \in \mathcal{B}(L^{1,s})$ , and, by Proposition 2.2,  $T_1'(L_0^\infty) \subseteq L_0^\infty$ . Set  $T'(g) = T_1'(g)$  for  $g \in L_0^\infty$ . Then clearly

$$\langle Tf, g \rangle = \langle f, T'g \rangle, \quad f \in L^{1,s}, g \in L_0^\infty.$$

Therefore  $T \in \mathcal{A}(L^{1,s}, L_0^\infty)$ .

Now we verify (3). We use the natural bilinear form on  $L^{p,\infty} \times (L_0^\infty)'$  given by

$$\langle f, \alpha \rangle = \alpha(f), \quad f \in L^{p,\infty}, \alpha \in (L_0^\infty)'$$

(note that  $L^{p,\infty} \subseteq L_0^\infty$ ). For  $T \in \mathcal{B}_{p,\infty}$ , set  $T' = T'_\infty$ , so

$$\langle Tf, \alpha \rangle = \langle f, T'\alpha \rangle, \quad f \in L^{p,\infty}, \alpha \in (L_0^\infty)'.$$

Since  $T \in B(L^{p,\infty})$  and  $T' \in B((L_0^\infty)')$ , we have  $T \in \mathcal{A}(L^{p,\infty}, (L_0^\infty)')$ .  $\square$

The next result clarifies the role played by the Banach algebras  $\mathcal{B}_{p,s}$  in the study of properties of the operators  $T_r, r \in [p, s]$ .

**THEOREM 3.3.** *Assume  $1 \leq p < s \leq \infty$  and  $T \in \mathcal{B}_{p,s}$ . Assume  $p \leq r \leq s$ .*

- (1) *If  $T \in \text{Inv}(\mathcal{B}_{p,s})$ , then  $T_r \in \text{Inv}(L^r)$  ( $\text{Inv}(L_0^\infty)$  when  $r = \infty$ ).*
- (2) *If  $T \in \Phi(\mathcal{B}_{p,s})$ , then  $T_r \in \Phi(L^r)$  ( $\Phi(L_0^\infty)$  when  $r = \infty$ ) and  $\text{ind}(T_p) = \text{ind}(T_r) = \text{ind}(T_s) = \text{ind}(T_{p,s})$ . In addition, when  $p = 1$ ,  $(T_1'|L_0^\infty) \in \Phi(L_0^\infty)$  and  $\text{ind}(T_{1,s}) = -\text{ind}(T_1'|L_0^\infty)$ .*

PROOF. The assertion in (1) is elementary since  $T \rightarrow T_r$  is an algebra monomorphism of  $\mathcal{B}_{p,s}$  into  $B(L^r)$  ( $B(L_0^\infty)$  when  $r = \infty$ ).

To prove (2), suppose that  $T \in \Phi(\mathcal{B}_{p,s})$ . Then  $\exists S \in \mathcal{B}_{p,s}$  and  $\exists G, F \in \mathcal{F}_{p,s}$  such that  $ST = I - F$  and  $TS = I - G$ . Thus on  $L^r$  ( $L_0^\infty$  if  $r = \infty$ ),  $S_r T_r = I - F_r$  and  $T_r S_r \in I - G_r$ . Therefore  $T_r \in \Phi(L^r)$  ( $\Phi(L_0^\infty)$  if  $r = \infty$ ).

Now assume  $1 \leq r < \infty$ , and let  $u$  be the conjugate exponent of  $r$ . By Proposition 3.2  $\mathcal{B}_{p,s} \subseteq \mathcal{A} = \mathcal{A}(L^{p,s}, L^u)$ . Also,  $F, G \in \mathcal{F}(L^{p,s}, L^u)$ . It follows that  $T \in \Phi(\mathcal{A})$ . Therefore [3, Theorem 2.5 (3)] implies that  $T_{p,s} \in \Phi(L^{p,s})$ ,  $T'_r \in \Phi(L^u)$ , and  $\text{ind}(T_{p,s}) = -\text{ind}(T'_r)$ . By standard Fredholm theory [16; Theorem 4.1, p. 120],  $\text{ind}(T_r) = -\text{ind}(T'_r)$ . Thus  $\text{ind}(T_{p,s}) = \text{ind}(T_r)$  whenever  $p \leq r \leq s$ ,  $r \neq \infty$ .

Again, by Proposition 3.2  $\mathcal{B}_{1,s} \subseteq \mathcal{A} = \mathcal{A}(L^{1,s}, L_0^\infty)$  whenever  $1 < s \leq \infty$ . Also, for  $T \in \mathcal{B}_{1,s}$ , the adjoint  $T'$  of  $T$  on  $L_0^\infty$  is  $(T'_1|L_0^\infty)$ . Thus, when  $T \in \Phi(\mathcal{B}_{1,s})$ ,  $T \in \Phi(\mathcal{A})$ ; so by [3, Theorem 2.5 (3)],  $T_{1,s} \in \Phi(L^{1,s})$ ,  $(T'_1|L_0^\infty) \in \Phi(L_0^\infty)$ , and  $\text{ind}(T_{1,s}) = -\text{ind}(T'_1|L_0^\infty)$ .

Now assume  $1 \leq p < \infty$ . By Proposition 3.2(3),  $\mathcal{B}_{p,\infty} \subseteq \mathcal{A} = \mathcal{A}(L^{p,\infty}, (L_0^\infty)')$ . An argument analogous to the previous ones shows that in this case when  $T \in \Phi(\mathcal{B}_{p,\infty})$ , then  $T \in \Phi(\mathcal{A})$ , and thus,  $\text{ind}(T_{p,\infty}) = \text{ind}(T_\infty)$ .

These three cases complete the proof of (2).  $\square$

The final result of this section is an elementary proposition which proves useful later.

PROPOSITION 3.4. *Assume  $1 \leq p < s \leq \infty$ .*

- (1) *If  $R \in \mathcal{A}(L^{p,s}, L^t)$ , then  $R$  is  $s$ -continuous on  $L^{p,s}$ .*
- (2) *If  $R \in \mathcal{A}(L^{p,s}, L^q)$ , then  $R$  is  $p$ -continuous on  $L^{p,s}$ .*
- (3) *If  $R \in \mathcal{A}(L^{1,s}, L_0^\infty)$ , then  $R$  is 1-continuous on  $L^{1,s}$ .*

PROOF. The proofs of (1), (2) and (3) are similar. We prove

- (1). For  $f \in L^{p,s}, g \in L^t$ ,

$$|\langle Rf, g \rangle| = |\langle f, R'g \rangle| \leq \|f\|_s \|R'g\|_t \leq \|f\|_s \|R'\| \|g\|_t.$$

Taking the sup over  $\{g \in L^t : \|g\|_t \leq 1\}$  we have  $\|Rf\|_s \leq \|R'\| \|f\|_s$ .



The proof of (3) uses the equality in (2.1).  $\square$

**4. The situation when  $\mu$  is finite or special discrete.** In some important cases the algebra  $\mathcal{B}_{p,s}$  can be identified with one of the algebras  $\mathcal{A}(X, Y)$ . We consider this situation next.

Assume  $\Omega$  is the set of positive integers and  $\mu$  is a positive measure defined on the  $\sigma$ -algebra of all subsets of  $\Omega$ . Set  $\mu_k = \mu(\{k\})$  for  $k \in \Omega$ . We call  $\mu$  special discrete if the situation just described holds,  $\mu_k$  is finite for all  $k$ , and the set of numbers  $\{\mu_k : k \geq 1\}$  is bounded away from zero. We write  $\ell^p(\mu)$  in place of  $L^p(\mu)$  in this case. The notations  $c_0$  and  $\ell^1$  are reserved for the usual classical Banach spaces of sequences. Note that when  $\mu$  is special discrete, then  $c_0 = L^\infty(\mu)$ .

**THEOREM 4.1.**

- (1) Assume  $1 \leq p < s \leq \infty$ . If  $\mu$  is finite, then  $\mathcal{B}_{p,s} = \mathcal{A}(L^s, L^q)$ .
- (2) Assume  $1 \leq p < s < \infty$ . If  $\mu$  is special discrete, then  $\mathcal{B}_{p,s} = \mathcal{A}(\ell^p(\mu), \ell^t(\mu))$ .
- (3) Assume  $1 \leq p < \infty$ . If  $\mu$  is special discrete, then  $\mathcal{B}_{p,\infty} = \mathcal{A}(\ell^p(\mu), \ell^1)$ .

**PROOF.** When  $\mu$  is finite, then  $L^\infty \subseteq L^s \subseteq L^p \subseteq L^1$ . Thus,  $L^{p,s} = L^s$ . By Proposition 3.2  $\mathcal{B}_{p,s} \subseteq \mathcal{A} = \mathcal{A}(L^{p,s}, L^q) = \mathcal{A}(L^s, L^q)$ . Now assume  $T \in \mathcal{A}(L^s, L^q)$ . Certainly  $T$  is  $s$ -continuous on  $L^{p,s} = L^s$ . But also, from Proposition 3.4,  $T$  is  $p$ -continuous on  $L^s$ . It follows that  $T \in \mathcal{B}_{p,s} = \mathcal{B}$ . Also note, for  $T \in \mathcal{B}$ ,

$$\|T\|_{\mathcal{B}} = \max\{\|T_s\|, \|T_p\|\} = \max\{\|T_s\|, \|T'_p\|\} = \|T\|_{\mathcal{A}}.$$

This proves (1).

The proof of (2) is essentially the same using the fact that when  $\mu$  is special discrete and  $1 \leq p < s \leq \infty$ , then  $\ell^1(\mu) \subseteq \ell^p(\mu) \subseteq \ell^s(\mu) \subseteq c_0$ .

Again, the proof of (3) is similar, but we outline it. Define the natural bilinear form on  $\ell^p(\mu) \times \ell^1$  by

$$\langle a, b \rangle = \sum_{k=1}^{\infty} a_k b_k \mu_k, \quad a = \{a_k\} \in \ell^p(\mu), b = \{b_k\} \in \ell^1.$$

If  $T \in \mathcal{B}_{p,\infty}$ , then  $T_\infty$  is defined on  $c_0$ , so  $T'_\infty$  is defined on  $c'_0 = \ell^1$ . Setting  $T'(b) = T'_\infty(b)$  for  $b \in \ell^1$ , we have

$$\langle Ta, b \rangle = \langle a, T'b \rangle, \quad a \in \ell^p(\mu), b \in \ell^1.$$

Thus,  $\mathcal{B}_{p,\infty} \subseteq \mathcal{A}(\ell^p(\mu), \ell^1)$ . If  $T \in \mathcal{A}(\ell^p(\mu), \ell^1)$ , then  $T$  is  $p$ -continuous on  $\ell^p(\mu)$ . An argument similar to the proof of Proposition 3.4 shows that  $T$  is also  $\infty$ -continuous on  $\ell^p(\mu)$ . This proves  $\mathcal{B}_{p,\infty} = \mathcal{A}(\ell^p(\mu), \ell^1)$ .  $\square$

Having determined in the cases under consideration that  $\mathcal{B}_{p,s}$  is of the form  $\mathcal{A}(X, Y)$ , we can use the spectral and Fredholm theory of these latter algebras as developed in [3]. Thus, the next theorem is an immediate application of [3, Theorem 2.5], using Theorem 4.1 and standard properties of the adjoint operator [16; Theorem 4.1, p. 120].

**THEOREM 4.2.** *Assume that  $\mu$  is either a finite or a special discrete measure. Assume  $1 \leq p < s \leq \infty$  and  $T \in \mathcal{B}_{p,s}$ .*

(1)  *$T \in \text{Inv}(\mathcal{B}_{p,s})$  if and only if  $T_p \in \text{Inv}(L^p)$  and  $T_s \in \text{Inv}(L^s)$  ( $\text{Inv}(c_0)$  when  $\mu$  is special discrete and  $s = \infty$ ).*

(2)  *$T \in \Phi^0(\mathcal{B}_{p,s})$  if and only if  $T_p \in \Phi^0(L^p)$  and  $T_s \in \Phi^0(L^s)$  ( $\Phi^0(c_0)$  when  $\mu$  is special discrete and  $s = \infty$ ).*

(3)  *$T \in \Phi(\mathcal{B}_{p,s})$  if and only if  $T_p \in \Phi(L^p)$ ,  $T_s \in \Phi(L^s)$  ( $\Phi(c_0)$  when  $\mu$  is special discrete and  $s = \infty$ ), and  $\text{ind}(T_p) = \text{ind}(T_s)$ .*

For an operator  $T \in \mathcal{B}(X)$  let

$$\omega(T) = \{\lambda \in \mathbf{C} : (\lambda I - T) \notin \Phi(X)\},$$

and

$$W(T) = \{\lambda \in \mathbf{C} : (\lambda I - T) \notin \Phi^0(X)\}.$$

The set  $\omega(T)$  is the Fredholm spectrum of  $T$ , and  $W(T)$  is the Weyl spectrum of  $T$ .

**THEOREM 4.3.** *Assume  $\mu$  is either a finite measure or a special discrete measure. Assume  $1 \leq p < s \leq \infty$ , and  $p < r < s$ . Suppose  $T \in \mathcal{B}_{p,s}$ .*

(1)  $\sigma(T_r) \subseteq \sigma(T_p) \cup \sigma(T_s)$ ;

- (2)  $W(T_r) \subseteq W(T_p) \cup W(T_s)$ ;
- (3)  $\omega(T_r) \subseteq \omega(T_p) \cup \omega(T_s) \cup \omega_0$

where  $\omega_0 = \{\lambda \notin \omega(T_p) \cup \omega(T_s) : \text{ind}(\lambda I - T_p) \neq \text{ind}(\lambda I - T_s)\}$ .

PROOF. This result is a direct application of Theorem 4.2 and Theorem 3.3.  $\square$

A much stronger conclusion than that in Theorem 4.3(1) holds in the special case where  $T_2$  exists and is selfadjoint,  $T_2^* = T_2$ .

**THEOREM 4.4.** *Assume that  $\mu$  is either a finite or a special discrete measure. Suppose  $2 < s \leq \infty, T \in \mathcal{B}_{2,s}$ , and  $T_2 = T_2^*$ . If  $2 \leq r \leq v \leq s$  then  $\sigma(T_r) \subseteq \sigma(T_v)$ . Suppose  $1 \leq p < 2, T \in \mathcal{B}_{p,2}$ , and  $T_2 = T_2^*$ . If  $p \leq v \leq r \leq 2$ , then  $\sigma(T_r) \subseteq \sigma(T_v)$ .*

PROOF. We give the proof when  $\mu$  is a finite measure. The proof when  $\mu$  is special discrete is similar. Note that in the first case to prove that  $\sigma(T_r) \subseteq \sigma(T_v)$  it suffices to prove  $\sigma(T_r) \subseteq \sigma(T_s)$  (since  $T \in \mathcal{B}_{2,s} \subseteq \mathcal{B}_{2,v}$ ). Similarly, in the second case it suffices to prove  $\sigma(T_r) \subseteq \sigma(T_p)$ .

Assume  $2 < s \leq \infty, T \in \mathcal{B}_{2,s}$ , and  $T_2 = T_2^*$ . Since  $\mu$  is finite  $L^s \subseteq L^2$ ,  $L^s$  is an inner product space with

$$|(f, g)| = \left| \int_{\Omega} f \bar{g} d\mu \right| \leq \|f\|_2 \|g\|_2 \leq c \|f\|_s \|g\|_s$$

for some  $c > 0$ , for all  $f, g \in L^s$ . In this situation a result of P. Lax [12] implies that  $\sigma(T_2) \subseteq \sigma(T_s)$ . Now suppose  $2 \leq r \leq s$ . By Theorem 4.3,  $\sigma(T_r) \subseteq \sigma(T_2) \cup \sigma(T_s) = \sigma(T_s)$ .

Assume  $1 \leq p < 2, T \in \mathcal{B}_{p,s}$ , and  $T_2 = T_2^*$ . For  $f \in L^q$ ,

$$\begin{aligned} \|T'_p(f)\|_2 &= \sup\{|\langle g, T'_p f \rangle| : g \in L^{p,2}, \|g\|_2 \leq 1\} \\ &= \sup\{|\langle T_p g, f \rangle| : g \in L^{p,2}, \|g\|_2 \leq 1\} \leq \|T_2\| \|f\|_2. \end{aligned}$$

Thus,  $T'_p$  is 2-continuous on  $L^q$ . This implies  $T'_p \in \mathcal{B}_{2,q}$ .

Next we verify that  $T'_2$  is selfadjoint. For  $f, g \in L^2$ ,

$$\begin{aligned} (T'_2(f), g) &= \langle T'_2(f), \bar{g} \rangle = \langle f, T_2(\bar{g}) \rangle = (\bar{f}, T_2(\bar{g}))^- = (T_2(\bar{f}), \bar{g})^- \\ &= \langle T_2(\bar{f}), g \rangle^- = \langle \bar{f}, T'_2(g) \rangle^- = (f, T'_2(g)). \end{aligned}$$

Assume  $p \leq r \leq 2$ . Let  $u$  be the conjugate exponent of  $r$ . Now  $T'_r$  coincides with  $T'_p$  on  $L^q$ . It follows that  $(T'_p)_u = T'_r$ . We have  $2 \leq u \leq q$ ,  $T'_p \in \mathcal{B}_{2,q}$ , and  $T'_2 = (T'_p)_2$  is selfadjoint. This implies by the previous case that  $\sigma((T'_p)_u) \subseteq \sigma(T'_p)$ . Thus  $\sigma(T'_r) \subseteq \sigma(T'_p)$ , so  $\sigma(T_r) \subseteq \sigma(T_p)$ .  $\square$

Next we prove a general Banach algebra result which is a small generalization of [1; Proposition 2.2, p. 276]. Here  $A$  and  $B$  are unital Banach algebras. It is easy to verify that when  $\varphi : B \rightarrow A$  is a unital (algebra) monomorphism, then

$$\sigma_A(\varphi(b)) \subseteq \sigma_B(b), \quad b \in B.$$

**THEOREM 4.5.** *Assume that  $\varphi : B \rightarrow A$  is a continuous unital algebra monomorphism (or anti-monomorphism). If  $b \in B$  and  $\Delta$  is a component of  $\sigma_B(b)$ , then  $\Delta \cap \sigma_A(\varphi(b))$  is nonempty.*

**PROOF.** First suppose that  $\Delta$  is a nonempty open and closed subset of  $\sigma_B(b)$ . If  $\Delta \cap \sigma_A(\varphi(b))$  is empty, then  $\sigma_B(b) = \Delta \cup \Gamma$  where  $\Gamma$  is open and closed in  $\sigma_B(b)$ ,  $\Delta$  and  $\Gamma$  are disjoint, and  $\sigma_A(\varphi(b)) \subseteq \Gamma$ . Choose  $U$  and  $V$  disjoint open subsets of  $\mathbf{C}$  with  $\Delta \subseteq V$  and  $\Gamma \subseteq U$ . Define

$$f(\lambda) = \begin{cases} 1, & \lambda \in U, \\ 0, & \lambda \in V, \end{cases}$$

$$g(\lambda) = 1 - f(\lambda), \quad \lambda \in U \cup V.$$

Using the usual operational calculus in a Banach algebra, we have  $f(b), g(b) \in B$  with  $1 = f(b) + g(b)$ . Note that  $g(b) \neq 0$  since  $\Delta$  is nonempty. Also,  $f(\varphi(b)) \in A$ , and since  $\varphi$  is continuous,  $\varphi(f(b)) = f(\varphi(b))$ . But as  $\sigma_A(\varphi(b)) \subseteq \Gamma$  and  $f \equiv 1$  on  $\Gamma$ , it follows that  $f(\varphi(b))$  is the unit of  $A$ . Thus,  $f(b) = 1$ , a contradiction.

Now assume that  $\Delta$  is a component of  $\sigma_B(b)$ . Suppose  $\Delta \cap \sigma_A(\varphi(b))$  is empty. Let  $\varepsilon = \text{dist}(\Delta, \sigma_A(\varphi(b))) > 0$ . By [13, Corollary 1, p. 83] there exists an open and closed subset  $\Omega$  of  $\sigma_B(b)$  such that  $\Delta \subseteq \Omega$  and  $\text{dist}(\omega, \Delta) < \varepsilon$  for all  $\omega \in \Omega$ . By the previous argument  $\Omega \cap \sigma_A(\varphi(b))$  is nonempty, a contradiction.  $\square$

Now we apply Theorem 4.5 in our situation.

**THEOREM 4.6.** *Assume  $\mu(\Omega) < +\infty$  or  $\mu$  is special discrete. Fix  $p$  and  $s$  with  $1 \leq p < s \leq \infty$ . Assume  $p \leq r \leq s$ . If  $T \in \mathcal{B}_{p,s}$ , then every component of  $\sigma(T_r)$  has nonempty intersection with both  $\sigma(T_p)$  and  $\sigma(T_s)$ . A component of either  $\sigma(T_p)$  or  $\sigma(T_s)$  has nonempty intersection with  $\sigma(T_r)$ .*

**PROOF.** Assume  $T \in \mathcal{B}_{p,s}$ . Suppose  $r \neq s$ , so that  $T \in \mathcal{B} = \mathcal{B}_{r,s}$ . By Theorem 4.2(1),  $\sigma_{\mathcal{B}}(T) = \sigma(T_r) \cup \sigma(T_s)$ . Suppose  $\Delta$  is a component of  $\sigma(T_r)$ . If  $\Delta \cap \sigma(T_s)$  is empty, then  $\Delta$  is a component of  $\sigma_{\mathcal{B}}(T)$ . But since the embedding  $\varphi : \mathcal{B} \rightarrow B(L^s)$  ( $B(L_0^\infty)$  if  $s = \infty$ ) given by  $\varphi(T) = T_s$  is a continuous monomorphism, this contradicts Theorem 4.4. Thus,  $\Delta \cap \sigma(T_s)$  must be nonempty. Now suppose  $\Delta$  is a component of  $\sigma(T_s)$ . The same argument as above, interchanging the roles of  $r$  and  $s$ , proves that  $\Delta \cap \sigma(T_r)$  must be nonempty.

Similar arguments prove the remaining assertions of the theorem.  $\square$

**COROLLARY 4.7.** *If  $\lambda_0$  is an isolated point of either  $\sigma(T_p)$  or  $\sigma(T_s)$ , then  $\lambda_0 \in \sigma(T_r)$  whenever  $p \leq r \leq s$ . Thus, if the isolated points of  $\sigma(T_j)$  are dense in  $\sigma(T_j)$  for  $j = p$  and  $j = s$ , then  $\sigma(T_p) = \sigma(T_r) = \sigma(T_s)$  whenever  $p \leq r \leq s$ .*

*If  $\sigma(T_p)$  and  $\sigma(T_s)$  are totally disconnected, then  $\sigma(T_p) = \sigma(T_r) = \sigma(T_s)$  whenever  $p \leq r \leq s$ .*

Results similar to Theorem 4.6 and Corollary 4.7 are proved in [2, 8, and 15]. A stronger result than Corollary 4.7 can be derived by using [9, Proposition 7.1].

Assume that  $\mu$  is either finite or special discrete. Suppose  $T \in \mathcal{B}_{p,s}$ . The next result implies that  $\text{ind}(T_r)$  (for  $r$  where this makes sense) is monotone in  $r$  on the interval  $[p, s]$ .

**THEOREM 4.8.** *Assume  $X$  and  $Y$  are Banach spaces, and  $X$  is a dense subspace of  $Y$  with  $X$  continuously embedded in  $Y$ . Assume  $T \in B(Y)$  and  $T(X) \subseteq X$ . Denote the restriction of  $T$  to  $X$  by  $T_r$ . Then  $T_r \in B(X)$ . If  $T \in \Phi(Y)$  and  $T_r \in \Phi(X)$ , then  $\text{ind}(T_r) \leq \text{ind}(T)$ .*

**PROOF.** It is easy to verify that  $T_r$  is closed on  $X$ , and hence

$T_r \in B(X)$ .

Since  $\mathcal{N}(T_r) \subseteq \mathcal{N}(T)$ , it follows that  $\text{nul}(T_r) \leq \text{nul}(T)$ . Now assume  $\{\alpha_1, \dots, \alpha_n\}$  is a linearly independent subset of  $\mathcal{N}(T)$ . Then

$$\alpha_k(T(Y)) = \{0\}, \quad 1 \leq k \leq n.$$

Also,  $\alpha_k \in X'$  for all  $k$  since  $X$  is continuously embedded in  $Y$ . Because  $X$  is dense in  $Y$ ,  $\{\alpha_1, \dots, \alpha_n\}$  is a linearly independent subset of  $X'$ . Finally,  $\alpha_k(T_r(X)) = \{0\}$ ,  $1 \leq k \leq n$ , and it follows that  $\text{def}(T_r) \geq \text{def}(T)$ . This proves  $\text{ind}(T_r) \leq \text{ind}(T)$ .  $\square$

The following corollary is an immediate consequence of the theorem.

**COROLLARY 4.9.** *Assume  $1 \leq p < s \leq \infty$  and  $T \in \mathcal{B}_{p,s}$ . Suppose  $p \leq r \leq u \leq s$  and  $T_r \in \Phi(L^r)$ ,  $T_u \in \Phi(L^u)$  ( $T_u \in \Phi(c_0)$  when  $u = \infty$  and  $\mu$  is special discrete).*

- (1) *When  $\mu$  is finite, then  $\text{ind}(T_u) \leq \text{ind}(T_r)$ .*
- (2) *When  $\mu$  is special discrete, then  $\text{ind}(T_r) \leq \text{ind}(T_u)$ .*

**5. The general situation.** In general the spectral and Fredholm theory of an operator  $T \in \mathcal{B}_{p,s}$  is more complicated than that presented in §4. In fact, the properties of  $T_p, T_s$ , and also of  $T_{p,s}$ , are involved in the general theory. This is already clear in the following result.

**THEOREM 5.1.** *Assume  $1 \leq p < s \leq \infty$  and  $T \in \mathcal{B} = \mathcal{B}_{p,s}$ .*

(1)  *$T \in \text{Inv}(\mathcal{B}_{p,s})$  if and only if  $T_p \in \text{Inv}(L^p)$ ,  $T_s \in \text{Inv}(L^s)$  ( $\text{Inv}(L_0^\infty)$  when  $s = \infty$ ), and  $T_{p,s} \in \text{Inv}(L^{p,s})$ .*

(2)  $\sigma_{\mathcal{B}}(T) = \sigma(T_p) \cup \sigma(T_s) \cup \sigma(T_{p,s})$ .

(3) *Let  $AP\sigma(T_{p,s})$  denote the approximate point spectrum of  $T_{p,s}$ . Then*

$$\partial\sigma(T_{p,s}) \subseteq AP\sigma(T_{p,s}) \subseteq \sigma(T_p) \cup \sigma(T_s).$$

(4)  $\partial\sigma_{\mathcal{B}}(T) \subseteq \sigma(T_p) \cup \sigma(T_s) \subseteq \sigma_{\mathcal{B}}(T)$ .

**PROOF.** The “only if” direction in (1) is easy to verify. Assume  $T_p, T_s$ , and  $T_{p,s}$  are invertible. To prove that  $T \in \text{Inv}(\mathcal{B}_{p,s})$  it suffices to show

that  $(T_{p,s})^{-1}$  is  $p$ -continuous and  $s$ -continuous on  $L^{p,s}$ . Since  $T_{p,s}$  is invertible,  $T$  maps  $L^{p,s}$  onto  $L^{p,s}$ . Then  $T_p^{-1}, T_s^{-1}$  and  $(T_{p,s})^{-1}$  coincide on  $L^{p,s}$  which proves the desired continuity property.

(2) follows immediately from (1).

To prove (3), assume  $\lambda \in AP\sigma(T_{p,s})$ . Then  $\exists \{f_n\} \subseteq L^{p,s}$  with  $\|f_n\|_{p,s} = \max(\|f_n\|_p, \|f_n\|_s) = 1$  for  $n \geq 1$  and  $\|(\lambda I - T_{p,s})f_n\|_{p,s} \rightarrow 0$ . There is a subsequence  $\{g_n\}$  of  $\{f_n\}$  such that  $\|g_n\|_k = 1, n \geq 1$ , for  $k$  either  $p$  or  $s$ . Suppose  $\|g_n\|_p = 1$  for all  $n$ . Then  $\|(\lambda I - T_p)g_n\|_p \leq \|(\lambda I - T_{p,s})g_n\|_{p,s} \rightarrow 0$ . Thus  $\lambda \in AP\sigma(T_p)$ . This proves

$$AP\sigma(T_{p,s}) \subseteq \sigma(T_p) \cup \sigma(T_s).$$

It is well-known that  $\partial\sigma(T_{p,s}) \subseteq AP\sigma(T_{p,s})$  [11, Theorem 4.1, p. 282]. This proves (3).

Finally, we prove (4). By (1) we have  $\sigma_{\mathcal{B}}(T) = \sigma(T_p) \cup \sigma(T_s) \cup \sigma(T_{p,s})$ . Therefore using (3) we have  $\partial\sigma_{\mathcal{B}}(T) \subseteq \partial\sigma(T_p) \cup \partial\sigma(T_s) \cup \partial\sigma(T_{p,s}) \subseteq \sigma(T_p) \cup \sigma(T_s)$ .  $\square$

**COROLLARY 5.2.** *Assume  $1 \leq p < s \leq \infty$  and  $T \in \mathcal{B}_{p,s}$ . Assume  $p \leq r \leq s$ . If  $\sigma(T_p)$  and  $\sigma(T_s)$  are totally disconnected, then  $\sigma(T_r) = \sigma(T_p) = \sigma(T_s)$ .*

**PROOF.** By Theorem 5.1(3)  $\partial\sigma(T_{p,s}) \subseteq \sigma(T_p) \cup \sigma(T_s)$  which is a totally disconnected set. Therefore  $\partial\sigma(T_{p,s})$ , and hence  $\sigma(T_{p,s})$ , is totally disconnected. Then  $\sigma_{\mathcal{B}}(T)$  is totally disconnected by Theorem 5.1(2). Now  $T \rightarrow T_r$  is a continuous monomorphism of  $\mathcal{B}_{p,s}$  into  $B(L^r)$  ( $B(L_0^\infty)$  when  $r = \infty$ ). It follows from Theorem 4.5 that  $\sigma(T_r) = \sigma_{\mathcal{B}}(T)$  for all  $r \in [p, s]$ .  $\square$

Corollary 5.2 also follows from results of D. Herrero [9] or H. Schaefer, [16].

For  $K$  a compact subset of  $\mathbf{C}$ , let  $\hat{K}$  denote the polynomial convex hull of  $K$ ; see [18, p. 23]. From the definition it is easy to see that when  $K$  and  $J$  are compact subsets of  $\mathbf{C}$  with  $\partial K \subseteq J \subseteq K$ , then  $\hat{K} = \hat{J}$ .

We have the following corollary of Theorem 5.1.

**THEOREM 5.3.** *Assume  $1 \leq p < s \leq \infty$  and  $p < r < s$ . Assume*

$T \in \mathcal{B}_{p,s}$ . Then

$$\sigma(T_r) \subseteq \sigma(T_p) \cup \sigma(T_s) \cup \sigma(T_{p,s}) \quad \text{and} \quad \sigma(T_r) \subseteq [\sigma(T_p) \cup \sigma(T_s)]^\wedge.$$

PROOF. Clearly  $\sigma(T_r) \subseteq \sigma_{\mathcal{B}}(T)$ , so the first inclusion follows from Theorem 5.1(2). But also, by Theorem 5.1(4),

$$\sigma(T_r) \subseteq [\sigma_{\mathcal{B}}(T)]^\wedge = [\sigma(T_p) \cup \sigma(T_s)]^\wedge. \quad \square$$

Theorem 5.3 can also be derived from [17, Lemma 1.7].

Now we prove a characterization of  $\Phi^0(\mathcal{B}_{p,s})$  which has as an application (Corollary 5.5) an inclusion relation for the Weyl spectrum of  $T_r$ ,  $r \in [p, s]$ .

THEOREM 5.4. Assume  $1 \leq p < s \leq \infty$ , and  $T \in \mathcal{B}_{p,s}$ .

(1) When either  $p \neq 1$  or  $s \neq \infty$ , then the following are equivalent:

- (i)  $T \in \Phi^0(\mathcal{B}_{p,s})$ ;
- (ii)  $T_p \in \Phi^0(L^p)$ ,  $T_s \in \Phi^0(L^s)$  ( $\Phi^0(L_0^\infty)$  when  $s = \infty$ ), and  $T_{p,s} \in \Phi^0(L^{p,s})$ ;
- (iii)  $\exists R \in \text{Inv}(\mathcal{B}_{p,s})$  and  $\exists G \in \mathcal{F}_{p,s}$  such that  $T = R + G$ .

(2) When  $p = 1$  and  $s = \infty$ , then (i) and (iii) are equivalent to (ii)'  $T_1 \in \Phi^0(L^1)$ ,  $(T_1|L_0^\infty) \in \Phi^0(L_0^\infty)$ ,  $T_\infty \in \Phi^0(L_0^\infty)$ , and  $T_{1,\infty} \in \Phi^0(L^{1,\infty})$ .

PROOF. Assume  $s \neq \infty$ . First we prove that (ii)  $\Rightarrow$  (iii) in this case. By Proposition 3.2,  $\mathcal{B}_{p,s} \subseteq \mathcal{A} = \mathcal{A}(L^{p,s}, L^t)$ . By hypothesis,  $T_{p,s} \in \Phi^0(L^{p,s})$  and  $T_s \in \Phi^0(L^s)$ . By [15; Theorem 4.1, p. 120],  $T_s' \in \Phi^0(L^t)$ . It follows from [3, Corollary 2.6] that  $T \in \Phi^0(\mathcal{A})$  and that  $\exists S \in \text{Inv}(\mathcal{A})$  and  $\exists F \in \mathcal{F}(L^{p,s}, L^t)$  such that  $T = S + F$ . Now  $F$  has the form,  $F = \sum_{k=1}^n h_k \otimes f_k$  where  $f_k \in L^{p,s}$  and  $h_k \in L^t$ ,  $1 \leq k \leq n$ . Given  $\varepsilon > 0$  we can choose  $G$  of the form  $G = \sum_{k=1}^n g_k \otimes f_k$ , where  $g_k \in L^{q,t}$  such that  $\|F - G\|_{\mathcal{A}} < \varepsilon$  (this is possible since  $L^{q,t}$  is dense in  $L^t$  and  $\|F - G\|_{\mathcal{A}} \leq \sum_{k=1}^n \|h_k - g_k\|_t \|f_k\|_{p,s}$ ). Choose such a  $G$  with  $\|F - G\|_{\mathcal{A}}$  sufficiently small that  $T - G \in \text{Inv}(\mathcal{A})$ . Set  $R = T - G$ , so  $T = R + G$ . Note that  $G \in \mathcal{F}_{p,s}$  as required. Also,  $R$  is  $p$ -continuous and  $s$ -continuous



on  $L^{p,s}$ . It remains to be shown that  $R \in \text{Inv}(\mathcal{B}_{p,s})$ . Since  $R \in \text{Inv}(\mathcal{A})$ , then, by [3, Theorem 2.5],  $R_{p,s} \in \text{Inv}(L^{p,s})$  and  $R_s \in \text{Inv}(L^s)$  (since  $R'_s \in \text{Inv}(L^t)$ ). Now  $R_p = T_p - G_p \in \Phi^0(L^p)$  and  $R_p$  maps  $L^{p,s}$  onto  $L^{p,s}$ . This implies  $R_p \in \text{Inv}(L^p)$ . Therefore, by Theorem 5.1,  $R \in \text{Inv}(\mathcal{B}_{p,s})$ . This completes the proof that (ii)  $\Rightarrow$  (iii) when  $s \neq \infty$ .

Now assume  $p \neq 1, s = \infty$ . Then, by Proposition 3.2,  $\mathcal{B}_{p,\infty} \subseteq \mathcal{A} = \mathcal{A}(L^{p,\infty}, L^q)$ . The argument proceeds just as in the previous case to show that  $T \in \Phi^0(\mathcal{A})$ , so that  $T = S + F$  where  $S \in \text{Inv}(\mathcal{A})$  and  $F \in \mathcal{F}(L^{p,\infty}, L^q)$ . Again choose  $G \in \mathcal{F}_{p,\infty}$  such that  $R = T - G \in \text{Inv}(\mathcal{A})$ . The proof that  $R \in \text{Inv}(\mathcal{B}_{p,\infty})$  proceeds as in the previous argument.

Next we prove that when  $p = 1$  and  $s = \infty$ , then (ii)'  $\Rightarrow$  (iii). In fact, the argument is similar to the two previous arguments. In this case  $\mathcal{B}_{1,\infty} \subseteq \mathcal{A} = \mathcal{A}(L^{1,\infty}, L_0^\infty)$  by Proposition 3.2. By hypothesis  $T_{1,\infty} \in \Phi^0(L^{1,\infty})$  and  $(T'_1|L_0^\infty) \in \Phi^0(L_0^\infty)$ . Thus, by [3, Corollary 2.6],  $T \in \Phi^0(\mathcal{A})$  and  $\exists S \in \text{Inv}(\mathcal{A})$  and  $F \in \mathcal{F}(L^{1,\infty}, L_0^\infty)$  such that  $T = S + F$ . Then choose  $G \in \mathcal{F}(L^{1,\infty}, L^{1,\infty})$  such that  $\|F - G\|_{\mathcal{A}}$  is sufficiently small that  $R = T - G \in \text{Inv}(\mathcal{A})$ . Thus  $T = R + G$  and  $G \in \mathcal{F}_{1,\infty}$ . By construction  $R$  is both 1-continuous and  $\infty$ -continuous on  $L^{1,\infty}$ , and so  $R \in \mathcal{B}_{1,\infty}$ . Now  $R_{1,\infty} \in \text{Inv}(L^{1,\infty})$  (since  $R \in \text{Inv}(\mathcal{A})$ ). By hypothesis  $R_1 = T_1 - G_1 \in \Phi^0(L^1)$  and  $R_\infty = T_\infty - G_\infty \in \Phi^0(L_0^\infty)$ . Since  $R$  maps  $L^{1,\infty}$  onto  $L^{1,\infty}$ , we have  $R_1 \in \text{Inv}(L^1)$  and  $R_\infty \in \text{Inv}(L_0^\infty)$ . Therefore, by Theorem 5.1,  $R \in \text{Inv}(\mathcal{B}_{1,\infty})$ .

Now (i) and (iii) are equivalent by [5; Theorem F.2.11, p. 33]. Finally, when (iii) holds,  $T_p = R_p + G_p$  where  $R_p \in \text{Inv}(L^p)$  and  $G_p$  has finite dimensional range in  $L^p$ . Thus,  $T_p \in \Phi^0(L^p)$ . Similarly,  $T_s \in \Phi^0(L^s)$  ( $\Phi^0(L_0^\infty)$  when  $s = \infty$ ) and  $T_{p,s} \in \Phi^0(L^{p,s})$ .  $\square$

**COROLLARY 5.5.** *Assume  $1 \leq p < s \leq \infty$  with either  $p \neq 1$  or  $s \neq \infty$  and  $p < r < s$ . Assume  $T \in \mathcal{B}_{p,s}$ . If  $T_p \in \Phi^0(L^p), T_s \in \Phi^0(L^s)$  ( $\Phi^0(L_0^\infty)$  when  $s = \infty$ ) and  $T_{p,s} \in \Phi^0(L^{p,s})$ , then  $T_r \in \Phi^0(L^r)$ . Thus,*

$$W(T_r) \subseteq W(T_p) \cup W(T_s) \cup W(T_{p,s}).$$

Finally, we look at the general Fredholm properties of an operator  $T \in \mathcal{B}_{p,s}$ .

**THEOREM 5.6.** *Assume  $1 \leq p < s \leq \infty$ .*

(1) Assume either  $s \neq \infty$  or  $p \neq 1$ . Then  $T \in \Phi(\mathcal{B}_{p,s})$  if and only if  $T_p \in \Phi(L^p), T_s \in \Phi(L^s)$  ( $\Phi(L_0^\infty)$  if  $s = \infty$ ),  $T_{p,s} \in \Phi(L^{p,s})$ , and  $\text{ind}(T_p) = \text{ind}(T_s) = \text{ind}(T_{p,s})$ .

(2)  $T \in \Phi(\mathcal{B}_{1,\infty})$  if and only if  $T_1 \in \Phi(L^1), (T'_1|L_0^\infty) \in \Phi(L_0^\infty), T_\infty \in \Phi(L_0^\infty), T_{1,\infty} \in \Phi(L^{1,\infty})$  and  $\text{ind}(T_1) = \text{ind}(T_\infty) = \text{ind}(T_{1,\infty}) = -\text{ind}(T'_1|L_0^\infty)$ .

PROOF. Assume  $T_p, T_s$ , and  $T_{p,s}$  satisfy the conditions stated in (1). We prove that  $T \in \Phi(\mathcal{B}_{p,s})$ . First assume  $s \neq \infty$ . By Proposition 3.2 we have  $\mathcal{B}_{p,s} \subseteq \mathcal{A} = \mathcal{A}(L^{p,s}, L^t)$  and  $\mathcal{B}_{p,s} \subseteq \tilde{\mathcal{A}} = \mathcal{A}(L^{p,s}, L^q)$ . By assumption  $T_{p,s} \in \Phi(L^{p,s}), T_s \in \Phi(L^s)$  and  $\text{ind}(T_{p,s}) = \text{ind}(T_s)$ . Thus,  $T'_s \in \Phi(L^t)$  and  $\text{ind}(T_{p,s}) = -\text{ind}(T'_s)$ . Therefore, by [3, Theorem 2.5],  $T \in \Phi(\mathcal{A})$ . An exactly analogous argument shows that  $T \in \Phi(\tilde{\mathcal{A}})$ . Since  $T \in \Phi(\mathcal{A}), \exists R \in \mathcal{A}$  and  $\exists F, G \in \mathcal{F}(L^{p,s}, L^t)$  such that

$$RT = I - F \quad \text{and} \quad TR = I - G.$$

Choose  $E \in \mathcal{F}(L^{p,s}, L^{t,q})$  such that  $\|F - E\|_{\mathcal{A}} < 1$ . Then  $\|I - RT - E\|_{\mathcal{A}} = \|F - E\|_{\mathcal{A}} < 1$ . By standard Banach algebra theory,  $\exists W \in \mathcal{A}$  such that

$$(3) \quad W(RT + E) = I = (RT + E)W.$$

By proposition 3.4  $WR$  is  $s$ -continuous on  $L^{p,s}$ . Also

$$(4) \quad (WR)T = I - WE \text{ and } WE \in \mathcal{F}(L^{p,s}, L^{t,q}).$$

Since  $T \in \Phi(\tilde{\mathcal{A}}), \exists V \in \tilde{\mathcal{A}}$  and  $\exists K \in \mathcal{F}(L^{p,s}, L^q)$  with  $TV = I - K$ . By Proposition 3.4,  $V$  is  $p$ -continuous on  $L^{p,s}$ . Now, by (4),

$$(WR)(I - K) = (WR)TV = V - WEV.$$

Thus,

$$WR = WRK + V - WEV.$$

Since all of the operators on the right are  $p$ -continuous,  $WR$  is  $p$ -continuous. Therefore  $WR \in \mathcal{B}_{p,s}$ . By (4),  $(WR)T = I - WE$  and  $WE \in \mathcal{F}_{p,s}$ . By (3),  $(RT+E)W = I$ . Thus,  $TRT(WR) = TR - TEWR$ . Using the fact that  $TR = I - G$ , we have

$$T(WR) = I - G + GTWR - TEWR.$$

Then  $-G + GTWR - TEWR$  is a finite rank operator in  $\mathcal{B}_{p,s}$ . This completes the proof that  $T \in \Phi(\mathcal{B}_{p,s})$ .

Now assume  $p \neq 1, s = \infty$ . Let  $\mathcal{A} = \mathcal{A}(L^{p,\infty}, L^q)$  and  $\tilde{\mathcal{A}} = \mathcal{A}(L^{p,\infty}, (L_0^\infty)')$ . Then, by Proposition 3.2,  $\mathcal{B}_{p,\infty} \subseteq \mathcal{A}$  and  $\mathcal{B}_{p,\infty} \subseteq \tilde{\mathcal{A}}$ . As in the previous case, the assumptions imply  $T \in \Phi(\mathcal{A})$  and  $T \in \Phi(\tilde{\mathcal{A}})$ . The argument then follows exactly the argument in the first case. This completes the proof of the “if” direction of (1).

Now assume  $T_1, T_\infty$ , and  $T_{1,\infty}$  satisfy the conditions stated in (2). By Proposition 3.2,  $T \in \mathcal{A} = \mathcal{A}(L^{1,\infty}, L_0^\infty)$  and  $T \in \tilde{\mathcal{A}} = \mathcal{A}(L^{1,\infty}, (L_0^\infty)')$ . Again, the argument proceeds just as in the proof of (1). The assumptions that  $(T_1'|L_0^\infty) \in \Phi(L_0^\infty)$  and  $\text{ind}(T_{1,\infty}) = -\text{ind}(T_1'|L_0^\infty)$  are used to show  $T \in \Phi(\mathcal{A})$ .

The “only if” assertions in (1) and (2) follow from Theorem 3.3.  $\square$

**COROLLARY 5.7.** *Assume  $1 \leq p < s \leq \infty$ , with  $p \neq 1$  or  $s \neq \infty$  and  $p < r < s$ . If  $T \in \mathcal{B}_{p,s}$ , then*

$$\omega(T_r) \subseteq \omega(T_p) \cup \omega(T_s) \cup \omega(T_{p,s}) \cup \omega_1,$$

where  $\omega_1$  is the set of all  $\lambda \notin \omega(T_p) \cup \omega(T_s) \cup \omega(T_{p,s})$  such that  $\text{ind}(\lambda I - T_p), \text{ind}(\lambda I - T_s)$ , and  $\text{ind}(\lambda I - T_{p,s})$  are not all three equal.

#### REFERENCES

1. W. Arendt, *On the  $\sigma$ -spectrum of regular operators and the spectrum of measures*, Math. Z. **178** (1981), 271–278.
2. J. Auerhoff, *Interpolationseigenschaften des Spektrums linearer Operatoren auf  $L^p$ -Räumen*, Math. Z. **184** (1983), 397–406.
3. B. Barnes, *Fredholm theory in a Banach algebra of operators*, Proc. Roy. Irish Acad., Sect. A **87** (1987), 1–11.
4. ———, *The spectrum of integral operators on Lebesgue spaces*, J. Operator Theory **18** (1987), 115–132.
5. ———, G. Murphy, R. Smyth and T. West, *Riesz and Fredholm theory in Banach algebras*, Pitman, Boston-London-Melbourne, 1982.
6. D. Boyd, *The spectrum of the Cesàro operator*, Acta Sci. Math. **29** (1968), 31–34.
7. N. Dunford and J. Schwartz, *Linear Operators*, Part I, Interscience, New York-London, 1964.
8. C. Halberg and A. Taylor, *On the spectra of linked operators*, Pacific J. Math. **6** (1956), 283–290.

9. D.A. Herrero, *On limits of nilpotents in  $L^p$ -spaces and interpolation of spectra*, in *Operator Theory: Advances and Applications*, Vol. II, Birkhäuser, Basel, 1983, 191–232.
10. K. Jörgens, *Linear Integral Operators*, Pitman, Boston-London-Melbourne, 1982.
11. D. Lay and A. Taylor, *Introduction to functional analysis*, Second Edition, John Wiley & Sons, New York, Toronto, 1980.
12. P. Lax, *Symmetrizable linear transformations*, *Comm. Pure Appl. Math.* **7** (1954), 633–647.
13. M. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, Cambridge, 1954.
14. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.
15. H. Schaefer, *Interpolation of spectra*, *Integral Equations Operator Theory*, **3** (1980), 463–469.
16. M. Schechter, *Principles of functional analysis*, Academic Press, New York, London, 1971.
17. J.D. Stafney, *Set approximation by lemniscates and the spectrum of an operator on an interpolation space*, *Pacific J. Math.* **60** (1975), 253–265.
18. E. Stout, *The theory of uniform algebras*, Bogden & Quigley, New York, 1971.