ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 2, Spring 1990

CAUCHY PRODUCTS OF POSITIVE SEQUENCES

M. BADRI AND P. SZEPTYCKI

ABSTRACT. Two elementary properties of positive sequences (of weights) are studied that correspond to properties of the reproducing kernels (being generalized Bergman kernels), and of the weighted shifts (being hyponormal), in the space of analytic functions in the disk determined by the sequence. Both properties are inherited by the Cauchy products (corresponding to the products of reproducing kernels).

1. Introduction. The Cauchy product of two sequences $(a_n), (b_n)$ is defined by

$$c_n = \sum_{l=0}^{n} a_{n-l} b_l. \tag{1.1}$$

It is of interest to study various properties of sequences $(a_n), (b_n)$ which are inherited by the sequence (c_n). In this note we consider two such properties of positive sequences (t_n):

$$\lim_{n\to\infty} \inf \left(\inf_{k\geq 1} \frac{t_k}{t_{k+n}} \right)^{1/n} \geq 1 \tag{1.2}$$

and, logarithmic concavity,

$$t_k^2 \geq t_{k-1} t_{k+1}, \quad (k = 1, 2, \dots). \tag{1.3}$$

Properties (1.2) and (1.3) arise in the following Hilbert setting.

For a sequence (t_n) of positive numbers such that $\sup t_n/t_{n+1} < \infty$, let $H(t_n)$ be the space of analytic functions in the unit disk, defined by $H(t_n) = \{f = \sum_0^\infty \hat{f}(n) z^n; \sum_0^\infty |\hat{f}(n)|^2/t_n < \infty\}$.

$H(t_n)$ is a Hilbert space with scalar product $(f,g) = \sum_{n=0}^\infty \hat{f}(n)\overline{\hat{g}(n)}/t_n$, with the orthonormal basis $e_n(z) = \sqrt{t_n} z^n$ and the reproducing kernel $K_{(t_n)}(z,w) = \sum_{n=0}^\infty t_n (z\overline{w})^n$.

Received by the editors on August 15, 1987.

The operator T_z of multiplication by z in $H(t_n)$ can be written in the form $T_z e_n = \sqrt{t_n/t_{n+1}} e_{n+1}$ and is therefore a weighted shift.

Properties of $H(t_n)$, of $K_{(t_n)}$ and of T_z in $H(t_n)$, correspond to properties of (t_n). The Cauchy product of sequences corresponds to the pointwise product of reproducing kernels, and the questions about sequences are equivalent to ones about reproducing kernels.

(1.2) (for a sequence with bounded quotients) corresponds to $K_{(t_n)}$ being a generalized Bergman kernel [**1, 2**]. It can be shown that this property is inherited by products [**1**]. In this case the question about sequences can be settled in a Hilbert space setting, nevertheless an elementary solution is of some interest.

We also note that the sequence $s_n = \inf_{k\geq 1} t_k/t_{k+n}$ satisfies the inequality $s_{n+m} \geq s_n s_m$ and, as in [**3**], one can conclude that $\lim_{n\to\infty} s_n^{1/n}$ exists, possibly as $+\infty$. Hence limit inferior in (1.2) could be replaced by limit.

Logarithmic convexity of (t_n) corresponds to T_z being hyponormal—it is not clear, at least for the time being, that this property is preserved by products of reproducing kernels. Thus, in this case, a positive answer of the question concerning sequences yields a contribution towards understanding of the Hilbert space setting.

Some additional comments about the operator theoretical aspects and more details can be found in [**1**].

2. The main result. This section is devoted to the proof of the following theorem.

THEOREM. *Let (a_n) and (b_n) be two sequences of positive numbers.*

(i) *If both (a_n) and (b_n) satisfy* (1.2), *then so does their Cauchy product.*

(ii) *If (a_n) and (b_n) are logarithmically concave and if (c_n) is the Cauchy product of (a_n) and (b_n), then $c_n^2 - c_{n-1}c_{n+1} \geq a_0 a_n b_0 b_n$ for all $n \geq 0$. In particular, (c_n) is logarithmically concave. The above inequality is the best possible.*

PROOF. (i). Let $\varepsilon > 0$. (1.2) implies existence of $n(\varepsilon)$ such that, for all

k and for all $n \geq n(\varepsilon)$,

$$a_{n+k} \leq (1-\varepsilon)^{-n} a_k \quad \text{and} \quad b_{n+k} \leq (1-\varepsilon)^{-n} b_k. \tag{2.1}$$

We will establish a similar inequality for c_{n+k} for large n. Let $n \geq 2n(\varepsilon)$ and suppose first that $k \geq n(\varepsilon)$.

Write c_{k+n} in the form

$$c_{k+n} = \sum_{l=0}^{k+n} a_{k+n-l} b_l = \Big(\sum_{l=0}^{k} + \sum_{l=k+1}^{n-1} + \sum_{l=n}^{n+k}\Big) a_{k+n-l} b_l.$$

In the first and in the third sum we write $a_{k+n-l} b_l \leq (1-\varepsilon)^{-n} a_{k-l} b_l$ and $a_{k+n-l} b_l \leq (1-\varepsilon)^{-n} a_{k+n-l} b_{l-n}$; both sums can be bounded by $2(1-\varepsilon)^{-n} c_k$.

The middle sum is 0 when $k+1 < n-1$, otherwise let s be the least integer such that $n-1 \leq sn(\varepsilon)$ and write the middle sum in the form

$$\sum_{l=k+1}^{n-1} a_{k+n-l} b_l \leq \sum_{r=1}^{s-2} \sum_{l=rn(\varepsilon)+1}^{(r+1)n(\varepsilon)} a_{k+n-l} b_l + \sum_{l=(s-1)n(\varepsilon)+1}^{sn(\varepsilon)} a_{k+n-l} b_l \tag{2.2}$$

(here we use the inequality $k \geq n(\varepsilon)$). The double sum is 0 when $s = 2$ (notice that $s \geq 2$).

In the double sum in (2.2) we use (2.1):

$$a_{k+n-l} \leq (1-\varepsilon)^{-n+rn(\varepsilon)} a_{k+rn(\varepsilon)-l}, \qquad b_l \leq (1-\varepsilon)^{-rn(\varepsilon)} b_{l-rn(\varepsilon)}$$

adding up to the bound $(s-2)(1-\varepsilon)^n c_k$.

In the second term in (2.2) we write

$$b_l \leq (1-\varepsilon)^{-(s-1)n(\varepsilon)} b_{l-(s-1)n(\varepsilon)},$$
$$a_{k+n-l} \leq (1-\varepsilon)^{-n+(s-1)n(\varepsilon)} a_{k+(s-1)-l},$$

(here again we use $k \geq n(\varepsilon)$), getting the bound $(1-\varepsilon)^n c_k$.

Adding these up,

$$c_{k+n} \leq (s+1)(1-\varepsilon)^n c_k \leq \left(\left[\frac{n-1}{n(\varepsilon)}\right] + 2\right)(1-\varepsilon)^n c_k, \tag{2.3}$$

where $[x]$ denotes the largest integer contained in x.

When $k < n(\varepsilon)$ we take $n \geq 3n(\varepsilon)$ and, using (2.3), can write

$$\begin{aligned} c_{k+n} &= c_{k+n(\varepsilon)+n-n(\varepsilon)} \\ &\leq \left(\left[\frac{n-n(\varepsilon)-1}{n(\varepsilon)}\right] + 2\right)(1-\varepsilon)^{-n+n(\varepsilon)} \frac{c_{k+n(\varepsilon)}}{c_k} c_k \\ &\leq \lambda_n(\varepsilon) c_k (1-\varepsilon)^{-n}, \end{aligned} \tag{2.4}$$

where

$$\lambda_n(\varepsilon) = \left(\left[\frac{n-n(\varepsilon)-1}{n(\varepsilon)}\right] + 2\right)(1-\varepsilon)^{n(\varepsilon)} \max_{1\leq k\leq n(\varepsilon)} \frac{c_{k+n(\varepsilon)}}{c_k}.$$

(2.3) and (2.4) give

$$c_{n+k} \leq \mu_n(\varepsilon)(1-\varepsilon)^{-n} c_k \quad \text{for } n \geq 3n(\varepsilon) \quad \text{and all } k \geq 1, \tag{2.5}$$

where $\mu_n(\varepsilon) = \max\big(\lambda_n(\varepsilon), (n-1)/n(\varepsilon) + 2\big)$.

Observe that $\lim_{n\to\infty} \mu_n(\varepsilon)^{1/n} = 1$. It follows that

$$\frac{c_k}{c_{n+k}} \geq (1-\varepsilon)^n \mu_n(\varepsilon)^{-1} \quad \text{for all } k \geq 1 \quad \text{and } n \geq 3n(\varepsilon)$$

and that

$$\liminf_{n\to\infty} \left(\inf_{k\geq 1} \frac{c_k}{c_{n+k}}\right)^{1/n} \geq 1-\varepsilon.$$

(ii). Observe that c_n^2 is a sum of n^2 terms and $c_{n+1}c_{n-1}$ is a sum of n^2-1 terms. Not all of the terms in these sums can be compared one with one, some of them have to be compared in pairs: these comparisons leave an extra term as indicated in the statement. To carry out the details of the argument we note that $t_k^2 \geq t_{k-1}t_{k+1}$ implies

$$t_{r-1}t_{s+1} \leq t_r t_s \tag{2.6}$$

for $1 \leq r \leq s$.

To prove that $c_n^2 \geq c_{n-1}c_{n+1}$, we write

$$c_n^2 = \sum a_l^2 b_{n-l}^2 + 2 \sum_{0 \leq k < l \leq n} a_k a_l b_{n-k} b_{n-l}, \tag{2.7}$$

$$c_{n-1}c_{n+1} = \sum_{k=0}^{n-1} \sum_{l=0}^{n+1} a_k a_l b_{n-k-1} b_{n-l+1} \tag{2.8}$$

and record a consequence of (2.6),

$$\begin{aligned} a_k a_l b_{n-k} n_{n-l} &+ a_{k-1} a_{l+1} b_{n-l-1} b_{n-k+1} \\ &\geq a_k a_l b_{n-k+1} b_{n-l-1} + a_{k-1} a_{l+1} b_{n-l} b_{n-k}, \end{aligned} \tag{2.9}$$

valid whenever $1 \leq k \leq l \leq n-1$.

Indeed, letting $a_k a_l = a_{k-1} a_{l+1} + \varepsilon, b_{n-k} b_{n-l} = b_{n-k+1} b_{n-l-1} + \eta$, $e, \eta > 0$, we find that the difference between the left and the right sides of (2.9) is $\varepsilon\eta$.

We can now compare term by term (2.7) with (2.8). The terms in (2.7) of the form $a_0 a_k b_n b_{n-k}$ or $a_n a_k b_{n-k} b_0$ can be compared with terms in (2.8) as follows (using 2.6)):

$$\begin{aligned} a_0 a_k b_n b_{n-k} &\geq a_0 a_k b_{n+1} b_{n-k-1}, \qquad 0 \leq k \leq n-1, \\ a_n a_k b_{n-k} b_0 &\geq a_{n+1} a_{k-1} b_{n-k} b_0, \qquad 1 \leq k \leq n. \end{aligned} \tag{2.10}$$

(2.10) takes care of all terms in (2.8) which contain a_{n+1} or b_{n+1}, of the two terms in (2.7) of the form $a_0^2 b_n^2, a_n^2 b_0^2$ and of half of the terms $a_0 a_k b_n b_{n-k}, 1 \leq k \leq n-1$, and $a_n a_k b_0 b_{n-k}, 1 \leq k \leq n-1$.

The term $2a_0 a_n b_0 b_n$ is not affected by this step.

Next, by (2.9),

$$a_k^2 b_{n-k}^2 + a_{k-1} a_{k+1} b_{n-k-1} b_{n-k+1} \leq a_k^2 b_{n-k-1} b_{n-k+1} + a_{k-1} a_{k+1} b_{n-k}^2, \tag{2.11}$$

which includes the squares in (2.7) remaining from (2.10), one half of the terms $a_{k-1} a_{k+1} b_{n-k+1} b_{n-k-1}$ in (2.7) and all of the terms containing a_k^2 or b_{n-k}^2 in (2.8).

For every term $a_k a_{k+1} b_{n-k} b_{n-k-1}$ in (2.7) there is a corresponding identical term appearing once in (2.8), $0 \le k \le n-1$, and, since the terms $a_0 a_1 b_n b_{n-1}$ and $a_{n-1} a_n b_0 b_1$ have already appeared once in (2.10), they are gone from (2.7). We are left with one of each of $a_k a_{k+1} b_{n-k} b_{n-k-1}$, $1 \le k \le n-2$, in (2.7) to be accounted for in the next step.

We next write, for $s = 1, 2, \dots, n-2$, using (2.9),

$$\begin{aligned} &a_k a_{k+s} b_{n-k} b_{n-k-s} + a_{k-1} a_{k+s+1} b_{n-k-s-1} b_{n-k+1} \\ &\quad \ge a_k a_{k+s} b_{n-k-s-1} b_{n-k+1} + a_{k-1} a_{k+s+1} b_{n-k} b_{n-k-s}, \\ &\qquad 1 \le k \le n-s-1, \end{aligned} \tag{2.12}$$

which includes all terms in (2.8) with indices of a differing by s, those of b differing by $s+2$, and those with indices of b differing by s and those of a by $s+2$.

On the left-hand side we see all the terms in (2.7) with indices of a and b differing by $s+2$, each appearing once, and with indices differing by s and taking values between 1 and $n-1$. This includes all terms left over from the step corresponding to $s-2$ except for those with indices 0 and n—those are taken care of in (2.10).

The step before the last is

$$\begin{aligned} a_1 a_{n-2} b_{n-1} b_2 + a_0 a_{n-1} b_1 b_n &\ge a_1 a_{n-2} b_1 b_n + a_0 a_{n-1} b_{n-1} b_2 \\ a_2 a_{n-1} b_{n-2} b_1 + a_1 a_n b_0 b_{n-1} &\ge a_2 a_{n-1} b_0 b_{n-1} + a_1 a_n b_{n-2} b_1, \end{aligned}$$

and the last step is

$$a_1 a_{n-1} b_{n-1} b_1 + a_0 a_n b_0 b_n \ge a_1 a_{n-1} b_0 b_n + a_0 a_n b_{n-1} b_1.$$

The term $a_0 a_n b_0 b_n$ appearing twice in (2.7) has been used only once in the last step, and is left over as claimed in (ii). □

REFERENCES

1. M. Badri, *On perturbations and products of generalized Bergman kernels,* Thesis 1987.

2. R. Curto and N. Salinas, *Generalized Bergman kernels and Cowen Douglas theory,* Amer. J. Math. **106** (1984), 447–488.

3. A Shields, *Weighted shifts operators and analytic function theory,* Surveys 13: Topics in Operator Theory (1974), 49–128.

Department of Mathematics, Umm Al-Qurah University, Makkah, Suadi Arabia

Department of Mathematics, University of Kansas, Lawrence, KS 66044