ROCKY MOUNTAIN JOURNAL
OF MATHEMATICS
Volume 20, Number 2, Spring 1990

PROBLEMS ABOUT REFLEXIVE ALGEBRAS

KENNETH R. DAVIDSON

The content of this paper was the subject of a talk given at the 1987 GPOTS meeting in Lawrence, Kansas. It was our purpose to try to survey some of the most important problems extant regarding reflexive algebras with a commutative invariant subspace lattice. In most cases, the motivation for these problems is based on a nice theory for nest algebras. However, the desired generalizations rarely turn out to be trivial; and often are not true in full generality.

We begin with two problems about nest algebras. Then we will briefly describe commutative subspace lattices (CSL's) and give a few examples before considering the remaining eight problems. Recall that a nest $\mathcal{N}$ is a family of closed subspaces of a Hilbert space containing $\{0\}$ and $\mathcal{H}$ which is totally ordered by inclusion and is complete with respect to intersection and closed span. The corresponding nest algebra $\mathcal{T}(\mathcal{N})$ is the algebra of all operators leaving each element of $\mathcal{N}$ invariant. The simplest example is given by taking an orthonormal basis $\{e_n, n \geq 1\}$ and forming $P_n = \operatorname{span}\{e_k, k \leq n\}$ for $n \geq 0$. Then $\mathcal{P} = \{P_n, n \geq 0; \mathcal{H}\}$ is a nest. Even for this simplest of all nests there is an interesting open problem.

PROBLEM 1. Is $\mathcal{T}(\mathcal{P})^{-1}$ connected?

It is surprising that such a simple sounding problem should remain open in this context. It is conjectured by many that the invertibles in $\mathcal{T}(\mathcal{P})$ are not connected. One reason is based on an analogy with function theory. The algebra H^∞ of bounded analytic functions on the unit disc is a nonselfadjoint subalgebra of L^∞ which is analogous to $\mathcal{T}(\mathcal{P})$ in $\mathcal{B}(\mathcal{H})$ in a number of ways. The set of Toeplitz operators $\{T_h^* : h \in H^\infty\}$ is a weak* closed abelian subalgebra of $\mathcal{T}(\mathcal{P})$ equal to the intersection of $\mathcal{T}(\mathcal{P})$ with the set of all Toeplitz operators. But the invertibles in H^∞ are not connected. Indeed, as in any commutative Banach algebra, the connected component of the identity is the set of

Received by the editors on Aug. 17, 1987 and, in revised form, on Jan. 4, 1988.

exponentials $\{e^h : h \in H^\infty\}$. It is easy to construct an invertible H^∞ function which has an unbounded logarithm and hence is not connectable to 1. For example, take

$$h(z) = \frac{2i}{\pi} \log \frac{1+z}{1-z}.$$

This is a conformal map of the disc onto $\{z : |\mathrm{Re} z| < 1\}$ such that $h(0) = 0$. Hence e^h is invertible in H^∞, but has no bounded analytic logarithm. Therefore, $T^*_{e^h}$ belongs to $\mathcal{T}(\mathcal{P})^{-1}$ and cannot be connected to T in $\mathcal{T}(\mathcal{P})^{-1} \cap \{\text{Toeplitz operators}\}$.

Another approach uses finite dimensional matrices. There exist nilpotent $k \times k$ matrices V_k such that $\lim_{k\to\infty} ||V_k|| = \infty$ and $\lim_{k\to\infty} ||V_k - \mathrm{Re}\ V_k|| = 0$. Put V_k in strictly upper triangular form, and set $T_k = e^{iV_k}$. Then $T = \sum_{k\geq 1} \oplus T_k$ is invertible in $\mathcal{T}(\mathcal{P})$, and in fact is a small compact perturbation of the unitary operator $U = \sum \oplus e^{i\mathrm{Re} V_k}$ (which is not in $\mathcal{T}(\mathcal{P})$). The spectrum of $\mathrm{Re} V_k$ cannot have any big gaps, so $\sigma(U)$ is the whole unit circle; and $\sigma_e(T)$ is thus the unit circle. If T were connected to I in $\mathcal{T}(\mathcal{P})^{-1}$, then each T_k is connected to I by a path $T_{k,t}$ with a uniform bound on (i) $||T_{k,t}||$, (ii) $||T_{k,t}^{-1}||$, and (iii) the arc length of $t \to T_{k,t}$. There are easy paths which control two of these. For example, the straight line $(1-t)T_k + tI$ controls (i) and (iii). A path suggested by Vern Paulsen is given by

$$T_s = (t_{ij} s^{j-i}), \qquad 0 \leq s \leq 1.$$

The map taking an *upper triangular* matrix T to T_s is a contractive homomorphism. Thus it controls both invertibility and the norm of $||T_s||$ and $||T_s^{-1}||$. Unfortunately, it generally "blows up" in length as the dimension k increases.

A third approach involves K-theory. The group $K_1(\mathcal{T}(\mathcal{P}))$ measures the connectedness of $(\mathcal{T}(\mathcal{P}) \otimes \mathcal{M}_n)^{-1}$ as n increases. If this group is nonzero, then $\mathcal{T}(\mathcal{P}^{(n)})^{-1}$ fails to be connected for some n. Almost certainly, $n = 1$ would suffice. This group has not been computed. However David Pitts [37] has computed K_0 of every nest algebra to be equal to K_0 of the atomic part of the diagonal algebra. In particular, $K_0(\mathcal{T}(\mathcal{P})) = K_0(\ell^\infty)$. The diagonal of a nest algebra is a von Neumann algebra. Since the invertibles in a von Neumann algebra are always connected, it follows easily that they have trivial K_1 groups. So it is possible that K_1 will not be sufficient to solve this problem.

PROBLEM 2. Is every (separably acting) maximal nest equal to the invariant subspace lattice of a single operator?

An operator T such that Lat T is a nest is called *unicellular*. Examples of such operators are quite rare. The Volterra operator on $L^2(0,1)$ given by $Vf(t) = \int_t^1 f(x)dx$ has the continuous nest of subspaces $N_t = \{f : \text{supp}(f) \subseteq [0,t]\}$, $0 \leq t \leq 1$, as its only invariant subspaces [**15**]. Weighted shifts $Se_1 = 0, Se_{n+1} = w_n e_n$, $n \geq 1$ such that w_n decreases monotonely to zero have the nest $\mathcal{P}$ as its invariant subspaces [**17, 42**]. Other examples are more difficult to come by. Domar [**16**] showed that certain bilateral weighted shifts have only the obvious invariant subspaces, while many others do not. Harrison and Longstaff [**21**] were able to "glue" two weighted shifts together to produce a nest order isomorphic to $\mathcal{P} + \mathcal{P}$. Then Barria and I [**7**] constructed unicellular operators with lattice order isomorphic to any countable ordinal.

All of these examples except for the Volterra operator essentially build up the unicellular operators from "smaller" ones. A useful test case for problem 2 is the nest on $\ell^2(\mathbf{Q})$ given by the subspaces $Q_t^+ = \{f \in \ell^2(\mathbf{Q}) : \text{supp}(f) \subseteq (-\infty, t]\}$ and $Q_t^- = \{f \in \ell^2(\mathbf{Q}) : \text{supp} f \subseteq (-\infty, t)\}$. The nest is uncountable, even though the Hilbert space is the sum of the countably many atoms. Moreover, every infinite piece of this nest is identical to the whole. So it is impossible to build it up from smaller pieces.

Now let us turn to commutative subspace lattices. A CSL is a complete lattice $\mathcal{L}$ of subspaces such that the corresponding projections $\{P(L) : L \in \mathcal{L}\}$ is commutative. The corresponding algebra, Alg $\mathcal{L}$, consists of all operators leaving each element of $\mathcal{L}$ invariant. This algebra contains the commutant $\mathcal{L}' = \{P(L) : L \in \mathcal{L}\}'$ which is the von Neumann algebra $\operatorname{Alg} \mathcal{L} \cap (\operatorname{Alg} \mathcal{L})^*$. This has abelian commutant $\mathcal{L}''$. Nest algebras (corresponding to chains) and von Neumann algebras with abelian commutant (complemented lattices) are the two extreme types of CSL algebras.

A more typical example is given as follows. Let $X = 2^{\aleph_0}$ be the Cantor set, thought of as sequences $x = (x_n)$ of 0's and 1's with the product topology. Let μ be a finite regular Borel measure on X. Form $L^2(\mu)$ and let $L_n = \{f \in L^2(\mu) : \text{supp}(f) \subseteq X_n\}$ where $X_n = \{x \in X : x_n = 0\}$. The lattice $\mathcal{L}$ generated by $\{L_n, n \geq 1\}$ is a CSL. In fact, a theorem of Arveson [**3**] shows that the most general CSL is given by the lattice generated by $\{L_{2n}, n \geq 1\}$.

A final example is the prototype for width 2 lattices (those generated by two commuting nests). Take the Hilbert space to be $\mathcal{H} = \ell^2(\mathbf{N}\times\mathbf{N})$, and consider the two nests

$$P_n^{(1)} = \{f : \operatorname{supp}(f) \subseteq E_n \times \mathbf{N}\}$$

$$P_n^{(2)} = \{f : \operatorname{supp}(f) \subseteq \mathbf{N} \times E_n\},$$

where $E_n = \{1, 2, \dots, n\}$. One can realize $\mathcal{H}$ as a tensor product $\ell^2(\mathbf{N}) \otimes \ell^2(\mathbf{N})$. Then the two nests become $\mathcal{P} \otimes \ell^2$ and $\ell^2 \otimes \mathcal{P}$. So we denote the lattice generated by these two nests by $\mathcal{P} \otimes \mathcal{P}$.

Arveson's seminal paper [**3**] on CSL's is a penetrating analysis of those lattices and their algebras. In particular, he shows that CSL's are reflexive. That is,

$$\operatorname{Lat}(\operatorname{Alg}\mathcal{L}) = \mathcal{L}$$

for every commutative subspace lattice. Later in this article, we will discuss some of the other implications of this paper. In the sequel, $\mathcal{L}$ will always denote a CSL.

Problem 3. What is the Jacobson radical of Alg $\mathcal{L}$?

In the study of any Banach algebra, the structure of the radical is fundamental to an understanding of the irreducible representations. In the two extreme cases, there is a complete description. Every C^*-algebra is semisimple, so the complemented case is trivial. Consider a nest $\mathcal{N}$. For a finite nest $\mathcal{F}, \mathcal{T}(\mathcal{F})$ consists of block upper triangular operators. It is routine to verify that the radical consists of the strictly upper triangular algebra, and the quotient $\mathcal{T}(\mathcal{F})/\operatorname{rad}\mathcal{T}(\mathcal{F})$ is isomorphic to the diagonal algebra $\mathcal{F}'$.

For an arbitrary nest $\mathcal{N}$, take any finite subnest $\mathcal{F}$. Then $\operatorname{rad}\mathcal{T}(\mathcal{F})$ is contained in $\mathcal{T}(\mathcal{N})$, and thus in $\operatorname{rad}\mathcal{T}(\mathcal{N})$. Now the C^*-algebra $C^*(\mathcal{N}) = \overline{\operatorname{span}}\{P(N) : N \in \mathcal{N}\}$ is commutative. Each multiplicative functional ϕ in its maximal ideal space is determined by its restriction to $\{P(N) : N \in \mathcal{N}\}$ which is necessarily an increasing map of $\mathcal{N}$ onto

$\{0,1\} = 2$. Thus $\phi|\mathcal{N}$ is an element of $\operatorname{Hom}(\mathcal{N}, 2)$, the set of lattice homomorphisms onto 2. Each ϕ gives rise to a seminorm on $\mathcal{T}(\mathcal{N})$:

$$||T||_\phi = \inf\{||ETE|| : E^- = P(N_1) - P(N_2), \phi(E) = 1\}$$

Now $\Phi_E(T) = ETE$ is a contractive homomorphism for each *interval* $E = P(N_1) - P(N_2)$. Thus the set

$$\mathcal{I}_\phi = \{T : ||T||_\phi = 0\}$$

is a closed two sided ideal of $\mathcal{T}(\mathcal{N})$.

For example, consider the nest $\mathcal{P} = \{P_n, n \geq 1; \mathcal{H}\}$. For each $n \geq 1$, there is a functional $\delta_n(P_k) = 1$ if $k \geq n$. Let A_n be the projection onto $P_n - P_{n-1}$. Then $||T||_{\delta_n} = ||A_n T A_n||$. The ideal $\mathcal{I}_{\delta_n}$ consists of those T in $\mathcal{T}(\mathcal{N})$ with $A_n T A_n = 0$. There is also the functional $\phi(P_k) = 0, k \geq 1$ and $\phi(\mathcal{H}) = 1$. In this case,

$$||T||_\phi = \lim_{n\to\infty} ||P(P_n^\perp)TP(P_n^\perp)|| = ||T||_e.$$

The kernel $\mathcal{I}_\phi$ consists of the compact operators in $\mathcal{T}(\mathcal{N})$. In this example, one can verify directly that

$$\operatorname{rad}\mathcal{T}(\mathcal{P}) = \cap_{\operatorname{Hom}(\mathcal{P},2)}\mathcal{I}_\phi = \{K \in \mathcal{T}(\mathcal{P}) \cap \mathcal{K} : \Delta_\mathcal{P}(K) = 0\},$$

where $\Delta_\mathcal{P}(K)$ is the diagonal part of K with respect to $\mathcal{P}$.

The general result is a theorem of Ringrose [**39**].

THEOREM. *Let $\mathcal{N}$ be a nest. For T in $\mathcal{T}(\mathcal{N})$, the following are equivalent:*

(1) $T \in \operatorname{rad}(\mathcal{T}(\mathcal{N}))$;

(2) $T \in \cap\{\mathcal{I}_\phi : \phi \in \operatorname{Hom}(\mathcal{N}, 2)\}$;

(3) $T \in \overline{\cup\{\operatorname{rad}\mathcal{T}(\mathcal{F}) : \mathcal{F} \text{ finite subnest of } \mathcal{N}\}}$;

(4) For all $\varepsilon > 0$, there is a finite subnest $\mathcal{F}$ of $\mathcal{N}$ such that $||\Delta_\mathcal{F}(T)|| < \varepsilon$.

In the case of a CSL algebra, Hopenwasser and Larson [**22, 24**] observed that conditions (1)–(4) above can be formulated analogously.

They show that (2), (3) and (4) are equivalent for CSL algebras and imply membership in the radical. Moreover, as in the nest case, every irreducible representation contains exactly one $\mathcal{I}_\phi$ in its kernel. So Problem 3 can be reformulated: Does rad Alg$(\mathcal{L})$ equal $\cap\{\mathcal{I}_\phi : \phi \in \mathrm{Hom}(\mathcal{L}, 2)\}$?

PROBLEM 4. Is there a distance formula for Alg $(\mathcal{P} \otimes \mathcal{P})$?

For any reflexive algebra $\mathcal{A} = \mathrm{Alg}\,\mathcal{L}$, $\mathcal{L} \in \mathcal{L}$, and any operators T in $\mathcal{B}(\mathcal{H})$ and A in $\mathcal{A}$, one has

$$||P(L)^\perp TP(L)|| = ||P(L)^\perp (T-A)P(L)|| \le ||T-A||.$$

Thus

$$\sup_{L\in\mathcal{L}} ||P(L)^\perp TP(L)|| \le \mathrm{dist}(T, \mathcal{A}).$$

In [**4**], Arveson proves the remarkable result that, for nests $\mathcal{N}$,

$$\sup_{N\in\mathcal{N}} ||P\,(N)^\perp TP(N)|| = \mathrm{dist}(T, \mathcal{T}(\mathcal{N})).$$

This result is of crucial importance in developing the structure theory for nests. In his analysis of derivations and cohomology of von Neumann algebras, Christensen [**10**] proves that, for AF von Neumann algebras $\mathcal{A}$ (i.e., those algebras which are the weak* closure of an increasing sequence of finite dimensional subalgebras),

$$\mathrm{dist}(T, \mathcal{A}) \le 4 \sup_{L\in \mathrm{Lat}\,\mathcal{A}} ||P(L)^\perp TP(L)||.$$

An algebra for which there is a constant K (in lieu of 4) making the above formula valid is called *hyperreflexive*.

Unfortunately, Power and I [**14**] showed that many CSL algebras fail to be hyperreflexive, including all infinite tensor products of nests. Larson [**31**] has a different approach which produces similar examples. One hope remains—finite width lattices. The prototypical width 2 lattice is $\mathcal{P} \otimes \mathcal{P}$, so we ask the question in this context.

Aside. let us mention a related question outside the context of CSL's. Is the weakly closed algebra $W(S)$ generated by a subnormal operator always hyperreflexive? A deep theorem of Olin and Thomson

[**34**] generalizing the work of Scott Brown [**8**] shows that $W(S)$ is always reflexive. In [**13**], it is shown that, for the unilateral shift, $W(S)$ is hyperreflexive. The Olin-Thomson result gives quantitative control of the predual of $W(S)$. What is needed for hyperreflexivity is quantitative control of the preannihilator.

PROBLEM 5. Are there examples of close CSL algebras which are not similar? or even not isomorphic?

The distance between two subspaces is measured as the Hausdorff distance between their unit balls. Perturbation results for von Neumann algebras were first considered by Kadison and Kastler [**27**]. Then Christensen [**9**] showed:

THEOREM. *Let* $\mathcal{A}$ and $\mathcal{B}$ *be type* I *von Neumann algebras. Then the following are equivalent:*

(1) $\mathcal{A}$ *and* $\mathcal{B}$ *are close.*

(2) A' *and* $\mathcal{B}'$ *are close whence* Lat $\mathcal{A}$ *and* Lat $\mathcal{B}$ *are close.*

(3) *There is a unitary operator* U *such that* $U\mathcal{A}U^* = \mathcal{B}$ *and* $||U - I||$ *is small.*

Here we interpret close and small as $O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. The analogous result for nests was proved by Lance [**29**].

THEOREM. *Let* $\mathcal{N}$ *and* $\mathcal{M}$ *be nests. Then the following are equivalent:*

(1) $\mathcal{T}(\mathcal{N})$ *and* $\mathcal{T}(\mathcal{M})$ *are close.*

(2) $\mathcal{N}$ *and* $\mathcal{M}$ *are close.*

(3) *There is an invertible operator* S *such that* $S\mathcal{N} = \mathcal{M}$ *and* $||S - I||$ *is small.*

In both these results, the distance estimate is crucial. These results give important information about the unitary and similarity classes of these algebras. It is reasonable to expect that the failure of the distance formula should lead to some interesting pathology in CSL's. This might shed some light on the equivalence problem for these lattices. In [**14**], an example is given of close lattices which are similar but require a similarity far from I. Nevertheless, it is shown in [**12**] that close CSL algebras have close, isomorphic lattices. As yet, it is not clear what sort of breakdown to expect.

PROBLEM 6. Classify commutative subspace lattices up to approximate unitary equivalence.

Two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are *unitarily equivalent (similar)* if there is a unitary (invertible) operator S such that $S\mathcal{L}_1 = \mathcal{L}_2$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattice isomorphic, say via θ, then they are *approximately unitarily equivalent* if there is a sequence of unitary operators U_n such that

$$\lim_{n\to\infty} ||P(\theta(L)) - U_n P(L) U_n^*|| = 0$$

uniformly for L in $\mathcal{L}$. The notion of similarity is not convenient for CSL's as it does not preserve orthogonality.

In the case of nests, unitary equivalence reduces to the Hellinger-Hahn multiplicity theory for self-adjoint operators together with an order relation [**18**]. The corresponding result for CSL's must be possible along similar lines. For nests however, the other two equivalence relations have proven to be much more important. If $S\mathcal{N} = \mathcal{M}$, then S induces an order isomorphism θ_S of $\mathcal{N}$ onto $\mathcal{M}$. Moreover, because of the spatial nature of S, if $N_1 < N_2$, then $\dim\theta(N_2)/\theta(N_1) = \dim N_2/N_1$. That is, θ_S *preserves dimension.* The work of Andersen [**1**], Larson [**30**] and myself [**11**] culminated in the

SIMILARITY THEOREM. *Let $\mathcal{N}$ and $\mathcal{M}$ be separably acting nests. Then the following are equivalent:*

(1) *$\mathcal{N}$ and $\mathcal{M}$ are approximately unitarily equivalent*

(2) *$\mathcal{N}$ and $\mathcal{M}$ are similar.*

(3) *There is an order isomorphism θ of $\mathcal{N}$ onto $\mathcal{M}$ which preserves dimension.*

Moreover, given any such order isomorphism θ and $\varepsilon > 0$, there is a unitary operator U and a compact operator K with $||K|| < \varepsilon$ such that $\theta_{U+K} = \theta$.

In the case of von Neumann algebras, Christensen's theorem on perturbations implies that approximate unitary equivalence coincides with unitary equivalence. Moreover, by the Johnson-Parrott Theorem [**26**], even the compact perturbations of von Neumann algebras are rigid. Arveson [**5**] develops a perturbation theory valid for certain "homogeneous" CSL's which are compact in the strong operator topology. In particular, his results give a completely new approach to the approximate unitary equivalence of continuous nests. Is there some way to unify the theory for nests and complemented lattices?

PROBLEM 7. If $\mathcal{K} \cap \operatorname{Alg}\mathcal{L}$ is weak* dense in $\operatorname{Alg}\mathcal{L}$, is $\mathcal{L}$ completely distributive?

This question asks for a lattice condition on $\mathcal{L}$ equivalent to having sufficiently many compact operators in Alg $\mathcal{L}$. Erdos [**19**] showed that the span of the rank one operators in a nest algebra is weak* dense. On the other hand, the non-atomic masa $L^\infty(0,1)$ acting on $L^2(0,1)$ has no compact operators at all. A CSL is called *completely distributive* if the infinite distributive law

$$\bigwedge_{x\in X} \bigvee_{\alpha\in\Lambda_x} L_\alpha = \bigvee_{f\in\Pi\Lambda_x} \bigwedge_{x\in X} L_{f(x)}$$

holds for every collection $\{\Lambda_x : x \in X\}$ of subsets of $\mathcal{L}$. Fortunately, this unsightly condition has a number of more tractable formulations. Using a deep result of Arveson [**3**], Laurie and Longstaff [**32**] prove that the finite rank operators are weak* dense in Alg $\mathcal{L}$ precisely when $\mathcal{L}$ is completely distributive. This is also equivalent to the density of the Hilbert-Schmidt operators [**25**].

Froelich [**20**] has shown that CSL algebras may contain compact operators yet fail to have Hilbert-Schmidt operators. On the other hand, density of the compact operators implies the compactness of $\mathcal{L}$ in the strong operator topology. Wagner [**41**] has shown this latter condition to be equivalent to a weaker infinite distributive law. However, for CSL's, these conditions may be equivalent. Even a negative answer to Problem 7 should shed light on the correct answer.

PROBLEM 8. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are commuting synthetic lattices, is $\mathcal{L}_1 \vee \mathcal{L}_2$ synthetic?

In Arveson's original paper on CSL's [**3**], he shows that, for each lattice $\mathcal{L}$, there is a smallest algebra $\mathcal{A}_{\min}(\mathcal{L})$ such that Lat $\mathcal{A}_{\min}(\mathcal{L}) = \mathcal{L}$ and $\mathcal{A}_{\min}$ contains a masa. (In fact, $\mathcal{A}_{\min}$ contains $\mathcal{L}'$.) This result was motivated in part by a theorem of Radjavi and Rosenthal [**38**] which showed that if $\mathcal{A}$ was a WOT closed algebra containing a masa and Lat $\mathcal{A}$ is a nest $\mathcal{N}$, then $\mathcal{A} = \mathcal{T}(\mathcal{N})$. A lattice $\mathcal{L}$ is *synthetic* if $\mathcal{A}_{\min}(\mathcal{L}) = \operatorname{Alg}\mathcal{L}$. The surprising thing is that $\mathcal{A}_{\min}(\mathcal{L})$ is sometimes strictly less than $\operatorname{Alg}(\mathcal{L})$. All such examples are closely related to failure of spectral synthesis in harmonic analysis. Let G be a locally compact abelian group, and let E be a closed subset of G. Let $\mathcal{A}_E$ be the algebra acting on $L^2(G) \oplus L^2(G)$ consisting of all operators of the form $\begin{bmatrix} M_1 & T \\ 0 & M_2 \end{bmatrix}$, where M_i are multiplication operators in $L^\infty(G)$, and T is an operator "supported" on $\sum_E = \{(x,y) : x - y \in E\}$ (i.e., if $U \times V$ is an open rectangle disjoint from $\sum_E$, then $M_U T M_V = 0$ where M_U is multiplication by the characteristic function of U). Froelich [**20**] shows that $\operatorname{Lat}\mathcal{A}_E$ is synthetic if and only if E is a set of spectral synthesis.

On the positive side, $\mathcal{L}$ is a synthetic if it is either finite width or completely distributive. The present knowledge of synthetic lattices is generally mired in measure theoretic technicality. The real problem is to find an operator theoretic approach to determining synthetic lattices. Two related test questions are: If $\mathcal{L}_1$ and $\mathcal{L}_2$ are commuting synthetic lattices, are $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \otimes \mathcal{L}_2$ synthetic? Any operator theoretic result which yields a new result in harmonic analysis will be very interesting indeed.

PROBLEM 9. Does $\operatorname{Alg}(\mathcal{L}_1) \otimes \operatorname{Alg}(\mathcal{L}_2) = \operatorname{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2)$?

The tensor products here are spatial. $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the strongly closed subspace lattice of $\mathcal{H}_1 \otimes \mathcal{H}_2$ generated by $\{L_1 \otimes L_2 : L_i \in \mathcal{L}_i\}$, and $\mathrm{Alg}(\mathcal{L}_1) \otimes \mathrm{Alg}(\mathcal{L}_2)$ is the WOT closed span of $\{A_1 \otimes A_2 : A_i \in \mathrm{Alg}(\mathcal{L}_i)\}$. Clearly, $A_1 \otimes A_2$ belongs to $\mathrm{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2)$ so one containment is automatic.

If $\mathcal{L}_i$ are complemented, then $\mathcal{A}_i = \mathrm{Alg}(\mathcal{L}_i)$ are von Neumann algebras. Since $\mathcal{L}_i$ is the projection lattice of $\mathcal{A}_i'$, our problem in this context is resolved positively by a celebrated theorem of Tomita [**40**]:

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = (\mathcal{A}_1' \otimes \mathcal{A}_2')'.$$

Hopenwasser and Kraus [**23**] showed that the answer is affirmative if $\mathcal{L}_1$ is finite width, and in Kraus [**28**] it is also shown to be valid if $\mathcal{L}_1$ is completely distributive. Even in much more general situations, no counterexample is known.

Problem 10. Is every bounded representation of a nest algebra completely bounded with $||\phi||_{cb} = ||\phi||$?

This final problem deals with the representation theory of nest algebras. Every map ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ induces maps $\phi^{(n)}$ from $\mathcal{A} \otimes \mathcal{M}_n$ to $\mathcal{B}(H^{(n)})$. The map ϕ is *completely bounded* if $||\phi||_{cb} \equiv \sup ||\phi^{(n)}||$ is finite. Arveson [**2**] identified completely contractive maps of nonself adjoint algebras as being precisely those maps which can be dilated to a * representation of the enveloping C^*-algebra.

Paulsen, Power and Ward [**36**] considered weak* continuous contractive representations of nest algebras. They were able to show that all such maps are completely contractive, and hence can be dilated to a weak* continuous * representation of $\mathcal{B}(\mathcal{H})$. Such representation are just the maps $\pi(T) = T^{(n)}$. In finite dimension, these results were obtained by Ball-Gohberg [**6**] and McAsey-Muhly [**33**]. Paulsen and Power [**35**] were then able to prove lifting theorems for representations of nest algebras analogous to the Sz-Nagy-Foias dilation theory. Some of their results extend to completely distributive lattices but even finite dimensional CSL algebras have contractive representations which are not completely contractive [**43**]. When weak* continuity is dropped, problem 10 is open even for contractive maps.

Added in Proof. Y. Domar has settled positively the special case of Problem 2 of the Cantor nest in a short note "An invariant subspace lattice", preprint 1989.

D. Pitts and I make further partial progress on Problem 7 in "Compactness and complete distributivity for commutative subspace lattices," to appear in J. London Math. Soc.

J. Kraus has shown that the variant of Problem 9 fails for arbitrary lattices, even when one is complemented. See his 1989 preprint "The Slice Map Problem and Approximation Properties."

REFERENCES

1. N.T. Andersen, *Compact perturbations of reflexive albegras,* J. Funct. Anal. **38** (1980), 366–400.

2. W.B. Arveson, *Subalgebras of* C^**-algebras,* Acta. Math. **123** (1969), 141–224.

3. ———, *Operator algebras and invariant subspaces,* Ann. of Math. (2) **100** (1974), 433–532.

4. ———, *Interpolation problems in nest algebras,* J. Funct. Anal. **3** (1975), 208–233.

5. ———, *Perturbation theory for groups and lattices,* J. Funct. Anal. **53** (1983), 22–73.

6. J.A. Ball and I. Gohberg, *A commutant lifting theorem for triangular matrices with diverse applications,* Integral Equations Operator Theory **8** (1985), 205–267.

7. J. Barria and K.R. Davidson, *Unicellular operators,* Trans. Amer. Math. Soc. **284** (1984), 229–246.

8. S. Brown, *Some invariant subspaces for subnormal operators,* Integral Equations Operator Theory **1** (1978), 310–333.

9. E. Christensen, *Perturbations of type* I *von Neumann algebras,* J. London Math. Soc. (2), **9** (1975), 395–405.

10. ———, *Perturbations of operator algebras* II, Indiana Univ. Math. J. **26** (1977), 891–904.

11. K.R. Davidson, *Similarity and compact perturbations of nest algebras,* J. Reine Angew. Math. **348** (1984), 72–87.

12. ———, *Perturbations of reflexive operator algebras,* J. Operator Theory, **15** (1986), 289–305.

13. ———, *The distance to the analytic Toeplitz operators,* Illinois J. Math. **31** (1987), 265–273.

14. ——— and S.C. Power, *Failure of the distance formula,* J. London Math. Soc. (2) **32** (1984), 157–165.

15. J. Dixmier, *Les opérateurs permutables á l' opérateur integral,* Portugal. Math. **8** (1949), 73–84.

16. Y. Domar, *Translation invariant subspaces and weighted ℓ^p and L^p spaces,* Math. Scand. **49** (1981), 133–144.

17. W.F. Donoghue, *The lattice of invariant subspaces of a completely continuous quasinilpotent transformation,* Pacific J. Math. **7** (1957), 1031–1035.

18. J.A. Erdos, *Unitary invariants for nests,* Pacific J. Math. **23** (1967), 229–256.

19. ———, *Operators of finite rank in nest algebras,* J. London Math. Soc. **43** (1968), 391–397.

20. J. Froelich, *Compact operators, invariant subspaces and spectral synthesis,* Ph.D. thesis, Univ. Iowa, 1984. (also available as a preprint)

21. K.J. Harrison and W.E. Longstaff, *An invariant subspace lattice of order type $\omega + \omega + 1$,* Proc. Amer. Math. Soc. **79** (1980), 45–49.

22. A. Hopenwasser, *The radical of a reflexive operator algebra,* Pacific J. Math. **65** (1976), 375–392.

23. ——— and J. Kraus, *Tensor products of reflexive algebras* II, J. London Math. Soc. (2) **28** (1983), 359–362.

24. ——— and D.R. Larson, *The carrier space of a reflexive operator algebra,* Pacific J. Math. **81** (1979), 417–434.

25. ———, C. Laurie and R. Moore, *Reflexive algebras with completely distributive subspace lattices,* J. Operator Theory **11** (1984), 91–108.

26. B.E. Johnson and S.K. Parrott, *Operators commuting with a von Neumann algebra modulo the set of compact operators,* J. Funct. Anal. **11** (1972), 39–61.

27. R.V. Kadison and D. Kastler, *Perturbations of von Neumann algebras* I, *Stability of type,* Amer. J. Math. **94** (1972), 38–54.

28. J. Kraus, *Tensor products of reflexive algebras,* J. London Math. Soc. (2) **28** (1983), 350–358.

29. E.C. Lance, *Cohomology and perturbations of nest algebras,* Proc. London Math. Soc. (3) **43** (1981), 334–356.

30. D.R. Larson, *Nest algebras and similarity transformations,* Ann. of Math. **121** (1985), 409–427.

31. ———, *Hyperreflexivity and a dual product construction,* Trans. Amer. Math. Soc., to appear.

32. C. Laurie and W. Longstaff, *A note on rank one operators in reflexive algebras,* Proc. Amer. Math. Soc. **89** (1983), 293–297.

33. M.J. McAsey and P.S. Muhly, *Representations of nonselfadjoint crossed products,* Proc. London Math. Soc. (3) **47** (1983), 128–144.

34. R.F. Olin and J.E. Thomson, *Algebras of subnormal operators,* J. Funct. Anal. **37** (1980), 271–301.

35. V.I. Paulsen and S.C. Power, *Lifting theorems for nest algebras,* J. Operator Theory **20** (1988), 311–327.

36. ———, ——— and J. Ward, *Semidiscreteness and dilation theory for nest algebras,* J. Funct. Anal. **80** (1988), 76–87.

37. D.R. Pitts, *On the K_0 groups of nest algebras,* K-Theory **2** (1989), 737–752.

38. H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras,* Amer. J. Math. **91** (1969), 683–692.

39. J.R. Ringrose, *On some algebras of operators,* Proc. London Math. Soc. (3) **95** (1965), 61–83.

40. M. Tomita, *Standard forms of von Neumann algebras,* Vth Functional Analysis Symposium, Math. Soc. Japan, Sendai, 1967.

41. B. Wagner, *Weak limits of projections and compactness of subspace lattices,* Trans. Amer. Math. Soc. **304** (1987), 515–535.

42. B.V. Yakubovic, *Conditions for unicellularity of weighted shift operators,* Dokl. Akad. Nauk SSSR **278** (1984), 821–824.

43. V.I. Paulsen, S.C. Power, and R.R. Smith, *Schur products and matrix completions,* J. Funct. Anal. **85** (1989), 151–178.

Department of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1