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# LOCAL SEMIGROUPS IN LIE GROUPS AND LOCALLY REACHABLE SETS

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ABSTRACT. In this paper the relationship between locally reachable sets for a fixed set of controls in the Lie algebra of a Lie group and the local semigroups generated by the corresponding one-parameter semigroup is considered. It is convenient to carry out the investigation in the Lie algebra itself, and the appropriate machinery for doing this is first developed. It is shown that local semigroups contain locally reachable sets. A general criterion (rerouting) is developed for the converse inclusion, and it is shown that if the set of controls is contained in a proper cone or is a Lie wedge (i.e., the tangent of a local semigroup), then it is the case that locally reachable sets contain local semigroups.

S. Lie's Fundamental Theorem states that there is a bijection between the subalgebras of a given finite dimensional real Lie algebra $L$ and local subgroups of a fixed local group having $L$ as Lie algebra. In recent years there has been considerable interest in studying (local) subsemigroups of (local) Lie groups [**3, 4, 6, 7-9, 10, 11**], partly because of their relevance in geometric control theory, partly because of their occurrence in the theory of symmetric spaces and "causal" semigroups, and partly in order to complete S. Lie's original program.

For the local study of subsemigroups of a Lie group it is convenient to stay inside the given Lie algebra $L$ and to fix a convex symmetric open neighborhood on which the Campbell-Hausdorff-Baker multiplication $(x,y) \to x*y = x+y+[x,y]/2+\cdots$ is defined as a function $B\times B \to L$ through the absolute convergence of the Campbell-Hausdorff-Baker series. Such a neighborhood we will call a CHB-*neighborhood* (short for *Campbell-Hausdorff-Baker-neighborhood*). (This approach is no restriction in considering local theory since the exponential mapping is a local analytic isomorphism from $B$ with the CHB-multiplication

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to a neighborhood of the identity in the group. It also covers the case of infinite-dimensional Dynkin algebras, for which there may be no corresponding global group.)

*A local semigroup $S$ of $L$ with respect to $B$* is then simply a subset $S \subset B$ containing 0 with $S*S \cap B \subset S$. Its tangent object $L(S)$ is the set of all $x$ for which there is a sequence $x_n \in S$ such that $x = \lim nx_n$ (cf. [**7**]). This tangent object is a convex closed wedge $W \subset L$ (i.e., a closed subset stable under non-negative scalar multiplication and addition) whose edge $H = W \cap -W$ satisfies the condition $e^{\mathrm{ad}\ x} W \subset W$ for all $x \in H$. In particular this implies that $H$ is a subalgebra. That, conversely, all such wedges are tangent sets for the local Lie semigroup they generate was announced in [**11**] and independently proved in [**8, 9**] for the class of wedges $W$ which have a vector space complement $V$ of $H$ in $L$ with $[H, V] \subset V$, and in general in [**5**].

Whenever one actually constructs a local semigroup from a given set of tangent vectors, it is of great importance to have explicit methods to exhibit the local semigroup in terms of the given set of vectors. In [**7, 8**] this was done in the spirit of topological algebra. The first section presents an approach via differential equations inside the Lie algebra itself. This is parallel, although not identical, to the usual method of constructing local semigroups of matrices by solving equations of the form

$$\dot{X}(t) = X(t)U(t), \quad X(0) = E_n$$

for suitable functions $U(t)$ ranging through the set of generating tangent vectors, or more generally to finding solution curves beginning at the identity to a time varying system of left invariant vector fields on a Lie group (see, e.g., [**3, 6**].)

The differential equation which will play an analogous role for the CHB multiplication in $L$ is

$$X'(t) = g(\mathrm{ad}\ X(t))U(t),\ X(0) = 0,$$

where $g$ is an analytic function given once and for all through a fixed power series $g(T) = 1 + T/2 + \sum_{n=1}^{\infty} (b_{2n}/(2n)!)T^{2n}$ with the Bernoulli numbers $b_{2n}$, and where $U(t)$ is a suitable function ranging through the set of generating vectors. (See Theorem 1.3 below.) The reason that the function $g$ appears is that the translation function $\lambda_y : B \to L$ sending $x$ to $y * x$ has, as derivative at $0, d\lambda_y(0) = g(\mathrm{ad}\ y) : L \to L$.

Thus the vector fields given by $\Gamma_w : B \to L, \Gamma_w(y) = g(\mathrm{ad}\ y)(w)$ are the left invariant (with respect to $*$) vector fields on $B$.

The methods we propose here are reminiscent of the ones appearing in [**10**].

In the latter sections we use the results of the first section to study the relationship between local semigroups and locally reachable sets in dynamical systems on Lie groups. It is shown that local semigroups always contain locally reachable sets, and the reverse inclusion is obtained under certain circumstances. These results give important parallels and contrasts between the rapidly expanding theory of (local) Lie semigroups and notions of geometric control theory.

**1. The basic differential equation.** Let $L$ be a Dynkin algebra (i.e., a complete normed Lie algebra with continuous operations) and let $B$ be a Campbell-Hausdorff-Baker neighborhood of 0. We employ (ordinary) differential equations to describe the local semigroup generated by a set $\Phi$ of tangent vectors. To this end we give explicitly the differential equation satisfied by a curve $X: [0,T] \to L$ with $X(0) = 0$ which is defined by the property that, up to small terms (i.e., terms of order two or more), the point $X(t+h)$ is obtained from $X(t)$ by Campbell-Hausdorff-Baker multiplication by a vector $hU(t)$, where $U(t)$ ranges through $\Phi$ in a fashion that allows for sufficiently many discontinuities to accommodate sudden changes in direction. Such sudden changes occur in the context of the algebraic generation of a local semigroup and in applications in geometric control. A reasonable class of functions with which to operate, therefore, is the class of *regulated functions* in the sense of Bourbaki [**1**] (fonctions réglées) which may be characterized either as being uniform limits of step functions or else as having limits from the right and the left in all points of their domain of definition (wherever such limits make sense). This class includes the piecewise continuous functions, a class frequently used in geometric control theory.

We turn now to a standard lemma on the Campbell-Hausdorff multiplication (see, e.g., [**2**]).

LEMMA 1.1. *In the algebra of formal power series in one variable* $T$ (*over* $\mathbf{R}$) *we define*

(A) $f(T) = (1 - e^{-T})/T = \sum_{n=0}^{\infty}((-1)^n/(n+1)!)T^n$,

(B) $g(T)=1/f(T)=1+T/2+\sum_{n=1}^{\infty}(b_{2n}/(2n)!)T^{2n}$,

*where the* $b_{2n}$ *are the Bernoulli numbers. In the power series algebra of* (*non-commuting*) *variables* $U, V$ *we have that*

(a) $-U*(U+V)=(f(\mathrm{ad}\,U))(V)+R_1$,

(b) $U*V=U+(g(\mathrm{ad}\,U))(V)+R_2$,

where $R_1$ and $R_2$ denote the sums of the bihomogeneous terms whose degree in $V$ is greater than 1.

COROLLARY 1.2. *Let* $L$ *be a Dynkin algebra, and let* $B$ *be an open symmetric* CHB-*neighborhood of* 0. *Then, for* $u\varepsilon B, v\in L$, *and* $h\in\mathbf{R}$ *where* $hv\in B$,

($\alpha$) $-u*(u+hv)=hf(\mathrm{ad\ u})v+R_1(h)$,

($\beta$) $u*(hv)=u+hg(\mathrm{ad\ u})(v)+R_2(h)$,

*where* $\lim_{h\to 0}\|R_i(h)\|/h=0$ for $i=1,2$.

Equation (D) in the following theorem gives the basic differential equation arising in the generation of local semigroups (with respect to the Campbell-Hausdorff-Baker multiplication). Equation (M) gives a useful alternate.

THEOREM 1.3. *Let* $L$ *be a Dynkin algebra, and let* $B$ *be an open symmetric* CHB-*neighborhood of* 0 *in* $L$. *Let* $U:[0,T]\to L$ *be a regulated function which is continuous on the complement of some countable set* $C$. *Then, for any continuous function* $X:[0,\varepsilon]\to B$ *with* $0<\varepsilon\leqslant T$ *which is differentiable off of* $C$, *the following two statements are equivalent* (*with* $x_0\in B$):

(1) *For all* $t\in[0,\epsilon]\backslash C$ *and all* $h$ *with* $X(t+h), hU(t)\in B$, *we have*

$$(M)\qquad X(t+h)=X(t)*hU(t)+o(t,h),\quad X(0)=x_0$$

*with a suitable remainder function* $o$ *satisfying* $\lim_{h\to 0}\|o(t,h)\|/h=0$.

(2) *For all* $t\in[0,\varepsilon]\backslash C$ *we have*

$$(D)\qquad X'(t)=g(\mathrm{ad}\ X(t))U(t),\quad X(0)=x_0$$

*with* $g$ *the analytic function defined in Lemma* 1.1.

*The differential equation* (D) *is equivalent to the differential equation*

$$f(\mathrm{ad}\ X(t))X'(t) = U(t), \quad X(0) = x_0, \tag{D$'$}$$

*with $f$ as in Lemma* 1.1*, and to the integral equation*

$$X(t) = x_0 + \int_0^t g(\mathrm{ad}\ X(s))U(s)ds. \tag{I}$$

*The function $X$ is uniquely determined on $[0, \varepsilon]$ by (D), (D$'$), or (I).*

PROOF. (1) implies (2). By Corollary 1.2, there exists a suitable remainder function $R$ with $\lim_{h\to 0} \|R(t,h)\|/h = 0$ such that, for $h \neq 0$,

$$\begin{aligned} X(t) * hU(t) &= X(t) + g(\mathrm{ad}\ X(t))hU(t) + R(t,h) \\ &= X(t) + h(g(\mathrm{ad}\ X(t))U(t) + h^{-1}R(t,h)). \end{aligned} \tag{$\alpha$}$$

It then follows from (M) in condition (1) that

$$(X(t+h) - X(t))/h = g(\mathrm{ad}\ X(t))U(t) + h^{-1}(o(t,h) + R(t,h)).$$

Whenever $t \notin C$, we can pass to the limit by letting $h$ approach 0 and obtain (D) in condition (2).

(2) implies (1). Whenever $X$ is differentiable we have, in view of (D),

$$\begin{aligned} X(t+h) &= X(t) + hX'(t) + \delta(t,h) \\ &= X(t) + g(\mathrm{ad}\ X(t)hU(t) + \delta(t,h), \end{aligned}$$

with a suitable remainder term $\delta$ satisfying $\lim_{h\to 0} \|\delta(t,h)\|/h = 0$. Then condition $(\alpha)$ above yields

$$X(t+h) = X(t) * hU(t) + r(t,h)$$

with $o(t,h) = \delta(t,h) - R(t,h)$. This proves (1).

Equation (D$'$) is a simple transformation of (D) which is equivalent in light of Lemma 1.1, and (I) follows by straightforward integration.

The right-hand side of (D) satisfies a local Lipschitz condition on account of

$$g(\mathrm{ad}\ X)U(t) - g(\mathrm{ad}\ Y)U(t) = \frac{1}{2}[X - Y, U(t)] + \sum_{n\geq 0}(1/n!)b_{2n}(\mathrm{ad}\ X^{2n} - \mathrm{ad}\ Y^{2n})U(t).$$

Thus the solutions of (D) are uniquely determined by the initial condition $X(0) = x_0$ (see, e.g., [**1**, II]). $\square$

REMARK. For every $x_0 \in B$, equation (D) has locally a unique solution $X(t)$ with initial value $X(0) = x_0$ (see [**1**, II]). We recall additionally the dependence of initial values and parameters (see [**1**, II]) and record the following property:

PROPOSITION 1.4. *If $x_n$ is a sequence of vectors in $B$ converging to $x_0$ in $B$ and if $\lim U_n = U$ uniformly, then the solutions $X_n(t)$ of*

$$(\mathrm{D_n}) \qquad X'_n(t) = g(\mathrm{ad}\ X_n(t))U_n(t), U_n(0) = x_n$$

*will converge uniformly to a solution of* (D) (*assuming the solutions are all in $B$ on* $[0, T]$). $\square$

PROPOSITION 1.5. *Suppose that $X(t)$ is a solution of (*D*) and that $y \in B$. Then $t \to y * X(t)$ is a solution of (*D*) with $y * X(0) = y * x_0$.* (*We assume here that $y * X([0, \varepsilon]) \subseteq B$.*)

PROOF. We have

$$y * X(t + h) = y * (X(t) * hU(t) + o(t, h)),$$

where $\lim_{h\to 0} \|o(t, h)\|/h = 0$ from condition (M) of Theorem 1.3. Fix $t$ and consider the curves $\alpha(h) = X(t) * hU(t), \beta(h) = r(t, h)$, and the vector-valued function $A(v) = y * v$. Let $\Gamma(h) = y * (X(t) * hU(t) + o(t, h)) = A(\alpha(h) + \beta(h))$. By the chain rule, $\Gamma'(0) = A'(\alpha(0)+\beta(0))[\alpha'(0)+\beta'(0)] = A'(\alpha(0))(\alpha'(0))$ (since $\beta(0) = 0 = \beta'(0)$ from the conditions $o$ satisfies). Set $\Lambda(h) = y*(X(t)*hU(t)) = A(\alpha(h))$.

Then $\Lambda'(0) = A'(\alpha(0))[\alpha'(0)]$. Thus $\Gamma$ and $\Lambda$ have the same derivative at 0. It follows that the derivative of $\Gamma - \Lambda$ is 0, and hence there exists a function $s(t,h)$ satisfying $\lim_{h\to 0} \|s(t,h)\|/h = 0$ defined by

$$\begin{aligned} s(t,h) &= \Gamma(h) - \Lambda(h) \\ &= y * (X(t) * hU(t) + o(t,h)) - y * X(t) * hU(t) \\ &= y * X(t+h) - y * X(t) * hU(t). \end{aligned}$$

By (M) of Theorem 1.3, the proof is complete. □

We note that every function of the form $t \to tw$ is a solution of (D) for the constant function $U(t) = w$, which is another way of saying that $t \to tw$ is a one parameter semigroup with respect to $*$. After Proposition 1.5, the function $t \to v * tw$ is a solution of (D) with $U(t) = w$, too. By induction, from this observation we find

PROPOSITION 1.6. *Suppose that $w_1, \dots, w_n$ are vectors, $t_1, \dots, t_n$ are positive real numbers, and $t_1w_1 * \cdots * t_{k-1}w_{k-1} * tw_k \in B$, a* CHB-*neighborhood, for all $k$ and all $t, 0 \leqslant t \leqslant t_k$. We define $s_k = t_1 + \cdots + t_k$ and*

$$w_k(t) = (t - s_{k-1})w_k \quad \text{for } s_{k-1} \leqslant t < s_k, \quad k = 1, \dots, n,$$

*and $w_n(s_n) = t_n w_n$.*

*From these functions we construct a continuous, piecewise differentiable function $X$: $[0, s_n] \to B$ defined by*

$$X(t) = t_1w_1 * \cdots * t_{k-1}w_{k-1} * w_k(t) \quad \text{for } s_{k-1} \leqslant t < s_k$$

*(It is understood that $X(t) = w_1(t)$ on $[0, t_1)$!).*

*Furthermore we let $U$ be the piecewise constant function which is $w_k$ on $[s_{k-1}, s_k)$ and $U(s_n) = w_n$.*

*Then $X$ is a solution of (D) for $U$ with $X(0) = 0$. Moreover, this solution is contained in the local semigroup $S \subseteq B$ generated by*

$$\cup\{\mathbf{R}^+ w_k \cap B : k = 1, \dots, n\} \ (\textit{with } \mathbf{R}^+ \textit{the non–negative reals}).$$

(For a thorough discussion of local semigroups with respect to $B$ and local generation of local semigroups, we refer to [**7**].)

**2. Locally reachable sets.** Let $B$ be a CHB-neighborhood in a Dynkin algebra $L$, and let $\Phi \subseteq L$. For a regulated function $U\colon [0,T] \to \Phi$, we say that $U$ *admits a principal solution in* $B$ if there exists $X\colon [0,T] \to B$ satisfying (D) of Theorem 1.3 with initial condition $X(0) = 0$. We denote this principal solution by $X_U$; note that if it exists it is unique. For $T \geq 0$, we say that $x \in B$ is *reachable (from* $0$*) at time* $T$ *by means of the control function* $U$ if $U\colon [0,T] \to \Phi$ is a regulated function, admits a principal solution $X_U(t)$, and $X_U(T) = x$. The set of all points reachable at time $T$ by all regulated control functions with codomain $\Phi$ is denoted $\mathrm{Reach}_\Phi(T;B)$. We also define

$$\mathrm{Reach}_\Phi(<T;B) = \cup_{0\leqslant t<T}\mathrm{Reach}_\Phi(t;B).$$

If $x \in \mathrm{Reach}_\Phi(T;B)$, we say $x$ is $T$-*reachable*. If $\Phi$ is understood, the subscript is frequently dropped.

There are several important minor variants of the preceding notions. The class of control functions may be chosen smaller (e.g., piecewise constant or piecewise continuous functions) or larger (bounded measurable functions); slightly different reachable sets may arise for the different cases. We denote, for example, the points reachable at time $T$ using only piecewise constant controls by pc-Reach$(T,B)$. Also it is convenient to consider the closures of these sets. We denote the closure in $B$ of $\mathrm{Reach}(T;B)$ $(\mathrm{Reach}(<T;B))$ by $\mathrm{wReach}(T;B)(\mathrm{wReach}(<T;B))$. We say $x$ is *weakly reachable at time* $T$ if $x \in w\mathrm{Reach}(T;B)$.

PROPOSITION 2.1. *The point* $x \in B$ *is* $T$*-reachable by means of a piecewise constant control function* $U\colon [0,T] \to \Phi$ *if and only if there exists* $w_1,\dots,w_n \in \Phi$ *and positive real numbers* $t_1,\dots,t_n$ *such that* $x = t_1w_1 * \cdots * t_nw_n, T = \sum_{i=1}^n t_i$, *and* $t_1w_1 * \cdots * t_{k-1}w_{k-1} * tw_k \in B$ *for all* $k, 0 \leqslant t \leqslant t_k$. *If* $y \in B$ *is weakly* $T$*-reachable then it is a limit point of such points* $x$.

PROOF. That the second part of the "if and only if" statement implies the first part follows from Proposition 1.6. Conversely suppose $0 = s_0 < s_1 \cdots < s_n = T$ is a partition of $[0,T]$ with $U(t) = w_i$ on the interval $[s_{i-1}, s_i[$. Then, for $t_i = s_i - s_{i-1}$ and $X\colon [0,T] \to B$ defined as in Proposition 1.6, $X(t)$ is the (unique) solution of (D) for $U$ with initial condition $X(0) = 0$. Then $X(T) = t_1 w_1 * \cdots * t_n w_n$, so the second part follows.

If $y \in B$ is $T$-reachable by means of some regulated control function $U$, then $U$ is a uniform limit of piecewise constant functions $U_n : [0,T] \to L$ whose principal solutions converge uniformly to the solution for $U$ (Proposition 1.4). Since the $U_n$ are piecewise constant and close to $U$, they can be altered slightly so that their range is contained in $\Phi$ and they still converge uniformly to $U$; we assume this was originally true. Since the solution for $U$ lies entirely in $B$ and the convergence is uniform, eventually the solutions for the piecewise constant control functions lie entirely in $B$. Hence $y$ is in the closure of the set pc-Reach$(T; B)$.

Thus we have shown that Reach$(T; B)$ is contained in the closure of pc-Reach$(T; B)$, and so its closure in $B$, wReach$(T; B)$, is also. $\square$

There is an alternate notion of controllability which is convenient for our considerations. A point $x \in B$ is *reachable at norm cost $\delta$ by means of the control function $U$* if $U\colon [0,T] \to \Phi$ is regulated, admits a principal solution $X_U(t)$ with $X_U(T) = x$, and $\delta = c(U) = \int_0^T \|U(t)\| dt$. Let NReach$_\Phi(\delta; B)$ denote the set of all points reachable at norm cost $\delta$ by all regulated control functions with codomain $\Phi$. Modifications such as NReach$_\Phi(< \delta; B)$ are defined in a way analogous to time reachability. Of course, some fixed norm on $L$ is assumed throughout.

The following proposition compares the two notions.

PROPOSITION 2.2. *Suppose $\|x\| \leqslant M$ for all $x \in \Phi$. Then* Reach$(< T; B) \subseteq N$Reach$(< MT; B)$. *On the other hand, if, for some $\varepsilon > 0, x \in \Phi$ implies $x = ry$ for some $y \in \Phi$ with $\varepsilon \leqslant \|y\|$, then*

$$\mathrm{wNReach}(< \delta; B) \subseteq \mathrm{wReach}(< \varepsilon^{-1}\delta; B).$$

PROOF. Let $x$ be reachable at time $s < T$ by means of the control function $U\colon [0,s] \to \Phi$. Then $c(U) = \int_0^s \|U(t)\| dt \leqslant \int_0^s M dt = Ms < MT$.

Conversely, let $x$ be reachable at norm cost $\gamma < \delta$ by means of a piecewise constant control function $U : [0,T] \to \Phi$. There exists a partition $0 = s_0 < s_1 < \cdots < s_n = T$ such that $U(t) = w_i \in \Phi$ for all

$s_{i-1} < t < s_i$. Then $\gamma = c(U) = \sum_{i=1}^n \|w_i\|\Delta s_i$. For each $w_i$, there exists $v_i \in \Phi, r_i > 0$, with $w_i = r_i v_i$, and $\varepsilon \leqslant \|v_i\|$. From the proof of Proposition 2.1

$$x = t_1 w_1 * \cdots * t_n w_n = t_1 r_1 v_1 * \cdots * t_n r_n v_n,$$

where $t_i = s_i - s_{i-1} = \Delta s_i$. Let $S = \sum_{i=1}^n t_i r_i$ and define $V\colon [0,S] \to \Phi$ by $V(t) = v_i$ for $0 < t - \sum_{j=1}^{i-1} t_j r_j < t_i r_i$. Again, by Proposition 2.1, $x$ is reachable at time $S$ by means of the control function $V$. Note that $\gamma = \sum_{i=1}^n \|w_i\|\Delta s_i = \sum_{i=1}^n r_i \|v_i\| t_i \geq \varepsilon \sum_{i=1}^n r_i t_i = \varepsilon S$. Thus $S \leqslant \varepsilon^{-1}\gamma < \varepsilon^{-1}\delta$. We conclude pc-NReach$(<\delta; B) \subseteq$ pc-Reach$(\varepsilon^{-1}\delta; B)$.

Now suppose $x$ is reachable at norm cost $\gamma < \delta$ by means of a regulated function $U$. Then $\gamma = c(U) = \int_0^T \|U(t)\|dt$. Now $U$ is the uniform limit of a sequence of piecewise constant functions $U_n : [0,T] \to \Phi$ (see the proof of Proposition 2.1). Thus $c(U_n) \to c(U)$. Hence $c(U_n) < \delta$ for large $n$. Let $X_n : [0,T] \to B$ be the principal solution for $U_n$ and $x_n = X_n(T)$. By the first part, $x_n$ is reachable in time less than $\varepsilon^{-1}\delta$ for large $n$. Since $x_n \to x, x \in w$Reach$(<\varepsilon^{-1}\delta; B)$, the closure of the points reachable in less than time $\varepsilon^{-1}\delta$. Hence NReach$(<\delta; B) \subseteq$ wReach$(<\varepsilon^{-1}\delta; B)$, and the same containment holds for the closure wNReach$(<\delta; B)$. $\square$

The next two results demonstrate the intuitively plausible fact that if the control is bounded, then one is unable to get very far from the origin in short amounts of time.

Henceforth we fix a norm compatible with the Lie algebra structure in the usual sense, i.e., with $\|[x,y]\| \leqslant \|x\| \, \|y\|$, among other things. The following condition (#) is certainly satisfied for all sufficiently small $r > 0$ on account of the explicit form of the power series $g(T) = 1 + T/2 + \cdots$.

$$(\#) \qquad \|g(\mathrm{adx})\| \leqslant 1 + \|x\| \text{ for all } x \in B_r.$$

LEMMA 2.3. *Let $B_r \subseteq B$ be an open ball of radius $r$ around* 0 *which satisfies the condition* (#). *If $X : [0,\varepsilon] \to B$ is the principal solution for $U$ and $\delta = \int_0^\varepsilon \|U(t)\|dt$, then $\|X(t)\| < 2\delta$, i.e., $X(t) \in 2B_\delta$ for all $t \in [0,\varepsilon]$, provided that $2\delta < \min\{1, r\}$.*

PROOF. Assume the conclusion is false. Then $\|X(t)\| = 2\delta$ for some $t \in [0, \varepsilon]$, and we may consider the smallest such in view of the continuity of $X$. From the equivalent integral condition (I) of Theorem 1.3, we obtain the estimate

$$\begin{aligned}
\|X(t)\| &\leqslant \int_0^t \|g(\text{ad } X(s))\| \, \|U(s)\| ds \\
&\leqslant \int_0^t (1 + \|X(s)\|)\|U(s)\| ds \\
&\qquad\qquad \text{since} \|X(s)\| \leqslant 2\delta < r \quad \text{for} \quad 0 \leqslant s \leqslant t \\
&\leqslant (1 + 2\delta) \int_0^\varepsilon \|U(s)\| ds \\
&\leqslant \delta + (2\delta)\delta < 2\delta.
\end{aligned}$$

Thus we obtain $2\delta = \|X(t)\| < 2\delta$, a contradiction. This establishes the claim. ◻

PROPOSITION 2.4. *Let $\Phi \subseteq L$ be bounded* (*respectively $\Phi \subseteq L$ be arbitrary*), *let $B$ be* CHB-*neighborhood, and let $B'$ be open with $0 \in B' \subseteq B$. Then, for all sufficiently small $T$* (*respectively $\delta = \int_0^T \|U(t)\| dt$*), *if $U : [0, T] \to \Phi$ is regulated, then there exists a principal solution $X_U : [0, T] \to B$; furthermore, $X(t) \in B'$ for $0 \leqslant t \leqslant T$.*

PROOF. If we have the proposition for $\Phi$ arbitrary, it follows readily for bounded $\Phi$, for if $\Phi$ is bounded in norm by $M$ and $T$ is chosen less than $\delta/M$, then, for $U : [0, T] \to \Phi$, $\int_0^T \|U(t)\| dt \ll MT < \delta$.

If $\text{norm}_1$ is bounded by a scalar multiple of $\text{norm}_2$, then the $\text{norm}_1$ cost of any regulated function $U$ is bounded by the same scalar multiple of the $\text{norm}_2$ cost of $U$. Hence, if the proposition is true for some norm, it is true for any equivalent norm. So, as in Lemma 2.3, we assume a norm with $\|g(\text{ad } x)\| \leqslant 1 + \|x\|$ for $\|x\| < r$. Pick $\delta$ with $2\delta < \min\{1, r\}, 3B_\delta \subseteq B'$. Let $U : [0, T] \to \Phi$ be a regulated function with $\int_0^T \|U(t)\| dt = \delta$. Then $U$ is the uniform limit of piecewise limit of piecewise constant $U_n : [0, T] \to \Phi$. If $\delta_n = \int_0^T \|U_n(t)\| dt$, then $\delta_n \to \delta$. So, for large $n, 2\delta_n < \min\{1, r\}$ and $2B_{\delta_n} \subseteq B'$. Proposition 1.6 shows how to construct a solution $X_n$ of (D) for $U_n$, and Lemma

2.3 guarantees $X_n(t) \in 2B_\delta$ for all $t, 0 \leqslant t \leqslant T$. By Proposition 1.4 these solutions converge uniformly to a solution $X$ of (D) for $U$, and it follows that $X(t) \in 2\overline{B}_\delta \subseteq 3B_\delta \subseteq B'$ for $0 \leqslant t \leqslant T$. ◻

The definition of the sets Reach($<T; B$) and NReach($<|!\delta; B$) depend on a previous choice of an open ball $B$ of reference and might, even for very small $T$ (respectively $\delta$), change if $B$ is varied even slightly. Proposition 2.4 says that in fact this is not the case and thus allows us to make the following definition.

DEFINITION 2.5. We define Reach($< T$) in case $\Phi$ is bounded to be Reach($< T; B$), provided that there is a CHB-neighborhood $B$ such that the principal solution $X_U : [0, T] \to B$ exists for every regulated $U : [0, T] \to \Phi$. We note by Proposition 2.4 that Reach($<T$) is well-defined for all small enough $T$. Similar definitions and remarks apply to NReach($<!\, \delta$) for $\Phi$ arbitrary, and for wReach($<T$) and wNReach($<T$).

We come now to one of the main results announced in the introduction.

THEOREM 2.6. *Let $L$ be a Dynkin algebra, $B$ a* CHB*-neighborhood, $\Phi \subseteq L$, and $W = \mathbf{R}^+\Phi$. Let $S$ denote the local semigroup with respect to $B$ generated by $W \cap B$, and let $S^*$ be its closure in $B$. Then* pc–NReach($< \delta$) $\subseteq S$ *and* wNReach($< \delta$) $\subseteq S^*$ *for all $\delta$ sufficiently small. Similarly if $\Phi$ is bounded,* pc $-$ Reach($<T$) $\subseteq S$ *and* wReach($< T$) $\subseteq S^*$ *for all $T$ sufficiently small.*

PROOF. By Proposition 2.4 the principal solution exists and lies in $B$ for any piecewise constant $U : [0, T] \to \Phi$ such that $c(U) = \int_0^T \|U(t)\| dt$ is sufficiently small. It then follows from Proposition 1.6 that pc-NReach($< \delta$) $\subseteq S$ for all $\delta$ sufficiently small. The argument in Proposition 2.4 shows that any element of NReach($< \delta$) is a limit point of members of pc-NReach($< \delta$), and hence in $S^*$. It follows from regularity that wNReach($< \delta$) $\subseteq S^*$ for all $\delta$ sufficiently small.

An analogous argument establishes the version of the theorem for $\Phi$ bounded; alternatively the first part of Proposition 2.2 may be employed. ◻

**3. Rerouting.** In §2 we have shown that, for $\Phi \subseteq L$, the closed local semigroup $S^*$ generated by $\mathbf{R}^+\Phi \cap B$ (where $B$ is a CHB-neighborhood) contains wNReach($< \delta$) for small $\delta$. We consider now the converse problem of whether wNReach($< \delta$) contains a local semigroup generated by $\mathbf{R}^+\Phi \cap B$ for some CHB-neighborhood $B$ of 0. In general this is not the case, and the problem of determining precisely when it is appears quite difficult. In the remainder of the paper we employ results of [**8**] and [**5**] to obtain affirmative solutions in certain special cases.

DEFINITION 3.1. Let $\Phi \subseteq L$. The set $\Phi$ *admits rerouting locally* if there exists a Dynkin algebra norm $\| \cdot \|$ on $L$ such that, given $r > 0$, there exists $0 < \delta \leqslant r$ and a CHB-neighborhood $B$ of 0 of diameter less than $r$ satisfying NReach($< 2\delta$) $\cap B \subseteq$ NReach($< \delta$).

PROPOSITION 3.2. *Let $\Phi$ be a subset of $L$ which admits rerouting locally. Then given $r > 0$, there exists $\delta, 0 < \delta \leqslant r$ such that, for all sufficiently small convex neighborhoods $C$ of $0$, the set*

$$A_C(\delta) = \text{wNReach}(< \delta) \cap C$$

*is a local semigroup with respect to $C$ which contains $\mathbf{R}^+\Phi \cap C$ and is closed in $C$.*

PROOF. Let $r > 0$. Choose $\delta$, $0 < \delta \leqslant r$, and $B$ satisfying the conditions of Definition 3.1 for $r$. Let $C$ be any convex neighborhood of 0 such that $C \subseteq B$. (We also assume wNReach($< 2\delta$) is well-defined and in $B$; see Definition 2.5.)

Suppose $x_1, x_2 \in A_C(\delta)$ and $x_1 * x_2 \in C$. Let $\epsilon < 0$. Let $V_1$ and $V_2$ be neighborhoods of $x_1$ and $x_2$ respectively such that $V_1 * V_2 \subseteq B$ and every point of $V_1 * V_2$ is within distance of $\varepsilon$ of $x_1 * x_2$. Since $x_i$ is in wNReach($< \delta$), there exists a regulated function $U_i : [0, T_i] \to \Phi$ such that $\int_0^{T_i} \|U_i(t)\| dt < \delta$ and such that the principal solution $X_i(t)$ for $U_i$ satisfies $X_i(T_i) \in V_i, i = 1, 2$.

Define $U : [0, T_1 + T_2] \to W$ by $U(t) = U_1(t)$ if $0 \leqslant t \leqslant T_1, U(t) = U_2(t - T_1)$ for $T_1 < t \leqslant T_1 + T_2$. Then the function $X : [0, T_1 + T_2] \to L$, defined by $X(t) = X_1(t)$ if $0 \leqslant t \leqslant T_1$ and $X(t) = X_1(T_1) * X_2(t - T_1)$ for

$T_1 \leqslant t \leqslant T_1 + T_2$, is a solution of (D) for $U$ with initial value $X(0) = 0$ (see Proposition 1.5). Since $X(T_1 + T_2) = X_1(T_1) * X_2(T_2) \in V_1 * V_2 \subseteq B$, since $\int_0^{T_1+T_2} \|U(t)\| dt = \int_0^{T_1} \|U_1(t)\| dt + \int_0^{T_2} \|U_2(t)\| dt < 2\delta$ and since $\Phi$ admits rerouting locally, we conclude $X_1(T_1) * X_2(T_2) \in \mathrm{NReach}(< \delta)$. Since $\varepsilon$ was arbitrary, $x_1 * x_2 \in \mathrm{wNReach}(< \delta) \cap C = A_C(\delta)$. Thus $A_C(\delta)$ is a closed local semigroup with respect to $C$.

Let $x \in \Phi$. Define $X\colon [0, T] \to L$ by $X(t) = tx$. Then $X'(t) = x \in \Phi$. Hence there exists $0 < \varepsilon \leqslant T$ such that $tx \in A_C(T)$ for $0 \leqslant t \leqslant \varepsilon$. Since $A_C(T)$ is a local semigroup with respect to $C$, we conclude $\mathbf{R}^+ x \cap C \subseteq A_C(T)$ (since $\mathbf{R}^+ x \cap C$ is the local semigroup with respect to $C$ generated by $[0, \varepsilon] \cdot x$). This completes the proof. □

Corollary 3.3. *Let $\Phi$ be a subset of $L$ which admits rerouting locally. Let $W = \mathbf{R}^+ \cdot \Phi$. Given a* CHB*-neighborhood $B$, let $(S_B)^*$ denote the smallest local semigroup closed in $B$ containing $W \cap B$. Then there exists $\delta > 0$ and a basis of convex neighborhoods $C$ of* 0 *contained in $B$ such that*

$$(S_C)^* \subseteq \mathrm{wNReach}(< \delta) \subseteq (S_B)^*.$$

Proof. The first assertion is just Theorem 2.5. The second assertion follows from Proposition 3.2 by choosing $C$ sufficiently small. Then $(S_C)^* \subseteq A_C$ since the latter is a closed local semigroup containing $C \cap W$, and $A_C(T) \subseteq \mathrm{wNReach}(< \delta)$. □

Thus, loosely speaking, we have in this setting that local semigroups contain locally reachable sets and vice-versa.

**4. Achieving local rerouting.** Let $L$ be a Dynkin algebra, $\Phi \subseteq L$. We can make $L$ into an abelian Lie algebra by making the Lie multiplication trivial. In this case $g(\mathrm{ad}\ x)$ is just the identity, and the integral equation (I) in Theorem 1.3 simplifies to

$$Z_U(t) = \int_0^t U(s) ds,$$

where $Z_U(t)$ is the solution of (D) (for the abelian case) with initial condition $Z_U(0) = 0$ for the control function $U$. The Campbell-Hausdorff-Baker multiplication for this case is just addition.

Since $g(T) = 1 + T/2 + \cdots$, it follows (as in Lemma 2.3) that, for some open ball $B_r$ of small enough radius $r$,

$$(\#\#) \qquad \|g(\text{ad } x)(y) - y\| \leqslant \|x\| \ \|y\| \text{ for } x \in B_r$$

for some equivalent norm $\|\cdot\|$.

Suppose now that $U : [0, T] \to \Phi$ is a bounded regulated function that admits a principal solution $X_U : [0, T] \to B$. Suppose further that $\delta = c(U) = \int_0^T \|U(t)\| dt$ satisfies $2\delta < \min\{1, r\}$, where $r$ is chosen as in the preceding paragraph. Then we have, for $0 < t \leqslant T$,

$$\begin{aligned}
\|X_U(T) - Z_U(U)\| &= \Big\| \int_0^T (g(\text{ad } X_U(t))U(t) - U(t)dt \Big\| \\
&\leqslant \int_0^T \|g(\text{ad } X_U(t))U(t) - U(t)\| dt \\
&\leqslant \int_0^T \|X_U(t)\| \ \|U(t)\| dt \\
&\qquad \text{(by the preceding paragraph)} \\
&\leqslant \int_0^T \big(2 \int_0^t \|U(s)\| ds\big) \|U(t)\| dt \text{ (by Lemma 2.3)} \\
&= 2 \int_0^T \int_0^t \|U(s)\| \ \|U(t)\| ds \ dt \\
&= \frac{1}{2} \Big( 2 \int_0^T \int_0^T \|U(s)\| \ \|U(t)\| ds \ dt \Big) \\
&\qquad \text{(since the integrand is symmetric} \\
&\qquad \text{about the diagonal)} \\
&= \int_0^T \|U(s)\| ds \cdot \int_0^T \|U(t)\| dt = (c(U))^2.
\end{aligned}$$

Recall also from Proposition 2.4 that the principal solution $X_U(t)$ exists for small $\delta$. Thus we have

PROPOSITION 4.1. *Let $\Phi \subseteq L$, a Dynkin algebra, and let $B$ be a* CHB-*neighborhood. There exists $\varepsilon > 0$ such that, for any regulated bounded control function $U : [0, T] \to \Phi$ with $c(U) = \int_0^T \|U(t)\| dt < \varepsilon$,* *the principal solution* $X_U\colon [0,T] \to B$ *exists, and* $\|X_U(T) - Z_U(T)\| \leqslant (c(U))^2$, *where* $Z_U(T) = \int_0^T U(t)dt$.

We consider now the case that $\Phi \subseteq K$, a closed cone (= proper cone) in a finite dimensional Lie algebra $L$ or a strictly positive cone in a Dynkin algebra (see [**8**] for the definition). In [**8**] it is shown that the norm on $L$ can be chosen so that it is also additive on $K$. For any bounded regulated function $U : [0,T] \to \Phi$ we then have $\|Z_U(T)\| = \|\int_0^T U(t)d(t)\| = \int_0^T \|U(t)\|dt = c(U)$. (The verification follows directly from the additivity of the norm for piecewise constant functions, and in general by taking uniform limits of such.)

THEOREM 4.2. *Let* $K$ *be a closed proper cone in a finite-dimensional Lie algebra* $L$ (*or a strictly positive cone in a Dynkin algebra*). *Then* $\Phi \subseteq K$ *admits rerouting locally.*

PROOF. Let $r > 0$. Choose $2\delta$ smaller than $\varepsilon$ in Proposition 4.1, for an additive norm $\|\cdot\|$, $2\delta < 1/4$. Choose $B$ to be the ball of radius $\delta/2$ around 0. Suppose $x \in \mathrm{NReach}(< 2\delta) \cap B$. Then there exists a regulated control function $U\colon [0,T] \to \Phi$ with principal solution $X_U(t)$ such that $c(U) < 2\delta$ and $X_U(T) = x$. By Proposition 4.1 and the remarks preceding this proposition,

$$\|X_U(T) - Z_U(T)\| \leqslant (c(U))^2 < (2\delta)^2 < \frac{1}{4}(2\delta) = \frac{1}{2}\delta.$$

Thus

$$c(U) = \|Z_U(T)\| \leqslant \|Z_U(T) - X_U(T)\| + \|X_U(T)\| < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta. \quad \square$$

Note that, in actuality, no rerouting is necessary for the cone case. If a reachable point has a small norm, then control functions to reach it must have a proportionally small norm cost.

We consider finally the construction of Hilgert and Hofmann in [**5**]. We restrict to the finite-dimensional case (the results remain true in Dynkin algebras with appropriate modifications). Let $W$ be a Lie wedge in $L$, i.e., a closed wedge whose edge $H = W \cap -W$ satisfies the condition $e^{\mathrm{ad}\, x} W = W$ for all $x \in H$. Let $V$ be a vector space complement for $H$ in $L$. Then $K = V \cap W$ is a closed (proper) cone. For small CHB-neighborhoods, $B$, there exist unique continuous projections $p_H \colon B \to H$ and $p_V \colon B \to V$ such that $x = p_V(x) * p_H(x)$. Pick a closed cone $K_0$ in $V$ surrounding $K$ (see [**7**] for the definition and construction). Hofmann and Hilgert show, for CHB-neighborhoods $B$ small enough, that if $X \colon [0,T] \to B$ is a solution of differential equation (D) with $X(0) = 0$ for $U \colon [0,T] \to W$, then $p_V \circ X$ is also a solution for some $U_V \colon [0,T] \to W$. Pick a norm additive on $K_0$ that also satisfies the other requirements of this section. We show $\Phi = W$ admits local rerouting.

Pick a small $\delta > 0$, with the ball $B$ of radius $2\delta$ satisfying the previous "smallness" conditions. Pick a smaller open ball $C$ around 0 with $p_H(C)$ and $p_V(C)$ contained in the ball of radius $\delta/3$. Suppose $X : [0,T] \to B, X(T) \in C$, is the principal solution for some $U : [0,T] \to \Phi$. Let $Y = p_V \circ X$. Then it is shown in [**5**] that (i) $Y$ is the principal solution for some regulated $U_1 : [0,T] \to W$ (Proposition 3.8); (ii) $Y'(t) \in K_0$; and (iii) $t \to \|Y(t)\|$ is increasing on $[0,T]$ (see Lemma 4.2). Since the norm is additive on $K_0$, we have $\|Y(T)\| = \| \int_0^T Y'(t)dt\| = \int_0^T \|Y'(t)\|dt$. Thus

$$\begin{aligned}
c(U_1) - \|Y(T)\| &\leqslant \Big| \int_0^T \|U_1(t)\|dt - \int_0^T \|Y'(t)\|dt \Big| \\
&\leqslant \int_0^T \big| \|Y'(t)\| - \|U_1(t)\| \big| dt \\
&\leqslant \int_0^T \|Y'(t) - U_1(t)\|dt \\
&= \int_0^T \|g(\mathrm{ad}\; Y(t))U_1(t) - U_1(t)\|dt \\
&\leqslant \int_0^T \|Y(t)\|\; \|U_1(t)\|dt \text{ (by (\#\#))} \\
&\leqslant \|Y(T)\| \int_0^T \|U_1(t)\|dt \text{ by increasing property} \\
&= \|Y(T)\| c(U_1).
\end{aligned}$$

Solving for $c(U_1)$, we obtain

$$\begin{aligned} c(U_1) &< \|Y(T)\|/(1-\|Y(T)\|) \\ &< \frac{1}{3}\delta/\frac{2}{3}\delta = \frac{1}{2}\delta. \end{aligned}$$

Since $X(T) \in C, \|p_H(X(T))\| < \delta/3$. Thus, using the constant control $p_H(X(T))$, we see that $p_H(X(T)) \in \mathrm{NReach}(\delta/3)$. Pasting this control together with $U_1$ (as, e.g., in Proposition 3.8, the Rerouting Theorem, of [**5**]), we see that

$$X(T) = p_V(X(T)) * p_H(X(T)) \in \mathrm{NReach}(\delta).$$

Thus $\Phi = W$ admits local rerouting. Thus we have proved

THEOREM 4.3. *Let $\Phi = W$ be a Lie wedge. Then $\Phi$ admits local rerouting.* □

REMARK 4.4. Note that Corollary 3.3 says closed local semigroups generated by $W \cap C$ are contained in $\mathrm{wNReach}(< \delta)$ for small $C$. By Proposition 2.2 we have also the same conclusion for $\mathrm{wReach}_\Phi(< T)$. Hence, for Lie wedges, local semigroups contain locally reachable sets and vice-versa.

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