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CERTAIN POLYNOMIAL SUBORDINATIONS

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Let K denote the family of analytic functions f in the open unit disk Δ that are normalized by $f(0) = f'(0) - 1 = 0$ and map Δ onto a convex region. Some years ago, T. Basgöze, J. Frank, and F. Keogh [**1**] determined necessary and sufficient conditions on the complex numbers λ, μ such that, for all $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in $K, \lambda z + \mu a_2 z^2$ maps Δ onto a convex region and the subordinations $z/2 \prec \lambda z + \mu a_2 z^2 \prec f(z)$ hold for z in Δ. Later Chiba [**3**] considered the analogous problem with z^2 replaced by z^n for an integer $n \geq 2$ and obtained necessary conditions on λ, μ for the subordinations $z/2 \prec \lambda z + \mu a_n z^n \prec f(z)$ in Δ. With this paper we present a relatively simple and direct proof of the necessary and sufficient conditions for a slightly more general version of these results.

Our principal theorem is

THEOREM A. *For a given integer $n \geq 2$, let λ be a positive real number and μ a complex number such that $\lambda z + \mu z^n$ is locally univalent in Δ. The subordinations*

$$z/2 \prec \lambda z + \mu a_n z^n \prec f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (z \in \Delta) \tag{1}$$

are valid for all $f \in K$ if and only if $\lambda = 1/2 + (-1)^n \mu, 0 \leq (-1)^n \mu \leq 1/(2(n^2-1))$.

PROOF. Since $\lambda z + \mu z^n$ is locally univalent, we have $\lambda \neq -n\mu z^{n-1}$ for $z \in \Delta$. It follows that $\lambda \geq n|\mu|$ and, hence, $\lambda z + \mu z^n$ is univalent in Δ.

Suppose (1) holds for all $f \in K$. Since $f_0(z) = z/(1-z)$ is in K, the first subordination and the univalence of $\lambda z + \mu z^n$ implies $|\lambda e^{i\theta} + \mu e^{in\theta}| \geq 1/2$ for all real θ. When $\mu \neq 0$ and $\theta = (-\arg \mu + \pi)/(n-1)$, we have $|\lambda - |\mu|| = \lambda - |\mu| \geq 1/2$. Furthermore, from the second subordination and the fact that $\Re f_0(z) \geq -1/2$ for z in Δ, it follows that

$$\Re(\lambda z + \mu z^n) \geq -1/2 \quad (z \in \Delta). \tag{2}$$

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When $z = x, -1 < x < 0$, and $x \to -1$, we obtain from (2) that $\lambda - (-1)^n \Re\mu \le 1/2$. Since $\lambda \ge 1/2 + |\mu|$, this implies μ is real, $(-1)^n \mu = |\mu|$, and $\lambda = 1/2 + (-1)^n \mu$. Again, by (2) with $z = e^{i\theta}, -\pi < \theta < \pi$, we have $0 \le 1 + 2\lambda \cos\theta + 2\mu \cos n\theta = 1 + \cos\theta + 2(-1)^n \mu[\cos\theta + (-1)^n \cos n\theta]$ or, when $\mu \ne 0$,

$$-\frac{\cos\theta + (-1)^n \cos n\theta}{1 + \cos\theta} \le \frac{1}{2(-1)^n \mu}. \tag{3}$$

Let $\theta \to \pi$ to obtain $1/(2(-1)^n \mu) \ge n^2 - 1$, which completes the proof of the necessity of Theorem A.

For the converse, it is known [**2**] that

$$\left|\frac{\cos\theta + (-1)^n \cos n\theta}{1 + \cos\theta}\right| \le n^2 - 1 \quad (-\pi < \theta < \pi).$$

Hence (3) holds when $0 \le (-1)^n \mu \le (1/2)(n^2 - 1)$ and (2) is valid for z in Δ. Since $1 + 2\lambda z + 2\mu z^n$ has positive real part in Δ, there is a probability measure α on $|\sigma| = 1$ such that

$$\lambda z + \mu z^n = \int_{|\sigma|=1} \frac{\sigma z}{1 - \sigma z} d\alpha(\sigma),$$

by the Herglotz representation theorem [**5**, p. 96]. Therefore, for any integer $k > 1, k \ne n$,

$$\lambda = \int_{|\sigma|=1} \sigma d\alpha(\sigma), \quad \mu = \int_{|\sigma|=1} \sigma^n d\alpha(\sigma), \quad \int_{|\sigma|=1} \sigma^k d\alpha(\sigma) = 0,$$

and, for z in Δ,

$$\lambda z + \mu a_n z^n = \int_{|\sigma|=1} f(\sigma z) d\alpha(\sigma) \prec f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

whenever $f \in K$. Finally, $|a_n| \le 1$ when $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ is in K [**5**, p. 117], so, for $|z| = 1$,

$$|\lambda z + \mu a_n z^n| \ge \lambda - |a_n|\ |\mu| \ge \lambda - |\mu| \ge 1/2.$$

This proves $z/2 \prec \lambda z + \mu a_n z^n$ in Δ. □

REMARKS. 1. The conditions of the theorem imply that $\lambda z + \mu a_n z^n$ is a convex univalent function since [**7**] (See also [**5**, p. 128])

$$|\mu a_n|/\lambda \le |\mu|/\lambda = 2|\mu|/(2|\mu|+1) \le 1/n^2.$$

2. The assumption that λ is a positive real number can be replaced by λ complex since, for any analytic function f in the unit disk, $f(\sigma z) \prec f(z)$ for all complex $\sigma, |\sigma| = 1$.

3. We can appeal to a general result of Wilf [**9**] rather than the Herglotz representation theorem in the sufficiency proof of Theorem A.

In [**1**] the authors prove

$$z/2 \prec (1/2+\mu)z + \mu a_2 z^2 \prec (2/3)z + (1/6)a_2 z^2$$

when $0 \le \mu \le 1/6$, that is, the polynomials $(1/2+\mu)z + \mu a_2 z^2$ form a subordination chain when $0 \le \mu \le 1/6$. Based on the following lemma [**4**] (See also [**8**, p. 159]), we prove the corresponding result with z^2 replaced by $z^n, n \ge 2$.

LEMMA. *For z in Δ and t in the real interval $[a,b]$, let $\phi(z,t)$ be an analytic function of z for each t and let ϕ be continuously differentiable with respect to t in $[a,b]$ for each fixed z of Δ. Let $\phi(0,t) = 0$ and $\frac{d}{dz}\phi(0,t) > 0$. Then, for z in $\Delta, \phi(z,t_1) \prec \phi(z,t_2)$ whenever $a \le t_1 \le t_2 \le b$ if and only if, for $0 < |z| < 1, a \le t \le b$, we have*

$$\Re\left\{\frac{\frac{d\phi}{dt}}{z\frac{d\phi}{dz}}\right\} \ge 0 \quad \text{or} \quad \frac{d\phi}{dt} = 0. \tag{4}$$

THEOREM B. *The polynomials $\phi(z,t) = (1/2+t)z + (-1)^n atz^n$, where $|a| \le 1$, form a subordination chain when $0 \le t \le 1/(2(n-1))$.*

PROOF. We have, for $z \in \Delta$ and $0 \le t < 1/(2(n-1))$,

$$\frac{\frac{d\phi}{dt}}{z\frac{d\phi}{dz}} = \frac{1+(-1)^n az^{n-1}}{\frac{1}{2}+t+(-1)^n tnaz^{n-1}} = \frac{1+w}{\frac{1}{2}+t+ntw}, \quad w = (-1)^n az^{n-1}.$$

(The denominator is not zero for $|z| \le 1$.) Since the last expression is analytic for $|w| \le 1$, (4) holds if and only if it holds when $|w| = 1$. Now the real part of the ratio is nonnegative on $|w| = 1$ if and only if

$$\begin{aligned} 0 &\le \Re\biggl\{(1+w)\biggl(\frac{1}{2}+t+nt\overline{w}\biggr)\biggr\} \\ &= \frac{1}{2}\Re(1+w) + nt\biggl[\biggl|w+\frac{1}{2}+\frac{1}{2n}\biggr|^2 - \frac{(n-1)^2}{4n^2}\biggr]. \end{aligned}$$

But $|w+1/2+1/(2n)| \ge (n-1)/(2n)$ when $|w| = 1$. We conclude that (4) holds for $0 \le |z| < 1, 0 \le t < 1/(2(n-1))$, and, by a limiting process, on the closed interval $0 \le t \le 1/(2(n-1))$.

In particular, for all $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in K we have, by Theorem A, that

$$\frac{z}{2} \prec \bigl(\frac{1}{2} + (-1)^n \mu\bigr)z + \mu a_n z^n \prec \frac{n^2}{2(n^2-1)} z + \frac{(-1)^n}{2(n^2-1)} a_n z^n \prec f(z)$$

for z in Δ. The last subordination is 'best' even though the subordination chain in Theorem B with $t = (-1)^n \mu$ does not end at $t = 1/(2(n^2-1))$.

Keogh [**6**] determined necessary and sufficient conditions on λ, μ such that, for all $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in the class S^* of analytic univalent normalized starlike functions in Δ the subordinations $z/4 \prec \lambda z + \mu a_2 z^2 \prec f(z)$ hold in Δ. When z^2 is replaced by z^n, we can prove the necessity of the extension of Keogh's result. We conjecture that the stated conditions are sufficient. □

THEOREM C. *For a given integer $n \ge 2$, let λ be a positive real number and μ a complex number such that $\lambda z + n\mu z^n$ is locally univalent in Δ. If the subordinations*

$$z/4 \prec \lambda z + \mu a_n z^n \prec f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (z \in \Delta) \tag{5}$$

are valid for all $f \in S^*$, then $\lambda = 1/4 + (-1)^n n\mu, 0 \le (-1)^n \mu \le 3/(8n(n^2-1))$.

PROOF. The function $\lambda z + n\mu z^n$ is univalent in Δ and $\lambda \geq n^2|\mu|$. The first subordination in (5) implies for $z = e^{i\theta}, -\pi < \theta \leq \pi$, that

$$|\lambda e^{i\theta} + n\mu e^{in\theta}| \geq 1/4$$

when $f(z) = z/(1-z)^2 \in S^*$. Furthermore, equality holds if and only if $\lambda - n|\mu| = 1/4$ and $e^{i(n-1)\theta} = e^{-i(\beta+\pi)}$, where $\beta = +\arg\mu$ for $\mu \neq 0$. Since $z/(1-z)^2$ maps Δ onto the complex plane slit along the negative real axis from $-1/4$ to $-\infty$, we must therefore have a real θ_0 such that $\lambda e^{i\theta_0} + n\mu e^{in\theta_0} = -1/4$ by the two subordinations of (5). Hence $\theta_0(n-1) \equiv -(\beta+\pi)(\mathrm{mod} 2\pi)$ and

$$\begin{aligned}-1/4 &= e^{i\theta_0}(\lambda + n\mu e^{i(n-1)\theta_0}) = e^{i\theta_0}(\lambda - n\mu e^{-i\beta})\\ &= e^{i\theta_0}(\lambda - n|\mu|) = (1/4)e^{i\theta_0},\end{aligned}$$

that is, $\theta_0 \equiv \pi$ $(\mathrm{mod} 2\pi)$. It follows that $(n-1)\theta_0 \equiv 0$ or $\pi(\mathrm{mod}\, 2\pi)$ according to whether n is odd or even. Since $(n-1)\theta_0 \equiv -(\beta+\pi)(\mathrm{mod}\ 2\pi)$, we have $\beta \equiv 0(\mathrm{mod}\, 2\pi)$ when n is even and $\beta \equiv \pi$ when n is odd. Thus μ is real, $(-1)^n\mu \geq 0$, and $\lambda = 1/4 + (-1)^n n\mu$.

To determine the upper bound on $(-1)^n\mu$, consider the function in S^* given for a fixed $t, 0 < t < \pi$, by

$$f(z,t) = \frac{z}{1 - 2z\cos t + z^2} = z + \sum_{j=2}^{\infty} \frac{\sin jt}{\sin t} z^j.$$

This function maps Δ onto the complex plane slit along the real axis from $-1/(2(1+\cos t))$ through ∞ to $1/(2(1-\cos t))$. By the second subordination of (5) and the fact that μ is real, we have, for $z = x, -1 < x < 0$,

$$-\frac{1}{2(1+\cos t)} \leq \left(\frac{1}{4} + (-1)^n n\mu\right)x + \mu\frac{\sin nt}{\sin t}x^n.$$

Let $x \to -1$ to obtain the inequality

$$-\frac{1}{2(1+\cos t)} \leq -\frac{1}{4} - (-1)^n\mu\left[n - \frac{\sin nt}{\sin t}\right].$$

Since $\sin nt/\sin t < n$ when $t \in (0,\pi)$, we have

$$(-1)^n\mu \leq \frac{1-\cos t}{4(1+\cos t)} \cdot \frac{\sin t}{n\sin t - \sin nt}.$$

Now let $t \to 0$ to obtain $(-1)^n\mu \le 3/(8n(n^2-1))$.

As mentioned previously the restriction of λ to positive real values is without loss of generality. Furthermore, it is easily proved that $(1/4+(-1)^n n\mu)z+\mu a_n z^n$ maps Δ onto a starlike region when $|a_n| \le n$ and $0 \le (-1)^n\mu \le 3/(8n(n^2-1))$. (See [**7**] or [**5**, p. 128].) $\square$

By the proof similar to that of Theorem B, we can obtain

THEOREM D. *The polynomials* $\psi(z,t) = (\frac{1}{4}+nt)z+(-1)^n atz^n$, *where* $|a| \le n$, *form a subordination chain when* $0 \le t \le 1/(4(n^2-n))$.

Keogh [**6**] proved this result for $n = 2$ and $0 \le t \le 1/16$.

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