ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 4, Fall 1990

NORMALITY FOR THE PROBLEM OF BOLZA WITH AN INEQUALITY STATE CONSTRAINT

VERA ZEIDAN AND PIERLUIGI ZEZZA

1. Introduction and preliminary results. The existence of a strong relation between the normality assumption in optimal control and the controllability of the associated variational equation has long since been noted. Here we extend the previous results to the problem of Bolza with general endpoint conditions and in the presence of a state inequality constraint. While the restriction on the initial and final point becomes, through linearization, a linear boundary condition on the variational equation, the state constraint, which induces a forcing term in the adjoint equation, becomes an isoperimetric condition. This kind of correspondence between linear differential equations and its *adjoint* has been introduced in [**5**] but, for the optimal control problem considered here, is new and requires further investigation.

In this paper we consider the problem of minimizing the cost functional

$$J(x,u) = g(x(a),x(b)) + \int_a^b f_0(s,x(s),u(s))\,ds \tag{1.1}$$

over all absolutely continuous functions, $x(\cdot)$, and measurable functions, $u(\cdot)$, satisfying

$$\begin{gathered} \dot{x}(t) = f(t,x(t),u(t)), \quad \text{a.e. } t \in I, \\ u(t) \in U, \quad \text{a.e. } t \in I, \\ \psi(x(a),x(b)) = 0, \\ \phi(x(t)) \le 0, \quad t \in I, \end{gathered} \tag{1.2}$$

where, for given open sets $X \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$, $U \subset V$, and for

The authors wish to thank G.N.A.F.A. of C.N.R. (Italy) and N.S.E.R.C. (Canada) whose financial support made this research possible.
Research of the second author was partly supported by research grant M.P.I. (40%), Teoria del Controllo dei Sistemi Dinamici.

$I = [a, b]$,

$$\begin{aligned}
f_0 &: I \times X \times V \to \mathbf{R}, \quad (t, x, u) \to f_0(t, x, u), \\
f &: I \times X \times V \to \mathbf{R}^n, \quad (t, x, u) \to f(t, x, u), \\
g &: X \times X \to \mathbf{R}, \quad (x, y) \to g(x, y), \\
\psi &: X \times X \to \mathbf{R}^k, \quad (x, y) \to \psi(x, y), \\
\phi &: X \to \mathbf{R}, \quad x \to \phi(x).
\end{aligned}$$

In the remainder of this section, we introduce the terminology that will be needed later and state the Pontryagin Maximum Principle.

DEFINITION 1.1. A pair $(x(\cdot), u(\cdot))$ is said to be *feasible* it $x(\cdot) \in AC(I, X)$, $u(\cdot)$ is measurable and $(x(\cdot), u(\cdot))$ satisfies the constraints (1.2).

DEFINITION 1.2. A feasible pair $(\hat{x}(\cdot), \hat{u}(\cdot))$, with $u \in L^\infty(I, U)$, is a *weak local minimum* for (1.1)–(1.2) if, for some positive ϵ, $(\hat{x}(\cdot), \hat{u}(\cdot))$ minimizes (1.1) over all feasible pairs $(x(\cdot), u(\cdot))$ satisfying

$$\begin{aligned}
x(t) &\in \hat{x}(t) + \epsilon B_n, \quad t \in I, \\
u(t) &\in \hat{u}(t) + \epsilon B_m, \quad \text{a.e. } t \in I,
\end{aligned}$$

where B_N is the unit ball in $\mathbf{R}^N$.

In stating the maximum principle, the following smoothness assumption will be recalled:

(H_1) For $H = (f_0, f)$ and for all $t \in I$, $H(t, \cdot, \cdot)$ is C^1; for all $(x, u) \in X \times V$, $H(\cdot, x, u)$, $H_x(\cdot, x, u)$ and $H_u(\cdot, x, u)$ are measurable on I; there exists an integrable function $m : I \to \mathbf{R}$ such that

$$|H(t, x, u)| + |H_x(t, x, u)| + |H_u(t, x, u)| \leq m(t), \quad (t, x, u) \in I \times X \times V;$$

and g and ψ are C^1 on $X \times X$ and ϕ is C^1 on X.

For a given $\epsilon > 0$, we denote the ϵ-tube around $\hat{u}(\cdot)$ by

$$U_\epsilon(t) = \{u \in \mathbf{R}^m : u \in U \cap (\hat{u}(t) + \epsilon B_m)\}.$$

In what follows the accent "ˆ" denotes evaluation along the trajectory $(\hat{x}(\cdot), \hat{u}(\cdot))$.

A weak local minimum is characterized by

THEOREM 1.1. (PONTRYAGIN MAXIMUM PRINCIPLE). [**1**] *If the regularity assumption* (H_1) *holds and* $(\hat{x}(\cdot), \hat{u}(\cdot))$ *is a weak local minimum* (*with corresponding* ϵ), *then there exists an absolutely continuous function* $p : I \to \mathbf{R}^n$, *a vector* $\lambda \in \mathbf{R}^k$, *a number* λ_0 *and a nondecreasing function* μ *of bounded variation such that, if we denote also with* μ *the nonnegative Radon measure associated with* μ, *we have*

(i)
$$\lambda_0 \geq 0, \qquad \lambda_0 + \|\mu\| + \|p\| > 0,$$

(ii)
$$-\dot{p}(t) = \hat{f}_x^T(t)\left(p(t) + \int_a^t \hat{\phi}_x^T(s)\, d\mu(s)\right) + \lambda_0 \hat{f}_{0x}(t), \quad \text{a.e. } t \in I,$$

and μ *is supported on the set* $\{t \in I : \phi(\hat{x}(t)) = 0\}$,

(iii)
$$p(a) = -\psi_x^T(\hat{x}(a), \hat{x}(b))\lambda + \lambda_0 g_x^T(\hat{x}(a), \hat{x}(b)),$$

(iv)
$$p(b) = \psi_y^T(\hat{x}(a), \hat{x}(b))\lambda - \int_a^b \hat{\phi}_x^T(s)\, d\mu(s) + \lambda_0 g_y^T(\hat{x}(a), \hat{x}(b)),$$

(v)
$$\begin{aligned} \min\Bigg\{ \Big\langle p(t) + \int_a^t \hat{\phi}_x^T(s)\, d\mu(s), f(t, \hat{x}(t), u) \Big\rangle \\ + \lambda_0 f_0(t, \hat{x}(t), u) : u \in U_\epsilon(t) \Bigg\} \\ = \Big\langle p(t) + \int_a^t \hat{\phi}_x^T(s)\, d\mu(s), \hat{f}(t) \Big\rangle + \lambda_0 \hat{f}_0(t), \quad \text{a.e. } t \in I. \end{aligned}$$

If U is convex, then (v) implies

(vi)

$$\left\langle \left(\hat{f}_u^T(t) \left[p(t) + \int_a^t \hat{\phi}_x^T(s)\, d\mu(s) \right] + \lambda_0 \hat{f}_{0u}(t) \right), w \right\rangle \geq 0, \qquad w \in U - \hat{u}(t), \quad \text{a.e. } t \in I. \tag{1.3}$$

REMARK . It is worth noting that conditions (ii)–(vi) depend on $F_{0\mu}(t) := \int_a^t \hat{\phi}_x(s)\, d\mu(s)$ rather than on μ itself. Hence, they are unchanged if we replaced μ by $\mu + \bar{\mu}$, where $\bar{\mu}$ satisfies $F_{0\bar{\mu}}(t) \equiv 0$; this kind of invariance does not hold true for condition (i).

In order to pinpoint the class of possible candidates for a weak local minimum, let us introduce

DEFINITION 1.3. An *extremal* is an admissible pair $(\hat{x}(\cdot), \hat{u}(\cdot))$ such that there exists a set of multipliers $(p, \mu, \lambda_0, \lambda)$ with the above properties, satisfying (i)–(iv) and (vi).

Since extremals whose set of multipliers have $\lambda_0 = 0$ do not depend explicitly on the cost function, those for which $\lambda_0 \neq 0$ play an important role (e.g., in sufficient conditions).

DEFINITION 1.4. An extremal pair $(\hat{x}(\cdot), \hat{u}(\cdot))$ is said to be *normal* if there is no set of multipliers associated with it such that $\lambda_0 = 0$.

Our goal is to give an equivalent characterization of normal extremals in terms of the local controllability of the "associated" variational equation.

2. Controllability of linear systems. In this section we will state necessary and sufficient conditions for local controllability of a control problem subject to control constraints and to a general boundary condition. Let us now consider the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad \text{a.e. } t \in I, \tag{2.1}$$

where $A(\cdot), B(\cdot)$ are matrices, $n \times n$ and $n \times m$, respectively, with L^1 entries and the control function

$$u(\cdot) \in \mathcal{U} = \{u(\cdot) \in L^\infty : u(t) \in U - \hat{u}(t) \text{ a.e. } t \in I\},$$

where $\hat{u}(\cdot) \in L^\infty(I, U)$.

Let us consider the following boundary value problem

$$Mx(a) + Nx(b) - \int_a^b dF(s)x(s) = l, \tag{2.2}$$

where $l \in \mathbf{R}^r$, M, N are $r \times n$ matrices and $F(\cdot)$ is an $r \times n$ matrix valued function whose entries are of bounded variation and the integral is in the sense of Borel-Stieltjes.

When the controls are unconstrained, the above control problem has been studied in [**7, 8**]. Unfortunately, being an internal report, [**7**] is not easily accessible, and [**8**], which is the published version, is badly printed and has a large number of misprints and some missing parts that make statements and proofs hard to follow. In this section, while deriving the characterization (2.4) and proving Theorem 2.1 and Corollary 2.1, we follow closely the techniques used in [**7**].

If $X(t)$ is a fundamental matrix of the homogeneous part of system (2.1), principal at a, then, by the variation of constants formula, the boundary condition (2.2) can be written equivalently as

$$\begin{aligned} Mx(a) + NX(b)x(a) + \int_a^b NX(b)X^{-1}(s)B(s)u(s)\,ds - P(a)x(a) \\ - \int_a^b dF(\theta) \int_a^\theta X(\theta)X^{-1}(s)B(s)u(s)\,ds = l. \end{aligned}$$

By the Dirichlet formula for integrals (see, e.g., [**6**, p. 55 or [**2**]),

$$\begin{aligned} \int_a^b dF(\theta) \int_a^\theta & X(\theta)X^{-1}(s)B(s)u(s)\,ds \\ &= \int_a^b \int_s^b dF(\theta)X(\theta)X^{-1}sB(s)u(s)\,ds \\ &= \int_a^b P(s)X^{-1}(s)B(s)u(s)\,ds, \end{aligned}$$

where

$$P(t) = \int_t^b dF(s)X(s),$$

and, hence, condition (2.2) becomes

$$[M + NX(b) - P(a)]x(a) + \int_a^b [NX(b) - P(s)]X^{-1}(s)B(s)u(s)\,ds = l.$$

Since the initial point is not fixed, the initial condition plays the role of a control parameter. In order to examine its influence on the controllability of the system, let us write $\Gamma = M + NX(b) - P(a)$ and define

$$\Lambda : \mathcal{U} \to \mathbf{R}^r$$

as

$$\Lambda(u(\cdot)) = \int_a^b [NX(b) - P(s)]X^{-1}(s)B(s)u(s)\,ds.$$

The boundary condition (2.2) can be written equivalently as

$$\Gamma x(a) + \Lambda(u(\cdot)) = l. \tag{2.3}$$

Let

$$\Lambda_1 : \mathbf{R}^n \times \mathcal{U} \to \mathbf{R}^r$$

be defined by

$$\Lambda_1 \begin{pmatrix} \alpha \\ u(\cdot) \end{pmatrix} = \Gamma\alpha + \Lambda(u(\cdot)).$$

The controllability of the system can be phrased in terms of Λ_1 as follows.

DEFINITION 2.1. We will say that the system is (M, N, F)-locally controllable on the interval I if

$$O \in \operatorname{int} \operatorname{Im} \Lambda_1. \tag{2.4}$$

There are two possibilities:

Case (a). $\operatorname{rank} \Gamma = r$. In this case, by only varying the control parameter $x(a)$, the system is obviously (M, N, F)-locally controllable on I.

Case (b). $\operatorname{rank}\Gamma < r$. In this case, we will state some characterizations of this property.

The following result is a reformulation of the controllability condition in an equivalent but simpler form.

LEMMA 2.1. *Assume that U is convex, then the system* (2.1) *is* (M, N, F)*-locally controllable on I if and only if the only $\gamma \in \operatorname{Ker}\Gamma^T$ such that*

$$\langle \gamma, \Lambda(u(\cdot))\rangle \geq 0, \quad \forall u(\cdot) \in \mathcal{U},$$

is $\gamma = 0$.

PROOF. It follows immediately that $\operatorname{Im}\Lambda_1$ is convex and contains O. The system (2.1) is not (M, N, F)-locally controllable if and only if there exists $\gamma \in \mathbf{R}^r$, $\gamma \neq 0$, such that

$$\langle \gamma, (\Gamma\alpha + \Lambda(u(\cdot)))\rangle \geq 0, \quad \forall u(\cdot) \in \mathcal{U},\ \forall \alpha \in \mathbf{R}^n. \tag{2.5}$$

It must be that $\gamma \in \operatorname{Ker}\Gamma^T$. Otherwise, decompose $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 \in \operatorname{Ker}\Gamma^T$ and $\gamma_2 \in \operatorname{Im}\Gamma$ with $\gamma_2 \neq 0$. Write $\gamma_2 = \Gamma\zeta$ and get

$$\langle \gamma, \Lambda(u(\cdot))\rangle + \langle \Gamma\zeta, \Gamma\alpha\rangle \geq 0, \quad \forall u(\cdot) \in \mathcal{U}, \quad \forall \alpha \in \mathbf{R}^n,$$

but this is impossible for $\alpha = -t\zeta$, $t > 0$. Then (2.5) holds only for $\gamma \in \operatorname{Ker}\Gamma^T$ and becomes

$$\langle \gamma, \Lambda(u(\cdot))\rangle \geq 0, \quad \forall u(\cdot) \in \mathcal{U}. \ \square$$

Assume, without loss of generality, that $F(a) = 0$, i.e., $\int_a^t dF(s) = F(t)$. For any $\gamma \in \mathbf{R}^r$, consider the "adjoint" boundary value problem,

$$\dot{y}(t) = -A^T(t)\left[y(t) + F^T(t)\gamma\right], \quad \text{a.e. } t \in I, \tag{2.6}$$

with the boundary conditions

$$\begin{aligned} y(a) &= -M^T\gamma \\ y(b) &= [N^T - F^T(b)]\gamma. \end{aligned} \tag{2.7}$$

This adjoint problem is derived from the one proposed in [**5**], which can be obtained from ours by setting

$$z(t) = y(t) + F^T(t)\gamma.$$

Since the entries of $F(\cdot)$ are only of bounded variation, then in [**5**] the adjoint variable $z(\cdot)$ is not necessarily absolutely continuous and, therefore, the solution of the adjoint equation was considered in a generalized sense.

As it turns out below, if for a certain γ the above adjoint problem has a solution, then $\gamma \in \operatorname{Ker}\Gamma^T$.

LEMMA 2.2. *A pair $(y(\cdot), \gamma)$ solves* (2.6) *and* (2.7) *if and only if*

$$\gamma \in \operatorname{Ker}\Gamma^T \quad \textit{and} \quad y(t) = X^{T-1}(t)[-P^T(t) + X^T(b)N^T]\gamma - F^T(t)\gamma,$$

where $X(\cdot)$ is the fundamental matrix of the homogeneous part of system (2.1), *principal at a, used earlier.*

PROOF. The solution of (2.6) that satisfies the boundary condition at b is

$$\begin{aligned} y(t) &= X^{T-1}(t)X^T(b)\left[N^T - F^T(b)\right]\gamma + X^{T-1}(t) \\ &\qquad \left[\int_t^b X^T(s)A^T(s)F^T(s)\gamma\,ds\right] \\ &= X^{T-1}(t)X^T(b)\left[N^T - F^T(b)\right]\gamma + X^{T-1}(t) \\ &\qquad \left[\int_t^b \left(\gamma^T F(s)\,dX(s)\right)^T\right] \end{aligned}$$

(by integrating by parts)

$$
\begin{aligned}
&= X^{T-1}(t)X^T(b)[N^T - F^T(b)]\gamma + X^{T-1}(t) \\
&\qquad \left[X^T(b)F^T(b)\gamma - X^T(t)F^T(t)\gamma\right] \\
&\quad - X^{T-1}(t)\int_t^b X^T(s)\,dF^T(s)\gamma \\
&= X^{T-1}(t)X^T(b)[N^T - F^T(b)]\gamma + X^{T-1}(t)X^T(b)F^T(b)\gamma \\
&\quad - F^T(t)\gamma - X^{T-1}(t)\int_t^b X^T(s)\,dF^T(s)\gamma \\
&= X^{T-1}(t)\left[X^T(b)N^T - \int_t^b X^T(s)\,dF^T(s)\right]\gamma - F^T(t)\gamma,
\end{aligned}
$$

and then

$$y(t) = X^{T-1}(t)[-P^T(t) + X^T(b)N^T]\gamma - F^T(t)\gamma.$$

Moreover, from the boundary conditions at a, we get

$$y(a) = [-P^T(a) + X^T(b)N^T]\gamma = -M^T\gamma$$

or, equivalently,

$$[M^T + X^T(b)N^T - P^T(a)]\gamma = 0, \quad \text{that is, } \Gamma^T\gamma = 0. \quad \square$$

The relation between the local (M, N, F)-controllability of system (2.1) and the adjoint boundary value problem is given by

THEOREM 2.1. *Assume that U is convex. Then the system* (2.1) *is (M, N, F)-locally controllable on I if and only if the only pair $(y(\cdot), \gamma)$ satisfying the adjoint boundary value problem* (2.6)–(2.7) *with*

$$\langle B^T(t)[y(t) + F^T(t)\gamma], u\rangle \geq 0, \quad \forall u \in \mathcal{U}, \tag{2.8}$$

is $(y(\cdot), \gamma) = (0, 0)$.

PROOF. From Lemma 2.1, the system (2.1) is not (M, N, F)-locally controllable if and only if there exists a nonzero $\gamma \in \operatorname{Ker}\Gamma^T$ such that

$$\langle \gamma, \Lambda(u(\cdot))\rangle \geq 0, \quad \forall u(\cdot) \in \mathcal{U},$$

that is,

$$\left\langle \gamma, \int_a^b [NX(b) - P(s)]X^{-1}(s)B(s)u(s)\,ds \right\rangle \geq 0$$

or

$$\int_a^b \left\langle B^T(s)[X^{T-1}(s)[-P^T(s) + X^T(b)N^T]\gamma], u(s) \right\rangle\,ds \geq 0.$$

And, since $0 \in \mathcal{U}$,

$$\left\langle B^T(t)[X^{T-1}(t)[-P^T(t) + X^T(t)N^T]\gamma], u \right\rangle \geq 0, \qquad \forall u \in U - \hat{u}(t), \quad \text{a.e. } t \in I;$$

but, by Lemma 2.2, $X^{T-1}(t)[-P^T(t) + X^T(b)N^T]\gamma = y(t) + F^T(t)\gamma$, where $y(t)$ is the solution of the adjoint system satisfying

$$y(b) = [N^T - F^T(b)]\gamma \quad \text{and} \quad y(a) = -M^T\gamma. \ \square$$

To compare the results in this section with those in [**4**], we will state the condition of Theorem 2.1 in a different way.

COROLLARY 2.1. *Assume that* $B(\cdot) \in L^2$ *and* $\hat{u}(t) \in \operatorname{int} U$. *The system* (2.1) *is* (M, N, F)*-locally controllable on* I *if and only if*

$$\int_a^b [NX(b) - P(s)]X^{-1}(s)B(s)B^T(s)X^{T^{-1}}(s)[-P^T(s) + X^T(b)N^T]\,ds > 0 \tag{2.9}$$

on $\operatorname{Ker}\Gamma^T$.

PROOF. Since $\hat{u}(t) \in \operatorname{int} U$ a.e. $t \in I$, then Lemma 2.2 and Theorem 2.1 imply that the (M, N, F)-local controllability is equivalent to the nonsingularity, for almost all $t \in I$, of the matrix

$$B^T(t)X^{T^{-1}}(t)[-P^T(t) + X^T(b)N^T]$$

on $\operatorname{Ker}\Gamma^T$. Hence, the set $A \subset I$ such that, on $\operatorname{Ker}\Gamma^T$,

$$[NX(b) - P(t)]X^{-1}(t)B(t)B^T(t)X^{T^{-1}}(t)[-P^T(t) + X^T(b)N^T] > 0, \quad \text{for } t \in A,$$

is of positive Lebesgue measure. Integrating, (2.9) follows. The converse is trivial. □

A more detailed analysis of the constrained local controllability for the usual boundary condition is in [**10**] and [**11**]; for the nonlinear case, see [**3**] and [**9**].

3. Normality and controllability. To relate the results of the previous section to the normality of an extremal, let us first write

$$\hat{A}(t) = \hat{f}_x(t), \qquad \hat{B}(t) = \hat{f}_u(t),$$

$$F_{0\mu}(t) = \int_a^t \hat{\phi}_x(s)\, d\mu(s), \quad M_0 = \psi_x(\hat{x}(a), \hat{x}(b)), \quad N_0 = \psi_y(\hat{x}(a), \hat{x}(b)),$$

$$\hat{M} = \begin{pmatrix} 0 \\ M_0 \end{pmatrix}, \qquad \hat{N} = \begin{pmatrix} 0 \\ N_0 \end{pmatrix}, \qquad \hat{F}_\mu(t) = \begin{pmatrix} F_{0\mu}(t) \\ 0 \end{pmatrix},$$

where the $k \times 2n$-matrix $[M_0\ N_0]$ is assumed to be onto.

For these choices of $\hat{A}, \hat{B}, \hat{M}, \hat{N}, \hat{F}_\mu$, we consider the system (2.1) with boundary condition (2.2) which becomes

$$(3.1)_\mu \qquad \begin{gathered} M_0 x(a) + N_0 x(b) = (l_1, \dots, l_k)^T \\ -\int_a^b dF_{0\mu}(s) x(s) = l_0. \end{gathered}$$

Notice that, in this setting, $k < r = k+1$, and then Case (a) of the preceding section is never verified and the $(\hat{M}, \hat{N}, \hat{F}_\mu)$-local controllability is not automatic. If the extremal $(\hat{x}(\cdot), \hat{u}(\cdot))$ is abnormal, then there exist μ, $p(\cdot)$ and λ such that $\|p\| + \|\mu\| > 0$,

$$(3.2)_\mu \qquad \begin{gathered} \dot{p}(t) = -A^T(t)\left(p(t) + F_{0\mu}^T(t)\right) \\ p(a) = -M_0^T \lambda \\ p(b) = N_0^T \lambda - F_{0\mu}^T(b), \end{gathered}$$

and

$$\langle B^T(t)[p(t) + F_{0\mu}^T(t)], u\rangle \geq 0, \quad \forall u \in U(t) = U - \hat{u}(t), \text{ a.e. } t \in I. \tag{3.3$_\mu$}$$

Given an extremal $(\hat{x}(\cdot), \hat{u}(\cdot))$, define

$\Lambda = \{\mu : \mu$ is a nonnegative Radon measure such that, for some $\lambda_0, p(\cdot)$ and λ, $(\lambda_0, \mu, p(\cdot), \lambda)$ satisfies (i)–(iv) and (vi) of Theorem 1.1$\}$.

The main result of this paper is

THEOREM 3.1. *An extremal $(\hat{x}(\cdot), \hat{u}(\cdot))$ is normal if and only if the corresponding system* (2.1) *is $(M_0, N_0, 0)$-controllable and $(\hat{M}, \hat{N}, \hat{F}_\mu)$-locally controllable for all $\mu(\neq 0) \in \Lambda$.*

PROOF. Assume that the extremal $(\hat{x}(\cdot), \hat{u}(\cdot))$ is normal. The system (2.1) is $(M_0, N_0, 0)$-locally controllable since, otherwise, using Theorem 2.1, there exist $\gamma \neq 0$ and $y(\cdot)$ satisfying (2.6)–(2.7) with $(M, N, F) = (M_0, N_0, 0)$. But the full rank condition on $[M_0, N_0]$ yields that $y(\cdot) \neq 0$. Hence, $(p(\cdot) := y(\cdot), \lambda := \gamma, \mu = 0)$ satisfies $(3.2)_\mu$-$(3.3)_\mu$, which contradicts the normality of $(\hat{x}(\cdot), \hat{u}(\cdot))$. Consider $\mu \neq 0$, with $\mu \in \Lambda$. Let us show that (2.1) is $(\hat{M}, \hat{N}, \hat{F}_\mu)$-locally controllable. If not, by Theorem 2.1, there exists $\gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \neq 0$ and $y(\cdot)$ solving

$$\begin{aligned} \dot{y}(t) &= -A^T(t)\left[y(t) + F_{0\mu}^T(t)\gamma_1\right] \\ y(a) &= -M_0^T\gamma_2 \\ y(b) &= N_0^T\gamma_2 - F_{0\mu}^T(b)\gamma_1 \end{aligned} \tag{3.4}$$

and

$$\langle B^T(t)[y(t) + F_{0\mu}^T(t)\gamma_1], u\rangle \geq 0, \quad \forall u \in U(t) = U - \hat{u}(t), \text{ a.e. } t \in I. \tag{3.5}$$

If $\gamma_1 \neq 0$, then it could be taken equal to 1, and, hence, $(p(\cdot) := y(\cdot), \mu, \gamma_2)$ solves $(3.2)_\mu$ and $(3.3)_\mu$ with $\mu \neq 0$, contradicting the normality. If $\gamma_1 = 0$, then $\gamma_2 \neq 0$, and, thus, a contradiction with the $(M_0, N_0, 0)$-local controllability is obtained.

Conversely, assume that (2.1) is $(M_0, N_0, 0)$-locally controllable and $(\hat{M}, \hat{N}, \hat{F}_\mu)$-locally controllable for all $\mu (\neq 0) \in \Lambda$. If the extremal $(\hat{x}(\cdot), \hat{u}(\cdot))$ is not normal, then there exists $(p(\cdot), \mu, \lambda)$ solving $(3.2)_\mu$ and $(3.3)_\mu$ with $\|p\| + \|\mu\| > 0$. If $\mu = 0$, then $\|p\| \neq 0$, and, hence, the full rank condition on $[N_0, M_0]$ yields that $\lambda \neq 0$. Through Theorem 2.1, we obtain a contradiction with the $(M_0, N_0, 0)$-local controllability. If $\mu \neq 0$, it is clear that $\left(p(\cdot), \mu, \gamma = \binom{1}{\lambda}\right)$ satisfies equations (3.4) and (3.5), and, thus, by Theorem 2.1, the $(\hat{M}, \hat{N}, \hat{F}_\mu)$-local controllability is contradicted. □

When $0 \in \operatorname{int} U - \hat{u}(t)$, a.e. $t \in I$, Corollary 2.1 can be used to generalize the results in [**4**] concerning the relation between normality and controllability.

In [**1**] the concept of pseudonormality is used to prove directly the local controllability of the nonlinear system (1.2) along the reference trajectory without introducing the linearized system.

We hope that these results will stimulate further research on the controllability properties of nonautonomous linear systems with a general boundary condition like the one examined in §2.

REFERENCES

1. F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley Interscience, New York, 1983.

2. N. Dunford, J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

3. M. Furi, P. Nistri, M. P. Pera and P. Zezza, *Linear controllability by piecewise constant controls with assigned switching times*, J. Opti. Theory Appl. **45** (1985), 219–229.

4. V. B. Haas, *Normality and controllability for the problem of Bolza*, Internat. J. Control **30** (1979), 331–338.

5. A. Halanay, A. Moro, *A boundary value problem and its adjoint*, Ann. Mat. Pura Appl. (IV), **LXXIX** (1968), 399–412.

6. T. H. Hildebrant, *Theory of integration*, Academic Press, New York & London, 1969.

7. C. Marchió, (M, N, F) *Controllabilitá Completa*, in *Questioni di controllabilitá*, Rapporto Interno, Universitá di Firenze, 1973/2.

8. ———, *Complete* (M, N, F)*-controllability*, Rev. Roumaine Math. Pures Appl. **24** (1979), 273–280.

9. P. Nistri, *On a general notion of controllability for nonlinear systems*, Bull. Un. mat. Ital. C Serie VI, **5** (1986), 383–403.

10. W. E. Schmitendorf and B. R. Barmish, *Null controllability of linear systems with constrained controls*, SIAM J. Control Optim. **18** (1980), 327–345.

11. ——— and ———, *A necessary and sufficient condition for local constrained controllability of a linear system*, IEEE Trans. Automat. Control **AC-25** (1980), 97–100.

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario T6G 2G1, Canada

Dipartimento di Sistemi e Informatica, Via di S. Marta 3, 50139 Firenze, Italy