ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 4, Fall 1990

A SURVEY OF G.J. BUTLER'S RESEARCH IN THE QUALITATIVE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. In this paper we shall survey some of the main results obtained by our late colleague and friend, Professor Geoffrey James Butler. The outline is as follows. We shall review his work on periodic solutions for nonlinear differential equations (§2), oscillation and nonoscillation for second order linear and nonlinear equations (§3), comparison theorems and oscillation for higher order linear equations and systems (§4). In §5, we discuss some miscellaneous contributions in chaotic behavior of mappings and fixed point theorems and packing and covering problems.

2. Periodic solutions of second order nonlinear equations. Consider the second order nonlinear differential equations

$$x'' + f(x)h(x'^2) + g(x) = \mu p(t) \tag{2.1}$$

and

$$x'' + f(x)h(x'^2) + g(x) = 0, \tag{2.2}$$

where f, g, h and p are continuous with p periodic of period ω and μ is a parameter. Extending earlier results of Heidel [**A10**] and Utz [**A20**], Butler in [**9**] gave necessary and sufficient conditions for the existence of infinitely many periodic solutions. Furthermore, he also obtained sufficient conditions for the existence of periodic solutions of (2.2) with arbitrarily large periods. These are summarized in

THEOREM 2.1. [**9**]. *Let h be locally Lipschitz in a neighborhood of zero. Then* (2.2) *has infinitely many periodic solutions if and only if property* (P) *below holds*:

There exist $a, b,\ a < b$, *such that*

$$G(a) = G(b) > G(x), \quad x \in (a, b), \tag{P}$$

where $G(x) = \int_0^x g(t)\,dt$.

THEOREM 2.2. [**9**]. *Assume that* h *is locally Lipschitz in a neighborhood of zero,* $\lim_{x\to\infty} h(x) = +\infty$, *and* $f(x) > 0$ *for all* x. *Assume further that initial value problems for* (2.2) *are uniquely solvable and that there exist* $\alpha, \beta, \gamma, \delta, \quad \gamma < \alpha < \beta < \delta$ *such that* $G(\alpha) = G(\beta) > G(x)$ *for* $x \in (\alpha, \beta)$ *and* $g(\gamma) = g(\delta) = 0$. *Then* (2.2) *has periodic solutions with arbitrarily large period.*

The results included in the previous two theorems improve previous results by allowing a more general nonlinear damping term and also remove the usual sign restriction on the "restoring force" term $g(x)$. The methods of proof are topological in nature and involve continuous dependence of solutions on initial data along with some additional considerations. One then obtains, via the results of Bernstein and Halanay [**A2**], that the nonautonomous equation (2.1) has a periodic solution of period ω, for $|\mu|$ sufficiently small.

In another paper Butler and Freedman [**10**] showed that condition (P) is a necessary and sufficient condition for the existence (in the Carathéodory sense) of a periodic solution of the equation

$$x'' + g(x) = 0, \tag{2.3}$$

assuming only that $g(x)$ is locally integrable. These results were then applied to the problem

$$\begin{aligned} x'' + g(x) + \mu h(x) &= 0 \\ x(0) = A, \quad x'(0) &= 0 \end{aligned} \tag{2.4}$$

(with the assumption that g, h are continuous and (2.4) is locally uniquely solvable). The region of the (μ, A)-plane $(\mu > 0)$ was characterized for which problem (2.4) admits nontrivial periodic solutions. This region $\mathcal{A}$, termed "admissible," turns out to be an open set whose boundary may be quite complicated. These results were further extended in [**20**] to the equation

$$x'' + f(\mu, x) = 0, \tag{2.5}$$

and we refer to [**20**] for further details and to a discussion of certain converse theorems.

In several other papers [**17, 27**], Butler, in extending work of Jacobowitz [**A12**], obtained existence results for periodic solutions of the equation

$$x''(t) + f(t, x(t)) = 0, \tag{2.6}$$

where $f(t,x)$ is ω-periodic in t and $f(t,0) \equiv 0$ for all t. Jacobowitz had applied the Poincaré-Birkhoff "twist" theorem to show that (2.6) has infinitely many (nontrivial) periodic solutions. The form of the "twist" theorem used was

THEOREM 2.3. [**27**]. *Let A be a bounded topological annular region of $\mathbf{R}^2$ which includes its outer boundary S_1 but not its inner boundary S_0. Let T be a homeomorphism of the closure $\overline{A}$ of A in $\mathbf{R}^2$ such that $T(S_0) = S_0$, T preserves Lebesgue measure, and T is a "twist" map of $\overline{A}$. Then T has a fixed point in A.* (Here, a "twist" map, loosely speaking, is a map which "twists" S_0 and S_1 in opposite directions.)

The result obtained by Jacobowitz may be stated as

THEOREM 2.4. [**A12**]. *Assume that $f(t,x)$ satisfies the following*:

(i) *f is continuous, periodic in t with period $\omega > 0$, and initial value problems for* (2.6) *are uniquely solvable*;

(ii) *$xf(t,x) > 0$, for $x \neq 0$*;

(iii) *$\lim_{|x|\to\infty} \frac{f(t,x)}{x} = \infty$ uniformly in t*;

(iv) *$f(t,x)/x$ is bounded on $0 < |x| \leq 1$ uniformly in t.*

Then (2.6) *has infinitely many periodic solutions.*

Butler in [**17, 27**] was able to substantially improve this result and extend it also to sublinear equations. One of these results which he obtained was

THEOREM 2.5. [**27**]. *Assume hypothesis* (i) *of Theorem* 2.4 *holds and, further, that* f *satisfies*

(ii)$'$ $xf(t,x) > 0$ *for* $|x|$ *sufficiently large* (*independent of* t), *and* $f(t,0) \equiv 0$.

Assume further that there is a measurable set S *of positive measure such that either*

(a)

$$\begin{cases} \lim_{|x|\to\infty} \frac{f(t,x)}{x} = \infty, & \textit{for } t \in S, \\ f(t,x)/x & \textit{is bounded for } 0 < |x| \le 1 \textit{ uniformly in } t \end{cases}$$

or

$$\text{(b)} \qquad \begin{cases} \lim_{x\to 0} \frac{f(t,x)}{x} = \infty, & \textit{for } t \in S, \\ f(t,x)/x & \textit{is bounded for } |x| \ge 1 \textit{ uniformly in } t. \end{cases}$$

Then (2.6) *has infinitely many periodic solutions of period* ω. *Furthermore, for each sufficiently large integer* N *there exists a periodic solution of period* ω *with precisely* $2N$ *zeros on* $[0,\omega)$.

To establish the proof of the above theorem involves two technical problems. In the superlinear case ((a) above) there is a problem related to extendability of solutions of (2.6) to $[0,\omega]$ (in order to define the first return maps associated with (2.6) written as a system). In the sublinear case ((b)) one has the problem of uniqueness of solutions of initial value problems. Special cases of the above result may be found in [**14**] where Butler considered a generalized Emden-Fowler equation of the form

$$x'' + q(t)|x|^{\alpha}\operatorname{sgn} x = 0, \quad \alpha > 1. \tag{2.7}$$

We refer to [**14, 27**] for further details of this and additional remarks.

3. Oscillation and nonoscillation for second order equations. In this section we shall survey some of the most important results obtained by Butler in a series of papers relating to the oscillatory and nonoscillatory behavior of the equation

$$y''(t) + q(t)f(y(t)) = 0, \quad t \geq t_0, \tag{3.1}$$

where q and f are continuous and f is monotone increasing with $yf(y) > 0, y \neq 0$. Much of the original impetus for the study of (3.1) arose from equation (2.7) of the previous section, in particular, for the case that $q(t) \geq 0$. A number of later authors, in extending results due to Atkinson [**A1**] for (2.7), relaxed the nonnegativity assumption on the coefficient function $q(t)$. However, then there arises the problem of extendability of solutions. In particular, it was shown by Burton and Grimmer in [**A3**] that, if $q(t)$ takes on negative values and if f satisfies a certain growth condition, there will always exist solutions of (3.1) with a bounded maximal interval of existence. It is therefore of importance to demonstrate the existence of a solution for all $t \geq t_0$. In [**16**], Butler established a general result on continuability of solutions of (3.1). The following is a simpler version of the more general result.

THEOREM 3.1. [**16**]. *Let f be locally Lipschitz with $yf(y) > 0$, $y \neq 0$ and $\lim_{|y|\to\infty} f(y)/y = \infty$. Let q be continuous with isolated zeros and piecewise monotone on each bounded interval of $[t_0, \infty)$. Then* (3.1) *has infinitely many continuable solutions. Moreover, if q oscillates, then* (3.1) *has infinitely many oscillatory solutions.*

In two other papers [**18, 13**] Butler extended some known results for Hill's equation to a nonlinear analogue, namely,

$$y''(t) + (mp(t) - k)|y(t)|^\alpha \operatorname{sgn} y(t) = 0, \tag{3.2}$$

where m, k are constants, $\alpha > 0$ and $p(t)$ is a nonconstant periodic function of least period ω with zero mean. As a consequence of a more general result (somewhat complicated to state), the following results were obtained:

THEOREM 3.2. [**13**]. *Let $k \leq 0, m \neq 0$, and let p be a nonconstant periodic function with zero mean. Then all solutions of* (3.2) *oscillate.*

THEOREM 3.3. [**13**]. *Let $\alpha > 1, k > 0$, and let p be as in Theorem 3.2. Then* (3.2) *has a nontrivial nonoscillatory solution.*

If we denote by N_α the set of functions $q = mp - k$, for which (3.2) has a nontrivial nonoscillatory solution, and

$$R_\alpha = \{(m, k) \in \mathbf{R}^2 : mp - k \in N_\alpha\},$$

then the following result is obtained [**A14**].

THEOREM 3.4. *Let $\alpha > 1, p$ be a nonconstant periodic function with zero mean. Then R_α is the positive k half-plane together with the origin.*

Moore in [**A14**] had shown that R_1 is closed convex and, except for the origin, is entirely contained in the positive k half-plane. Thus, for $\alpha > 1, R_\alpha \supset R_1$ and neither of N_α, N_1 contains the other.

In [**18**] analogous results were obtained for the sublinear case, $0 < \alpha < 1$. These are summarized for both cases below:

THEOREM 3.4. [**18**]. *Let $q(t) \equiv mp(t) - k$ have isolated zeros and be sufficiently smooth to guarantee continuability* (*in case* $\alpha > 1$) *or local uniqueness of the zero solutions* (*if* $0 < \alpha < 1$) *on intervals I on which $q(t) \geq 0$. Then, for all $0 < \alpha \neq 1$:*

(i) *all solutions of* (3.2) *oscillate if and only if* (m, k) *belongs to the nonpositive k half-plane, excluding the origin*;

(ii) *all solutions of* (3.2) *are nonoscillatory if and only if $k \geq m \max p(t)$ and $m \geq 0$ or $k \geq m \min p(t)$ and $m < 0$*;

(iii) *for all other values of m and k* (*i.e., those for which q oscillates and $\int_0^\omega q < 0$*), *there exist both oscillatory and nonoscillatory solutions. Further, if $\alpha > 1$, the nonoscillatory solutions tend asymptotically to zero as $t \to \infty$, and, if $0 < \alpha < 1$, the nonoscillatory solutions are unbounded.*

With the above-cited results, the oscillatory/nonoscillatory behavior of solutions of (3.2) was more or less completely characterized. Additional properties of the solutions of (3.2) were also investigated—we refer to [**13, 18**] for further details.

In [**26**] Butler made important extensions of the well-known Wintner-Hartman oscillation tests for linear oscillation to the more general equation (3.1). We recall that, for the linear equation

$$\text{(L)} \qquad y'' + q(t)y = 0, \quad t \in [T, \infty),$$

three of the more important tests for oscillation are

$$(\mathrm{A}_1) \qquad \int_T^\infty q(s)\,ds = +\infty \qquad \text{(Fite [A8], Wintner [A25])}$$

$$(\mathrm{A}_2) \qquad \lim_{t\to\infty} \frac{1}{t} \int_T^t \int_T^s q(\tau)\,d\tau\,ds = +\infty \qquad \text{(Winter [A25])}$$

$$(\mathrm{A}_3) \qquad \begin{aligned} -\infty < \liminf_{t\to\infty} \frac{1}{t} \int_T^t \int_T^s q(\tau)\,d\tau\,ds \qquad \text{(Hartman [A9])} \\ < \limsup_{t\to\infty} \frac{1}{t} \int_T^t \int_T^s q(\tau)\,d\tau\,ds \le \infty. \end{aligned}$$

These tests were further refined and extended by Coles and Willett [**A4, A24**], who considered weighted averages of the integral of q of the form

$$A_\varphi(t,T) = \frac{\int_T^t \varphi(s) \left(\int_T^s q(\tau)\,d\tau\right)\,ds}{\int_T^t \varphi(s)\,ds}.$$

The main result obtained in [**26**] shows that, for a certain class $\mathcal{F}$ of functions f, both the Wintner condition (A_2) and the Hartman condition (A_3) are oscillation criteria for (3.1). (Waltman in [**A23**] had earlier extended (A_1) to (3.1)). This class includes all equations with $f(y) = |y|^\alpha \operatorname{sgn} y, \quad \alpha > 0$. The extension of more general averaging techniques to (3.2) is not as complete as in the linear case. However, refinements were obtained which dealt with the difficult case

$$(3.3) \qquad y''(t) + q(t)|y(t)|^\alpha \operatorname{sgn} y(t) = 0,$$

where $q(t) = t^\lambda \sin t$. It was shown that, for $\alpha > 1$, (3.3) is oscillatory if and only if $\gamma \geq -1$. For $0 < \alpha < 1$, Butler showed that (3.3) is oscillatory if $\lambda \geq 1$ or $\lambda = 0$. The cases $\lambda < 0$ and $0 < \lambda < 1$ were left open. However, Butler conjectured that (3.3) is oscillatory if and only if $\lambda \geq -\alpha, 0 < \alpha < 1$. In a later paper, Kwong and Wong [**A13**] showed that (3.3) is oscillatory for $\lambda > -\alpha$ and nonoscillatory if $\lambda < -\alpha$. The case $\lambda = -\alpha$ was settled by Onose in [**A18**], thus verifying completely Butler's conjecture.

To return to the more general equation (3.1), it was shown in [**11**] that it is possible to obtain necessary *and* sufficient conditions for the oscillation of (3.1) based on certain assumptions on the integral $\int_T^t q(s)\,ds$. The main result obtained in [**11**] is (We set $g_+(t) = \max(g(t), 0), g_-(t) = \max(-g(t), 0)$ in the results below.)

THEOREM 3.6. *Let $f(y)$ and $q(t)$ satisfy*

(A) *$f(y)$ is absolutely continuous on any bounded set with $yf(y) > 0$ for $y \neq 0$, and the essential infimum of $f'(y)$ over any closed set which excludes zero is positive.*

(B) $\int_{\pm 1}^{\pm\infty} \frac{dy}{f(y)} < \infty$.

(C) *There is a nontrivial compact interval I in which $yf(y) > 0$, for $y \neq 0$, and $f'(y)$ exists at each point, is nonnegative and bounded.*

(D) *$q(t)$ is locally integrable and $\int_t^\infty q(s)\,ds = Q(t)$ exists (possibly infinite) with $\lim_{T\to\infty} \int_t^T Q(s)\,ds > -\infty$ for all t.*

(E) $\int_t^\infty \left(\int_1^\infty Q_-^2(\sigma)\,d\sigma\right)\,ds < \infty$.

Then all extendable solutions of (3.1) *oscillate if and only if*

$$\int_t^\infty \left[Q(s) + \int_s^\infty Q^2(\sigma)\,d\sigma\right]\,ds = \infty.$$

It turns out that examples may be given to show that the assumptions on f and q in Theorem 3.6 may not be relaxed.

A further result on the existence of nonoscillatory solutions without (A), (B) is

THEOREM 3.7. [**11**]. *Let $f(y)$ and $q(t)$ satisfy conditions* (C), (D) *and* (E). *Then*

$$\int_t^\infty \left[Q(s) + \int_s^\infty Q^2(\sigma)\,d\sigma\right] ds < \infty$$

implies that there is a nontrivial, nonoscillatory solution of (3.1).

Examples of functions $q(t)$ for which one can determine the corresponding oscillation properties of (3.1) may be given, as in [**11**]:

(1) Let $q(\tau) = (\mu\cos\nu t)/t + (1+\sin\nu t)/t^2$, μ, ν nonzero constants. Suppose that $|\mu/\nu| > 1$. Then $Q(t)$ oscillates but, since $\int_t^\infty Q(s)\,ds = \infty$, all extendable solutions of (3.1) oscillate.

(2) Let $q(t) = (\mu\cos\nu t)/(t(\log t)^{1/2+\varepsilon})$, where μ, ν are nonzero and $\varepsilon \geq -1/2$. It may be shown on the basis of Theorems 3.6 and 3.7 that all solutions oscillate if $\varepsilon > 0$ and that there exists a nonoscillatory solution if $\varepsilon \leq 0$.

In [**22**], Butler considered the question of extending certain linear second order comparison theorems to the pair of equations

$$(Rx')' + pf(x) = 0, \tag{3.4}$$

$$(rx')' + qf(x) = 0. \tag{3.5}$$

The following result includes the well-known Hille-Wintner comparison theorem [**A11, A25**] (also considered by Taam [**A19**]).

THEOREM 3.8. *Let r, R, p, q be continuous on (t_0, ∞) such that $P(t) = \int_t^\infty p(s)\,ds$, $Q(t) = \int_t^\infty q(s)\,ds$ exists and such that $0 < r(t) \leq R(t)$, $|P(t)| \leq Q(t)$, $t \in [t_0, \infty)$. Assume that f satisfies*

(a) *f is continuously differentiable, $xf(x) > 0$ for all $x \neq 0$, $f'(x) > 0$ for all $x \neq 0$ and either*

(b) *f' is nondecreasing on $[0, \infty)$ and is nonincreasing on $(-\infty, 0]$ or*

(c) $\liminf_{|x|\to\infty} f'(x) > 0$ *and* $\int_{\pm 1}^{\pm\infty} \frac{du}{f(u)} < \infty$.

Then, if equation (3.4) *is oscillatory, so also is* (3.5).

Butler and the present author also considered the extension of oscillation criteria of Olech-Opial-Wazewski type in [**52**]. These are based on the asymptotic density of the set where $\int_0^t q\,ds$ is sufficiently positive. It was shown in [**A17**] that the linear equation

$$\text{(L)} \qquad y'' + q(t)y(t) = 0$$

is oscillatory if

$$\lim_{t\to\infty} \text{approx} \int_0^t q(s)\,ds = +\infty.$$

For the nonlinear equation (3.3) (and its generalization (3.2)), one can obtain a sharp result which generalizes the Olech-Opial-Wazewski theorem. For simplicity, we state the result only for (3.3). We define the density function $\rho_S(t)$ of a set $S \subset [0, +\infty)$ by

$$\rho_S(t) \equiv \frac{1}{2}\mu\{S \cap [0, t]\},$$

where μ denotes Lebesgue measure. We then have [**52**].

THEOREM 3.9. *Equation* (3.3) *is oscillatory if there exists a set* $S \subset [0, +\infty)$ *such that*

$$\limsup_{t\to\infty} t \left(\rho_S(t) - \left(\frac{\alpha - 1}{\alpha + 1}\right)^2\right) = +\infty$$

and

$$\lim_{\substack{t\to\infty \\ t\in S}} \int_0^t q(s)\,ds = +\infty.$$

To indicate sharpness, it was further shown that if S is any closed subset of $[0, +\infty)$ such that $\lim_{t\to\infty} \sup \rho_S(t) < ((\alpha-1)/(\alpha+1))^2$, $\alpha \neq 1$, then there exists a continuous function $q(t)$ with

$$\lim_{\substack{t\to\infty \\ t\in s}} \int_0^t q(s)\,ds = +\infty$$

and such that (3.3) has a nonoscillatory solution.

Additional results and details bearing on the oscillation problem may be found in the papers cited in this section.

4. Comparison theorems and systems oscillation. Higher order comparison theorems and oscillation properties of systems of equations were studied in a number of papers, jointly with the present author. In [**32, 33**] Butler and the present author considered the equations

$$L_n y + p(x)y = 0 \tag{4.1}$$

and

$$L_n y + q(x)y = 0, \tag{4.2}$$

where p, q are continuous on an interval $I \subset \mathbf{R}$ and L_n is an n-th order linear disconjugate differential operator on I (that is, the only solution of $L_n y = 0$ with n zeros on I counting multiplicities is $y \equiv 0$). It is well known that L_n can be written in factored form as

$$L_n y = \rho_n(\rho_{n-1}, \dots, (\rho_1(\rho_0 y)')', \dots)',$$

where $\rho_i > 0, \rho_i \in C^{n-i}(I)$. The *quasi-derivatives* $L_i y$ are defined by $L_0 y = \rho_0 y, L_i y = \rho_i(L_{i-1}y)', i = 1, \dots, n$. Elias, in a number of papers [**A5, A6, A7**], and Nehari, in [**A15, A16**], studied the oscillatory character of $L_n y + p(x)y = 0$ via a careful analysis of the distribution of zeros of the quasi-derivatives. In [**32, 33**] comparison theorems of generalized Hille-Wintner type for (4.1), (4.2) were obtained. If $\mathcal{I} = \{i_1, \dots, i_k\}, \quad \mathcal{J} = \{j_i, \dots, j_{n-k}\}$ are two arbitrary sets of indices from the set $\{0, 1, \dots, n-1\}$ with $0 \le i_1 \le i_2 < \cdots < i_k \le n-1, 0 \le j_1 < j_2 < \cdots \le n-1$, one considers boundary conditions of the form

$$L_i y(a) = 0, \quad i \in \mathcal{I}, \qquad L_j y(s) = 0, \quad j \in \mathcal{J}, \tag{4.3}$$

where $a, s \in I$. For $a \in I$ one defines (with respect to (4.1), cf. [**32**]) the i-th (right) extremal point $\theta_i(\mathcal{I}, \mathcal{J}; a)$ to be the i-th value of $s \in I \cap (a, +\infty)$ for which (4.1) has a nontrivial solution satisfying the boundary conditions (4.3). If $\mathcal{I} = \{0, 1, \dots, k-1\}, \quad \mathcal{J} = \{0, 1, \dots, n-k-1\}$ we denote these *conjugate-type* boundary conditions by $(\mathcal{I}_c, \mathcal{J}_i)$.

Similarly, if $\mathcal{I} = \{0, 1, \ldots, k-1\}$, $\mathcal{J} = \{k, \ldots, n-1\}$, these *focal-type* boundary conditions are denoted by $(\mathcal{I}_f, \mathcal{J}_f)$. Furthermore, we say that (4.1) is $(\mathcal{I}, \mathcal{J})$-disconjugate on I in case $\theta_i(\mathcal{I}, \mathcal{J}; a)$ does not exist in $(a, \infty) \cap I$ for any $a \in I$. If p is of one sign on I, then a necessary condition for the existence of $\theta_i(\mathcal{I}, \mathcal{J}; a)$ is that $(-1)^{n-k}p(x) < 0$. One also needs to restrict attention to the class $\mathcal{A}$ of *admissible* pairs of boundary conditions: $(\mathcal{I}, \mathcal{J}) \in \mathcal{A}$ if and only if, for any integer $l, 1 \leq l \leq n-1$, at least l terms of the sequence $i_1, \ldots i_k, j_1 \ldots, j_{n-k}$ are less than l. A partial order may be introduced on the set $\mathcal{A}$ and, it turns out, that one has the following results [**32**].

LEMMA 4.1.

(i) *Let* $(\mathcal{I}, \mathcal{J}) \in \mathcal{A}$. *Then* $(\mathcal{I}_c, \mathcal{J}_c) \prec (\mathcal{I}, \mathcal{J})$.

(ii) *Let* $(\mathcal{I}, \mathcal{J}) \in \mathcal{A}$ *with* $\mathcal{I} = \{0, 1, \ldots, k-1\}$. *Then* $(\mathcal{I}, \mathcal{J}) \prec (\mathcal{I}_f, \mathcal{J}_f)$.

One may then show that the first conjugate point $\theta_1(\mathcal{I}, \mathcal{J}; a)$ is a nonincreasing function with respect to the partial order on $\mathcal{A}$. That is, if $(\mathcal{I}, \mathcal{J}) \prec (\hat{\mathcal{I}}, \hat{\mathcal{J}})$, then $\theta_1(\hat{\mathcal{I}}, \hat{\mathcal{J}}; a) \leq \theta_1(\mathcal{I}, \mathcal{J}; a)$. On the basis of these and other considerations, one may then establish some generalized comparison theorems for $(\mathcal{I}, \mathcal{J})$-disconjugacy. As an example one has

THEOREM 4.2. *Let* $(\mathcal{I}, \mathcal{J}) \in \mathcal{A}$ *with* $\mathcal{I} \cap \mathcal{J} = \phi$ *and assume* $i_1 = 0$, $j_{n-k} = n-1$. *If* (4.2) *is* $(\mathcal{I}, \mathcal{J})$-*disconjugate on* (a, b) *and* $(-1)^{n-k}p(x) < 0, (-1)^{n-k}q(x) < 0$ *on* I *and*

$$\int_x^b \frac{|q(t)|\,dt}{\rho_0(t)\rho_n(t)} \geq \int_x^b \frac{|p(t)|\,dt}{\rho_0(t)\rho_n(t)}, \tag{4.4}$$

then (4.1) *is also* $(\mathcal{I}, \mathcal{J})$-*disconjugate on* (a, b).

This result extends in another way the linear Hille-Wintner comparison theorems referred to in Theorem 3.8. In the paper [**33**] the strict sign assumptions on the coefficients p, q were relaxed by considering a different approach based on nonlinear integral Riccati systems corresponding to the focal-type boundary conditions. This system technique was motivated by a technique introduced by Nehari [**A15**]. We refer to [**33**] for additional details along these lines.

Butler and the present author, in a series of papers [**37, 40, 43, 46**], studied the oscillation and comparison theory for second order systems. There are two natural types of extension of the classical theory for second order linear equations. The first of these (and the more studied) is in the context of B^*-algebras and uses the notion of positive operators in that setting. The second type of extension is in the Banach lattice context and utilizes the positivity induced by the lattice structure. In [**40, 43, 46**], oscillation theorems of the first type for the second order system

$$(R(t)Y')' + Q(t)Y = 0 \tag{4.5}$$

were obtained which generalized scalar oscillation criteria. Here R, Q, Y are $n \times n$ real continuous matrix functions with $R(t), Q(t)$ symmetric and $R(t)$ positive definite for $t \in [a, +\infty)$. The associated vector system is

$$(R(t)y')' + Q(t)y = 0, \tag{4.6}$$

where $y = \operatorname{col}(y_1, \ldots, y_n)$ is an n-vector. A solution $Y(t)$ of the matrix equation is *nontrivial* if $\det Y(t) \neq 0$ for at least one $t \in [a, \infty)$. A solution of (4.5) is prepared if $Y^*RY' - Y^{*\prime}RY \equiv 0$ on $[a, \infty)$, and equation (4.5) is said to be oscillatory in case the determinant of one (and hence every) prepared solution vanishes on $[b, +\infty)$ for each $b > a$. (This is equivalent to oscillation, i.e., the existence of conjugate points for each $b > a$, of the vector equation.) The results of [**46, 54**] involved extending well-known integral criteria (such as the Fite-Wintner test referred to in §2) and criteria based on averaging ideas and variational techniques. If one introduces the (extended real-valued) function L, defined on the class of $n \times n$ continuous real symmetric matrices given by

$$L(G) \equiv \liminf_{T \to \infty} \frac{1}{T} \int_a^T \int_a^t \operatorname{tr} Q(s)\, ds\, dt,$$

one may then obtain the following results which are analogues of scalar criteria (We let the real eigenvalues $\lambda_i(A)$ of a symmetric matrix be $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.).

THEOREM 4.3. *Assume* $L(Q) > -\infty$. *Then* (4.5) *is oscillatory in case any of the following holds*:

(i) $\lim_{T\to\infty} \sup \frac{1}{T} \int_a^T \lambda_1 \left(\int_a^t Q(s)\, ds \right) dt = +\infty$,

(ii) $\lim_{T\to\infty}$ approx sup $\lambda_1 \left(\int_a^T Q(s)\, ds \right) = +\infty$,

(iii) $\lim_{T\to\infty}$ approx inf $\lambda_1 \left(\int_a^T Q(s)\, ds \right) = -\infty$.

THEOREM 4.4. *Assume* $L(Q) = -\infty$. *Then* (4.5) *is oscillatory if*

$$\lim_{T\to\infty} \text{approx} \sup \lambda_n \left(\int_a^T Q(s)\, ds \right) > -\infty.$$

The proofs of Theorems 4.4 and 4.5 are based on Riccati techniques. The following result is most easily proved via variational ideas.

THEOREM 4.5. *Suppose that* $\lim_{t\to\infty} \sup \lambda_1 \left(\int_a^t Q(s)\, ds \right) = +\infty$. *Then* (4.5) *is oscillatory if either*

(A) $\lambda_1(Q(t))$ *is bounded above on* $[a, \infty)$

or

(B) $\lambda_n(Q(t))$ *is bounded below on* $[a, \infty)$.

This Theorem generalizes a result of Moore [**A14**] for the scalar case, and the following result may be considered as an extension of a result of Olech et al. [**A17**] for the scalar case.

THEOREM 4.6. *Suppose that, for each integer* $m \geq a$, *there exists a positive number* ε_m *and, for each positive integer* k, *there exists a unit vector* $x_{mk} \in \mathbf{R}^n$ *such that the set*

$$S_{mk} = \left\{ t \geq m : x_{mk}^* \left(\int_m^t Q(s)\, ds \right) x_{mk} \geq k \right\}$$

has measure at least ε_m. *Then* (4.5) *is oscillatory.*

In [**37**] some further extensions of the classical Hille-Wintner theorem were obtained in the Banach lattice context. We cite one result as an example of what may be obtained. We let B denote a Banach lattice with order continuous norm and let $\mathcal{L}(B)$ denote the Banach algebra of bounded linear operators $T : B \to B$. Let B_+ denote the positive cone and $\mathcal{L}_+(B)$ the corresponding positive cone in $\mathcal{L}(B)$, and we consider the second order operator-valued differential equations

$$Y'' + Q(t)Y = 0 \tag{4.7}$$

and

$$Y'' + Q_1(t)Y = 0, \tag{4.8}$$

where $Q, Q_1 : [a_1 + \infty) \to \mathcal{L}(B)$ are continuous in the uniform topology. We then have [**37**]

THEOREM 4.7. *Let B be a Banach lattice with order continuous norm, suppose the limits*

$$P(t) = \lim_{T\to\infty} \int_t^T Q(s)\,ds, \quad P_1(t) = \lim_{T\to\infty} \int_t^T Q_1(s)\,ds$$

exist (in the uniform operator topology of $\mathcal{L}(B)$), and, further,

(i) $$P(t), \quad P_1(t), \quad P(t) - P_1(t) \in \mathcal{L}_+(B), \quad t \in [a, \infty).$$

If there exists a nonoscillatory solution $Y(t)$ of (4.7) such that

(i) $$Z(t) \equiv Y'(t)Y^{-1}(t) \in \mathcal{L}_+(B), \quad t \in [a, +\infty),$$

then (4.8) has a nonoscillatory solution on $[a, +\infty)$.

In particular, Theorem 4.7 applies to $l_p, L_p, 1 \le p < \infty, c_0$ and any reflexive Banach lattice. Additional details and results may be found in [**37**].

5. Concluding remarks. Although Butler's original training was in convexity (packing and covering—see [**4, 6**]) his interests were very wide-ranging. In addition to his contributions in the areas outlined in the previous sections, he made substantial progress in other areas, as one may ascertain from his publication list. His work in mathematical biology is being reviewed separately so we shall briefly mention some additional results from other areas. In [**24**] Butler studied the problem of the convergence of successive approximations in a Banach space and showed, loosely speaking, that if E denotes a Banach space, I a closed interval of the real line, and if $f : I \times E \to E$ is continuous, then "almost all" initial value problems

$$y'(t) = f(t, y(t)), \quad y(\tau) = \eta, \tag{5.1}$$

are *uniquely* solvable by successive approximations. (That is, f, τ, η are allowed to vary.) This extended a result of Vidossich [**A21**] to the infinite dimensional case.

In [**21**] Butler studied the 1-set contractions and strict set contractions of a bounded closed convex subset C of a Banach space X (generalizations of nonexpansive mappings and contractions of C) and showed that "almost all" 1-set contractions have a fixed point. This extended a result of Vidossich [**A22**] for nonexpansive mappings.

In [**19**], Butler and Pianigiani investigated some properties of periodic points and chaotic functions in the unit interval. It was shown that the set of chaotic self-maps of the unit interval contains an open dense subset of the space of all continuous self-maps of the unit interval. Additional aspects of chaotic behavior of such maps were also considered.

In summary, it is hoped that this review of a portion of Butler's work will serve to demonstrate his wide-ranging interests and his untiring commitment to mathematics. His premature death is felt keenly by all those who knew him.

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