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# THE SUPPORTS OF MEASURES ASSOCIATED WITH ORTHOGONAL POLYNOMIALS AND THE SPECTRA OF THE RELATED SELF-ADJOINT OPERATORS

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Dedicated to Wolfgang J. Thron on the occasion of his 70th birthday

ABSTRACT. An elementary approach is given to prove Blumenthal's theorem describing the support of measures associated with orthogonal polynomials on the real line in case the recurrence coefficients associated with these polynomials tend to finite limits. Then the known approach using H. Weyl's theorem on compact perturbations of self-adjoint operators to Blumenthal's theorem is presented. Finally, using Weyl's theorem, Geronimus's result on the support is discussed when the recurrence coefficients with subscripts having the same residue $(\operatorname{mod} k)$ have finite limits. Instead of the usual approach of using continued fractions, the Hardy class $H^2$ is used to determine the spectrum of the self-adjoint operator arising in the study of this support.

**1. Introduction.** In what follows, the word measure will always refer to a positive Borel measure on the real line such that its support is an infinite set and all its moments are finite. Here the support of a measure $\alpha$ is the smallest closed set whose complement has $\alpha$-measure zero and is denoted by $\operatorname{supp}(\alpha)$, and the $n$-th moment of $\alpha$ is defined as

$$\int_{-\infty}^{\infty} x^n \, d\alpha(x), \qquad n \geq 0.$$

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These moments are said to be finite if these integrals are absolutely convergent.

As is well known, there is a unique system of orthonormal polynomials $p_n(x) = p_n(d\alpha, x) = \gamma_n(d\alpha)x^n + \cdots$, $n = 0, 1, \ldots$, associated with such a measure. These polynomials are defined by the requirements that $\gamma_n = \gamma_n(d\alpha) > 0$ for all $n \geq 0$ and

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)\, d\alpha(x) = \delta_{mn}, \qquad m, n \geq 0,$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise. They satisfy the three-term recurrence equation,

$$(1) \qquad a_{n+1}p_{n+1}(z) + (b_n - z)p_n(z) + a_n p_{n-1}(z) = 0, \qquad n \geq 0;$$

here $a_n = a_n(d\alpha) = \gamma_{n-1}/\gamma_n$ (for $n = 0$ put $\gamma_{-1} = 0$), $b_n = b_n(d\alpha)$, $p_0(x) = \gamma_0$, and $p_{-1}(x) = 0$ (see, for example, [**6**, formula (I.2.4), p. 17] or [**22**, formula (3.2.1), p. 42]).

Given real numbers $a_n, b_n$, $n \geq 0$, such that $a_0 = 0$ and $a_n > 0$ for $n > 0$, there is a measure $\alpha$ such that $a_n(d\alpha) = a_n$ and $b_n(d\alpha) = b_n$. This result is often attributed to J. Favard, but it goes back to T. J. Stieltjes; for a concise, modern discussion, see, for example, [**4**, §XII.10, pp. 1275–1276], or, for a more extensive discussion, [**21**, pp. 530–614]. Under fairly general conditions, the measure $\alpha$ is uniquely determined. This is certainly the case if the sequences of the numbers $a_n$ and $b_n$ are bounded, see, for example, [**6**, Theorem II.2.2, p. 64] (cf. also our Corollary 9 below).

Blumenthal's theorem (see Theorem 10 below) concerns the determination of the support of the measure $\alpha$ if the coefficients $a_n$ and $b_n$ have limits. In Sections 1–3 we discuss an elementary approach to this theorem. In Section 4 we show how a result of H. Weyl on compact perturbations of self-adjoint operators can be used to derive Blumenthal's theorem; this approach is not new, it is mentioned, for example, in [**13**, pp. 1023–1024]. In Section 5 we extend this approach to discuss the result of Geronimus [**8, 9, 10**, and **11**] (cf. also [**7**]) on the support of $\alpha$ when it is assumed that, for a fixed, positive integer $k$, the coefficients $a_n$ and $b_n$ with $n$ belonging to the same residue class $(\operatorname{mod} k)$ have limits when $n$ tends to infinity. In discussing this result, we use the

Hardy class $H^2$ to determine the spectrum of a self-adjoint operator rather than relying on the usual approach of continued fractions.

This paper started out as a survey of certain topics connected with Blumenthal's theorem, but in the end it went beyond its original goal in that some of the approaches and results are new. For example, Theorem 2 is not a consequence of the known theorem of Poincaré, and at least the formulation and method of proof, if not the content, of Theorem 13 appear to be new.

**2. Ratio asymptotics for the class $\mathbf{M}(\mathbf{a}, \mathbf{b})$.** The class $\mathbf{M}(2a, b)$ is the class of measures for which

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b. \tag{2}$$

In view of this limit relation, the recurrence equation in (1) enables one to derive an asymptotic formula for $p_{n-1}(z)/p_n(z)$, valid on most of the complex plane, with the aid of the following theorem of Poincaré (see [**17**], [**14**, §17.1, p. 526], or [**16**, §X.6, p. 300]):

**Theorem 1.** *Let $k$ be a positive integer. Suppose that, for every integer $n > 0$, the difference equation*

$$f(n+k) + \sum_{j=0}^{k-1} a_{jn} f(n+j) = 0 \tag{3}$$

*holds, where the limits*

$$\lim_{n\to\infty} a_{jn} = a_j, \qquad 0 \le j < k, \tag{4}$$

*exist and the roots of the "characteristic equation"*

$$z^k + \sum_{j=0}^{k-1} a_j z^j = 0 \tag{5}$$

*all have different absolute values. Write $\zeta_1, \ldots, \zeta_k$ for these roots. Then either $f(n) = 0$ for all $n$ large enough or there is an $l$ with $1 \le l \le k$ such that*

$$\lim_{n\to\infty} f(n+1)/f(n) = \zeta_l. \tag{6}$$

So as to make these notes more self-contained, we intend to avoid using this theorem in what follows. Instead, we will establish a special case for second order equations; this result and its proof are motivated by the approach given in [**23**, pp. 117 –119] to the proof of Theorem 3 below.

**Theorem 2.** *Assume that for every integer $n > 0$ we have*

$$f(n+2) + A_n f(n+1) + B_n f(n) = 0, \tag{7}$$

*where the limits*

$$\lim_{n\to\infty} A_n = A \quad \textit{and} \quad \lim_{n\to\infty} B_n = B \tag{8}$$

*exist. Let $\eta$ be a positive number, and assume that*

$$|f(n+1)/f(n)|^2 > |B| + \eta \tag{9}$$

*for each sufficiently large $n$. Let $\zeta_1$ and $\zeta_2$ denote the roots of the characteristic equation*

$$z^2 + Az + B = 0 \tag{10}$$

*and assume that $|\zeta_1| \geq |\zeta_2|$. Then*

$$\lim_{n\to\infty} f(n+1)/f(n) = \zeta_1. \tag{11}$$

The reason we mentioned Poincaré's theorem was to motivate this result. Note, however, that here we do not assume that the absolute values of $\zeta_1$ and $\zeta_2$ are different, and so this result is not a direct consequence of Poincaré's theorem. Since, however, $\zeta_1\zeta_2 = B$, the inequality $|\zeta_1| > |\zeta_2|$ follows from (9) and (11); the latter, however, has to be proved first. In fact, the role of assuming (9) is to ensure that this inequality holds and to exclude the possibility that the limit in (11) is $\zeta_2$.

*Proof.* Writing $L_n = f(n+1)/f(n)$, divide (7) through by $f(n+1)$. We obtain

$$L_{n+1} + A_n + \frac{B_n}{L_n} = 0.$$

Subtract this equation from the analogous equation obtained via replacing $n$ with $n+1$. We get

(12)

$$\begin{aligned} L_{n+2} - L_{n+1} &= A_n - A_{n+1} + \frac{B_n}{L_n} - \frac{B_{n+1}}{L_{n+1}} \\ &= A_n - A_{n+1} + \frac{B_n - B_{n+1}}{L_n} + \frac{B_{n+1}}{L_n L_{n+1}}(L_{n+1} - L_n). \end{aligned}$$

In view of (7), (8) and (9), there is a positive number $\rho < 1$ such that, given an arbitrary $\varepsilon > 0$, we have

$$\left| A_n - A_{n+1} + \frac{B_n - B_{n+1}}{L_n} \right| < \varepsilon$$

and

$$\left| \frac{B_{n+1}}{L_n L_{n+1}} \right| < \rho$$

for every $n$ large enough, say, for $n \geq N$. Thus, for $n \geq N$, (12) implies

$$|L_{n+2} - L_{n+1}| < \varepsilon + \rho |L_{n+1} - L_n|.$$

Using this for $n = N,\ N+1, \dots, N+\nu-1$, we obtain

$$\begin{aligned} |L_{N+\nu+1} - L_{N+\nu}| &< \varepsilon \sum_{j=0}^{\nu-1} \rho^j + \rho^\nu |L_{N+1} - L_N| \\ &< \frac{\varepsilon}{1-\rho} + \rho^\nu |L_{N+1} - L_N|, \qquad \nu \geq 1, \end{aligned}$$

that is,

$$\limsup_{n\to\infty} |L_{n+1} - L_n| \leq \frac{\varepsilon}{1-\rho}.$$

Since $\varepsilon$ here can be arbitrarily small, it follows that

$$\lim_{n\to\infty} |L_{n+1} - L_n| = 0. \tag{13}$$

Now let $\langle n_\nu \rangle$ be a sequence of integers such that the limit

$$L = \lim_{\nu\to\infty} L_{n_\nu}$$

exists or is $+\infty$ or $-\infty$. Then

$$L = \lim_{\nu\to\infty} L_{n_\nu+1},$$

as well, in view of (13). Hence, it is easy to see from (7) that $L$ is finite and it satisfies equation (10), that is, either $L=\zeta_1$ or $L=\zeta_2$. The latter alternative is impossible in view of (9). Hence, $L=\zeta_1$. That is, $\langle L_n\rangle$ has no subsequence that converges to any number (or $\pm\infty$) other than $\zeta_1$. Hence, (11) holds, completing the proof. □

Theorem 2 is sufficient to derive certain asymptotic estimates for $p_{n-1}(z)/p_n(z)$ outside the closure of the set of zeros of the denominator. To state these estimates, let $\mathbf{C}$ denote the complex plane, and let $\sqrt{z^2-1}$ denote the branch of the function $\pm\sqrt{z^2-1}$ that is holomorphic in the region $\mathbf{C}\backslash[-1,1]$ and is such that

$$|z-\sqrt{z^2-1}|<1 \quad \text{for} \quad z\notin[-1,1]. \tag{14}$$

Then we have

**Theorem 3.** *Assume that the measure $\alpha$ belongs to $\mathbf{M}(1,0)$ (cf. (2)). Let $S$ be an infinite set of integers, and let $\Omega$ be closure of the set of zeros of $p_n(z)$ as $n$ runs through the elements of $S$. Then*

$$\lim_{n\to\infty, n\in S} \frac{p_{n-1}(z)}{p_n(z)} = z-\sqrt{z^2-1}, \qquad z\in\mathbf{C}\backslash\Omega, \tag{15}$$

*and the convergence is uniform on compact subsets of $\mathbf{C}\backslash\Omega$.*

This is all we need for the proof of Theorem 8 below, but in itself the result is more useful in its stronger version: if $\alpha\in\mathbf{M}(1,0)$, then we have

$$\lim_{n\to\infty} \frac{p_{n-1}(z)}{p_n(z)} = z-\sqrt{z^2-1}, \qquad z\in\mathbf{C}\backslash\operatorname{supp}(\alpha) \tag{16}$$

and

$$\lim_{n\to\infty} \frac{p_{n-1}(z)}{p_n(z)} = z+\sqrt{z^2-1}, \qquad z\in\operatorname{supp}(\alpha)\backslash[-1,1]; \tag{17}$$

the convergence in (16) is uniform on compact subsets of the set indicated. To appreciate these formulas, one needs to know the structure of the support of $\alpha$ as described in Theorem 8. See [**15**, Theorems 4.1.13 and 4.1.18, pp. 33 and 36] for details.

For the proof, we need an upper estimate for $p_{n-1}(z)/p_n(z)$ so that we can use Theorem 3. This is furnished by the following simple lemma. As usual, in this lemma we write

$$x_{1n} > x_{2n} > \cdots > x_{nn} \tag{18}$$

for the zeros of $p_n$. As is well known, these zeros are real numbers, and they are interlaced in the sense that

$$x_{kn} > x_{k,n-1} > x_{k+1,n}, \qquad 1 \le k < n \tag{19}$$

holds for $n > 1$ (cf. e.g., [**6**, Theorem 1.2.3, p. 17] or [**22**, Theorem 3.3.2, p. 46]).

**Lemma 4.** *Let $n > 1$ be an integer, let $\eta$ be positive, and assume that*

$$|z - x_{kn}| \ge \eta, \qquad 1 \le k \le n. \tag{20}$$

*Then*

$$|p_{n-1}(z)/p_n(z)| \le a_n/\eta, \tag{21}$$

*where $a_n$ is the recurrence coefficient given occurring in equation (1).*

It is not of great importance that this result is also true for $n = 1$; however, it is somewhat inconvenient to pay attention to this case in the proof that follows.

*Proof.* We have

$$\left|\frac{p_{n-1}(z)}{p_n(z)}\right| = \left|\frac{\gamma_{n-1}\prod_{k=1}^{n-1}(z - x_{k,n-1})}{\gamma_n \prod_{k=1}^{n}(z - x_{kn})}\right| \le \frac{\gamma_{n-1}}{\gamma_n \min_{1\le k\le n}|z - x_{kn}|}. \tag{22}$$

The inequality here holds in view of (19), since this implies that we have

$$|z - x_{k,n-1}| < |z - x_{k+1,n}| \quad \text{or} \quad |z - x_{k,n-1}| < |z - x_{kn}|$$

according to whether $\Re z \le x_{k,n-1}$ or $\Re z \ge x_{k,n-1}$. Thus (20) and the relation $a_n = \gamma_{n-1}/\gamma_n$ imply (21), completing the proof. □

A different proof of the above lemma is given in [**23**, p. 118]. Next we turn to the

*Proof of Theorem* 3. Let $z$ be a nonreal complex number. Since every $x_{kn}$ is real, (20) is fulfilled with $\eta = |\mathfrak{F}\, z|$. Hence, we have

$$(23) \qquad |p_{n+1}(z)/p_n(z)| \ge |\mathfrak{F}\, z|/a_{n+1}$$

according to Lemma 4. Now equation (1) can be written in the form

$$(24) \qquad p_{n+2}(z) + A_n(z)p_{n+1}(z) + B_n p_n(z) = 0, \qquad n \ge -1,$$

where

$$(25) \qquad A_n = \frac{b_{n+1} - z}{a_{n+2}} \quad \text{and} \quad B_n = \frac{a_{n+1}}{a_{n+2}}.$$

Here

$$(26) \qquad \lim_{n\to\infty} A_n = -2z \quad \text{and} \quad \lim_{n\to\infty} B_n = 1,$$

since (2) holds with $a = 1/2$ and $b = 0$, as $\alpha \in \mathbf{M}(1,0)$ according to our assumptions. Therefore, (9) holds for large $n$ with $f(n) = p_n(z)$ in view of (23) provided, say, $|\mathfrak{F}\, z| > 2$. Hence, by Theorem 2,

$$\lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)} = z + \sqrt{z^2 - 1}$$

holds if $|\mathfrak{F}\, z| > 2$. Hence, a fortiori, (15) also holds in this case.

Now let $E$ be a connected regular compact subset (that is, it is the closure of its interior) of $\mathbf{C}\backslash\Omega$ such that it contains infinitely many points $z$ with $|\mathfrak{F}\, z| > 2$. Writing $\rho$ for Euclidean distance, let $\eta = \rho(E, \Omega)/2$; clearly, $\eta > 0$. Now, by definition of $\Omega$, there is an $N$ such that (20) holds with the above $\eta$ for every $z \in E$ and for every $n \in S$ with $n > N$. Hence, (21) holds for every $z \in E$ and $n \in S$ with $n > N$. Therefore, (15) holds for every $z \in E$, and the convergence is uniform on $E$ by Vitali's theorem (see, e.g., [**5**, §1.3, p. 9] or [**19**, §6.4.2, pp. 174–176]). The proof of Theorem 3 is now complete. □

**3. The set of limit points of zeros of polynomials in the class $\mathbf{M}(a, b)$.** The asymptotic result (Theorem 3) can be used to obtain information about the set of limit points of the zeros of orthogonal polynomials. Given a measure $\alpha$, denote by $\Xi = \Xi(d\alpha)$ the set of complex numbers $z$ such that, for every neighborhood $U$ of $z$, there is an $N$ such that, for every $n > N$, the polynomial $p_n(d\alpha)$ has a zero in $U$. Since all zeros of orthogonal polynomials are real, it is clear that $\Xi$ is a set of reals. Using Theorem 3, we obtain

**Theorem 5.** *Suppose that $\alpha$ belongs to $\mathbf{M}(1, 0)$. Then*

$$[-1, 1] \subset \Xi(\alpha) \tag{27}$$

*holds.*

*Proof.* Assume the contrary. As $\Xi$ is closed, there is then a $t \in (-1, 1)$ such that $t \notin \Xi$. Moreover, there is an infinite set $S$ of positive integers such that the set $\Omega$ of limit points of zeros of $p_n$ as $n$ runs through the elements of $S$ does not contain $t$. This is, however, impossible, for the following reason:

Let $E$ be a compact subset of $\mathbf{C} \backslash \Omega$ that contains $t$ in its interior. The sequence of continuous functions $p_{n+1}(z)/p_n(z)$ is uniformly convergent for $z \in E$ as $n \in S$ tends to $\infty$, according to Theorem 3. However, the limit of this sequence given by (15) is not continuous at $t$ (cf. (14)). This contradiction completes the proof of the theorem. □

In the opposite direction, we will show

**Theorem 6.** *Assume that $\alpha \in \mathbf{M}(1, 0)$, and let $\varepsilon > 0$. Then there are only finitely many elements of $\Xi(\alpha)$ that belong to the set*

$$\mathbf{C} \backslash (-1 - \varepsilon, 1 + \varepsilon).$$

To show this, we will first prove

**Lemma 7.** *Let $x$ be real and let $N$ be a nonnegative integer. Assume that*

$$|x| \geq \sup_{j \geq N} (a_j + a_{j+1} + |b_j|) \tag{28}$$

*and*

$$|p_N(x)| \geq |p_{N-1}(x)| \tag{29}$$

*hold. Then*

$$|p_{n+1}(x)| \geq |p_n(x)| \tag{30}$$

*and*

$$\operatorname{sgn}(x^{n+1} p_{n+1}(x)) = \operatorname{sgn}(x^n p_n(x)) \neq 0 \tag{31}$$

*hold for every $n \geq N$.*

We did not need to use absolute value signs for $a_j$ and $a_{j+1}$ in (28), since these are always positive. Note that (29) implies that $p_N(x) \neq 0$ since orthogonal polynomials of adjacent degree have no common zeros (cf. (19)). Thus, $p_n(x) \neq 0$ for $n \geq N$ in view of (30). Moreover, $x \neq 0$ by (28) since $a_j > 0$ for all $j \geq 0$. Therefore, (30) and (31) can be stated as a single formula:

$$\operatorname{sgn}(x p_n(x)) p_{n+1}(x) \geq |p_n(x)|, \qquad n \geq N. \tag{32}$$

*Proof.* Let $n \geq N$ and assume that (30) holds with $n-1$ replacing $n$. By the recurrence formula given in (1), we obtain

$$p_{n+1}(x) = \operatorname{sgn}(x) \frac{|x - b_n|}{a_{n+1}} p_n(x) - \frac{a_n}{a_{n+1}} p_{n-1}(x)$$

since $\operatorname{sgn}(x) = \operatorname{sgn}(x - b_j)$ in view of (28). Multiplying both sides by $\operatorname{sgn}(x p_n(x))$, we obtain

$$\begin{aligned} \operatorname{sgn}(x p_n(x)) p_{n+1}(x) &= \frac{|x - b_n|}{a_{n+1}} |p_n(x)| - \frac{a_n}{a_{n+1}} p_{n-1}(x) \operatorname{sgn}(x p_n(x)) \\ &\geq \frac{|x| - |b_n|}{a_{n+1}} |p_n(x)| - \frac{a_n}{a_{n+1}} |p_{n-1}(x)| \\ &\geq \frac{|x| - |b_n| - a_n}{a_{n+1}} |p_n(x)|. \end{aligned}$$

The last holds in view of the induction hypothesis, i.e., that (30) holds with $n-1$ replacing $n$. The coefficient of $|p_n(x)|$ on the right-hand side is $\geq 1$ according to (28). Thus, (32) follows for $n$, completing the proof. □

*Proof of Theorem* 6. We will show that the interval $[1+\varepsilon, \infty)$ contains only finitely many elements of $\Xi$; we can show this for the interval $(-\infty, -1-\varepsilon]$ in a similar fashion. Choose a positive integer $N$ such that

$$\sup_{j\geq N}(a_j + a_{j+1} + |b_j|) < 1 + \varepsilon/2. \tag{33}$$

Let $N_1 \geq N$ be such that $p_{N_1}(x)$ has a zero in the interval $(1+\varepsilon/2, 1+\varepsilon)$; if no such $N_1$ exists, then put $N_1 = N$. Let $N_2 \geq N_1$ be the least integer such that

$$|p_{N_2+1}(x_0)| \geq |p_{N_2}(x_0)| \tag{34}$$

for some $x_0 \in (1+\varepsilon/2, 1+\varepsilon)$. Such an $N_2$ must exist. Indeed, if $x_0$ is a zero of $p_{N_1}$ in this interval, then (34) holds with $N_2 = N_1$. If we had to choose $N_1 = N$ because the polynomial $p_n$ has no zero in this interval for any $n \geq N$, then (15) of Theorem 3 holds for every $z$ in this interval with $S$ being the set of all positive integers; thus, for every $x_0$ in this interval, (34) will be satisfied if $N_2$ is large enough.

We claim that $\Xi$ has no more than $N_2 + 1$ elements in the interval $[1+\varepsilon, \infty)$. Indeed, in view of (34), Lemma 7 shows that

$$\operatorname{sgn}(p_{n+1}(x_0)) = \operatorname{sgn}(p_n(x_0)) \neq 0, \qquad n \geq N_2 + 1,$$

according to (31), since $x_0$ is positive. As $\lim_{x\to+\infty} p_n(x) = +\infty$ (because the leading coefficient of $p_n$ is positive), it follows from here that for $n \geq N_2 + 1$ the number of zeros (i.e., of sign changes, these zeros being all simple) in the interval $[x_0, \infty)$ of $p_{n+1}$ has the same parity as does that of $p_{n+1}$. As these zeros are interlaced in the sense of (19), the difference between the number of zeros of $p_{n+1}$ and that of $p_n$ in this interval is either 0 or 1; therefore, for $n \geq N_2 + 1$ the number of zeros of $p_{n+1}$ and that of $p_n$ in this interval are equal.

That is, the number of zeros of $p_n$ for $n \geq N_2 + 1$ in this interval is the same as the number of zeros of $p_{N_2+1}$ in this interval; this number, however, cannot be more than $N_2+1$, the degree of the latter polynomial. Now the claim on the number of elements of $\Xi$ easily follows from the definition of this set. The proof of Theorem 6 is complete. □

The consequence of these results for the support of $\alpha$ can be summarized as follows.

**Theorem 8.** *Assume that* $\alpha \in \mathbf{M}(1,0)$. *Then*

$$[-1,1] \subset \operatorname{supp}(\alpha), \tag{35}$$

*and for every* $\varepsilon > 0$ *the set*

$$\operatorname{supp}(\alpha) \backslash (-1-\varepsilon, 1+\varepsilon) \tag{36}$$

*is finite.*

*Proof.* If $I$ is a real interval that is disjoint from the support of $\alpha$, then for each $n$ the polynomial $p_n$ has at most one zero in this interval (cf. [**22**, Theorem 3.41.2, p. 50]). Therefore, it follows from Theorem 5 that $\operatorname{supp}(\alpha)$ is dense in $(-1,1)$. Since $\operatorname{supp}(\alpha)$ is closed by its definition, (35) follows.

On the other hand, given an open real interval $I$ that intersects $\operatorname{supp}(\alpha)$, there is an integer $N$ such that for each $n > N$ the polynomial $p_n$ must have at least one zero in $I$. That is,

$$\operatorname{supp}(\alpha) \subset \Xi(\alpha). \tag{37}$$

Therefore, the finiteness of the set in (36) follows from Theorem 6. The proof is complete. □

A further consequence of Lemma 7 is

**Corollary 9.** *Given an arbitrary measure* $\alpha$, *we have*

$$\operatorname{supp}(\alpha) \subset [-\sup_{j\geq 0}(a_j + a_{j+1} + |b_j|),\ \sup_{j\geq 0}(a_j + a_{j+1} + |b_j|)]. \tag{38}$$

*Proof.* For $x$ outside the interval in (38), we have

$$|p_{n+1}(x)| \geq |p_n(x)|, \qquad n \geq 0, \tag{39}$$

according to (30). Indeed, $|p_0(x)| \geq |p_{-1}(x)|$ $(=0)$; hence, Theorem 6 is applicable with $N = 0$. As $p_0(x) = \gamma_0 > 0$, (39) implies that $p_n(x)$ is not zero for any $x$ outside the interval given in (38), $n \geq 0$. Thus, $\Xi$ is a subset of this interval. Hence, (38) follows from (37). □

Finally, by a simple linear transformation of $x$, we obtain Blumenthal's theorem (see [**1**], [**2**, §§IV.3–4, pp. 113–124, esp. after formula (4.2), p. 121], or [**15**, Lemma 3.3.6 and Theorem 3.3.7, p. 23]; note that the last paper uses the notation $\mathbf{M}(a,b)$ differently from us: what we mean by $\mathbf{M}(a,b)$ is written as $M(b,a)$) in [**15**]:

**Theorem 10.** *Let $a > 0$ and $b$ be real numbers, and assume that $\alpha \in \mathbf{M}(a,b)$. Then*

$$[b-a, b+a] \subset \operatorname{supp}(\alpha), \tag{40}$$

*and for every $\varepsilon > 0$ the set*

$$\operatorname{supp}(\alpha) \backslash (b-a-\varepsilon, b+a+\varepsilon) \tag{41}$$

*is finite.*

Blumenthal formulated his result in terms of the limit points of zeros of $p_n(d\alpha)$ rather than in terms of measures, and his result was slightly weaker in the sense that he considered the set of all limit points rather than the set $\Xi(d\alpha)$ considered above. The paper [**3**] surveys various other results connecting the support of $\alpha$ with the behavior of the recurrence coefficients, mostly for measures outside the class $\mathbf{M}(a,b)$ for finite $a$ and $b$.

**4. Application of Weyl's theorem to the study of the support.** With a system of orthogonal polynomials on the real line, one can associate the following matrix acting as an operator on the space $l^2$ of sequences:

$$\mathbf{A} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \cdots \\ 0 & 0 & a_3 & b_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{42}$$

where the numbers $a_n$ and $b_n$ are the coefficients in the recurrence formula given in (1). In other words, $\mathbf{A} = (a_{mn})_{0 \le m,n < \infty}$, where $a_{nn} = b_n, a_{n,n+1} = a_{n+1,n} = a_n$, $n \ge 0$, and $a_{mn} = 0$ for $|m - n| > 1$. In case the set of the coefficients $a_n$ and $b_n$ is bounded, the operator $\mathbf{A}$ is self-adjoint. If this set is not bounded, then the situation is more complicated; in this case the operator $\mathbf{A}$ is symmetric, and it can be extended to a self-adjoint operator, but this extension may not be unique. It will be unique exactly when the moment problem associated with the recurrence coefficients $a_n$ and $b_n$ has a unique solution. In any case, in what follows we will not be concerned with the unbounded case.

The connection between the operator $\mathbf{A}$ and the support of the measure is worked out in detail in Stone [**21**, §X.4, pp. 530–614]; also see [**4**, pp. 1275–76 in §XII.10] for a brief outline. The connection is particularly simple when $\mathbf{A}$ is bounded. Namely, in this case, the support of the measure $\alpha$ equals the spectrum of the operator $\mathbf{A}$.

The reason for this connection is that there is a canonical isomorphism $\iota$ between the Hilbert spaces $l^2$ and $L^2_\alpha(-\infty, \infty)$. Under this isomorphism the sequence $\langle \sigma_n \rangle_{n=0}^\infty$ in $l^2$ corresponds to the function

$$f = \sum_{n=0}^{\infty} \sigma_n p_n \in L^2_\alpha(-\infty, \infty),$$

where the convergence on the right is meant in the sense of the metric of $L^2_\alpha(-\infty, \infty)$. It is easy to see from (1) that, under this isomorphism, the operator $\mathbf{A}$ on $l^2$ corresponds to the operator $\mathbf{M} : f \mapsto xf$, and in the bounded case it is easy to relate the spectrum of $\mathbf{M}$ to the support of $A$.

The essential spectrum of an operator is defined as the set of limit points of its spectrum. An important result relating the essential spectrum of a self-adjoint operator to that of its perturbation plays a key role in discussing Theorem 8 with the aid of Hilbert space operators. The result we have in mind is a Theorem of H. Weyl; (see, e.g., [**18**, §134, p. 367]) and can be stated as follows.

**Theorem 11.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be self-adjoint operators of a Hilbert space, and assume that* $\mathbf{B}$ *is compact. Then the essential spectra of* $\mathbf{A}$ *and* $\mathbf{A}+\mathbf{B}$ *are the same.*

Recall that a linear operator is called *compact* if it maps the unit ball onto a set whose closure is compact; compact operators used to be called *completely continuous* (*vollstetig* in German). In [**18**, *loc. cit.*], this theorem is stated only for bounded $\mathbf{A}$, but it is remarked that it can easily be extended for unbounded $\mathbf{A}$; in any case, we will only be concerned below with the case of bounded $\mathbf{A}$. There is also a version of Weyl's theorem that is valid for nonself-adjoint Hilbert space operators; see, for example, [**12**, pp. 91–92, 145].

*Alternative Proof of Theorem 7.* For the matrix $\mathbf{A}$ we have $\mathbf{A} = \frac{1}{2}\mathbf{T}+\mathbf{B}$, where

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \text{and} \tag{43}$$

$$\mathbf{B} = \begin{pmatrix} b_0 & a_1-\frac{1}{2} & 0 & 0 & \cdots \\ a_1-\frac{1}{2} & b_1 & a_2-\frac{1}{2} & 0 & \cdots \\ 0 & a_2-\frac{1}{2} & b_2 & a_3-\frac{1}{2} & \cdots \\ 0 & 0 & a_3-\frac{1}{2} & b_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Now if $\alpha \in \mathbf{M}(1,0)$, that is, if (2) holds with $a = 1/2$ and $b = 0$, then the entries along the diagonals of the matrix $\mathbf{B}$ tend to 0. Since $\mathbf{B}$ is a tri-diagonal matrix (i.e., all its elements except for those in the main diagonal or immediately adjacent to it are 0), this implies that $\mathbf{B}$ is a compact operator. Thus, the essential spectrum of $\mathbf{A}$ is the same as that of $\frac{1}{2}\mathbf{T}$, according to Weyl's theorem (Theorem 11).

Therefore, it is sufficient to show that the spectrum of $\frac{1}{2}\mathbf{T}$ is the interval $[-1,1]$. We will show this in the next lemma. The proof is complete. $\square$

**Lemma 12.** *The spectrum of the matrix* $\mathbf{T}$ *given in* (43), *considered as an operator on the Hilbert space* $l^2$, *is the interval* $[-2,2]$.

A simple way to see the validity of this lemma is to notice that $\frac{1}{2}\mathbf{T}$ is the matrix associated with the orthonormal Chebyshev polynomials of the second kind; hence, its spectrum is the same as the support of the measure associated with the Chebyshev polynomials of the second kind. This support is well known to be the interval $[-1,1]$. We will, however, present a different proof using the Hardy class $H^2$, that is, the Hilbert space of power series

$$f(z)=\sum_{j=0}^{\infty} c_j z^j$$

under the norm

$$\|f\|=\left(\sum_{j=0}^{\infty}|c_j|^2\right)^{1/2}.$$

If $f\in H^2$, then $f(z)$ is holomorphic for $|z|<1$, and for almost every $z$ with $|z|=1$ we can define $f(z)$ as the radial limit $\lim_{r\nearrow 1} f(rz)$, and we have

$$\|f\|^2=\frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})|^2\,dt, \tag{44}$$

see, e.g., [**24**, Vol. 1, §VII.7, pp. 271–277]. The advantage of a proof using the class $H^2$ is that the ideas in this proof can be generalized to handle the asymptotically periodic case, to be discussed in the next section. The reason the class $H^2$ is useful for this purpose is that the unilateral shift in $H^2$ can be represented as multiplication by $z$.

*Proof of Lemma* 12. Under the isomorphism

$$\langle c_j\rangle_{j=0}^{\infty}\mapsto\sum_{j=0}^{\infty}c_j z^j \tag{45}$$

between $l^2$ and $H^2$, the operator $\mathbf{T}$ can be represented on $H^2$ as

$$(\mathbf{T}f)(z) = \left(z + \frac{1}{z}\right) f(z) - \frac{1}{z} f(0). \tag{46}$$

We will use the definition of the spectrum: a complex number $\lambda$ belongs to the spectrum of $\mathbf{T}$ if the operator $\mathbf{T} - \lambda\mathbf{I}$ does not have a bounded inverse ($\mathbf{I}$ is the identity operator). First we will show that if $\lambda \notin [-2, 2]$, then $\lambda$ is not in the spectrum of $\mathbf{T}$. Before we do this, we outline a simpler reasoning why such a $\lambda$ is not in the spectrum (however, the simpler reasoning is not usable under the more general circumstances considered in the next section): as $\mathbf{T}$ is self-adjoint, its spectrum is on the real line, and, as $\|\mathbf{T}\| = 2$, its spectrum is a subset of $\{z : |z| \le 2\}$. But, as we promised, we will give an argument applicable under more general circumstances.

Fix $\lambda \notin [-2, 2]$, and consider the equation $(\mathbf{T} - \lambda\mathbf{I})f = g$ for $f$. This equation can be written as

$$\left(z + \frac{1}{z} - \lambda\right) f(z) - \frac{1}{z} f(0) = g(z). \tag{47}$$

This equation can be solved for $f(z)$ as

$$f(z) = \frac{zg(z) + f(0)}{z^2 - \lambda z + 1}. \tag{48}$$

Solving equation (47) in this way leaves several questions unresolved. First, how to determine $f(0)$? This question must be answered since the solution must be unique. Second, is the $f$ obtained in this solution a member of $H^2$, and, even if this is the case, is the operator represented by this solution bounded?

As $\lambda \notin [-2, 2]$, the denominator of the right-hand side has two distinct zeros $\zeta_1$ and $\zeta_2$ with $|\zeta_1| < 1$ and $|\zeta_2| > 1$ (note that, clearly, $\zeta_1\zeta_2 = 1$). The zero in the denominator at $\zeta_1$ must be compensated for a zero at the same point in the numerator, since $f$, being in $H^2$, must not have a singularity inside the unit circle. That is, we must have

$$f(0) = -\zeta_1 g(\zeta_1). \tag{49}$$

This equation determines $f(0)$. The fact that, given $g \in H^2$, we can uniquely determine an $f \in H^2$ (via equations (48) and (49)) such that equation (47) is satisfied shows that the operator determined by this latter equation is one-to-one and onto. Hence, it has a bounded inverse, according to the Open Mapping Theorem (see, e.g., Rudin [**20**, Corollary 2.12(b), p. 49]).

Alternatively, it is also easy to see directly that the inverse defined by equations (48) and (49) is continuous and, therefore, bounded. Indeed, the denominator in (48) is bounded away from 0 for $|z| = 1$. Furthermore, in (49), $f(0)$ depends continuously on $g$; this can be seen by expressing $g(\zeta_1)$ with the aid of the Cauchy integral formula. Hence, $\lambda$ does not belong to the spectrum of $\mathbf{T}$.

Now let $\lambda \in [-2, 2]$. In this case, the zeros $\zeta_1$ or $\zeta_2$ of the denominator of (48) both have absolute value 1 (we may have $\zeta_1 = \zeta_2$). We will then show that $\lambda$ belongs to the spectrum of $\mathbf{T}$. We will do this essentially by showing that, in this case, there is no way to compensate for both singularities on the right-hand side of (48); that is, for an appropriate choice of $g$, the function $f$ defined by this equation is not in $H^2$ (because it is not square integrable on the circumference of the unit circle) no matter how we choose $f(0)$. Assume that $g$ is a polynomial, so that it is smooth on the circumference of the unit circle. In this case, to make the integral on the right-hand side of (44) exist, we have to compensate for both singularities on the circumference of the unit circle in a way similar to (49). In the case that $\zeta_1 \neq \zeta_2$, this means that we must have

$$f(0) = -\zeta_1 g(\zeta_1) = -\zeta_2 g(\zeta_2),$$

and if $\zeta_1 = \zeta_2$, this means that

$$f(0) = -\zeta_1 g(\zeta_1) \quad \text{and} \quad g(\zeta_1) + \zeta_1 g'(\zeta_1) = 0.$$

Given an appropriate choice of $g$, these requirements are impossible to satisfy, showing that the $\lambda$ in question belongs to the spectrum of $\mathbf{T}$. The proof is complete. $\square$

**5. Asymptotically periodic recurrence coefficients.** In what follows, we are going to discuss the case when the recurrence coefficients $a_n$ and $b_n$ are periodic in limit. That is, we are going to discuss the spectrum of the matrix $\mathbf{A}$ given in (42) when the entries satisfy the following: there exist $k > 1$, positive reals $\eta_0, \eta_1, \dots, \eta_{k-1}$ and arbitrary reals $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ such that

$$(50)\qquad \lim_{\substack{n\to\infty \\ n\equiv\nu \pmod k}} a_n = \eta_\nu \quad\text{and}\quad \lim_{\substack{n\to\infty \\ n\equiv\nu \pmod k}} b_n = \lambda_\nu, \qquad 0 \le \nu < k.$$

In view of Weyl's Theorem 11, to this end we need to study the spectrum of the matrix

$$(51)\qquad \mathbf{D} = \begin{pmatrix} \lambda_0 & \eta_1 & 0 & 0 & \cdots \\ \eta_1 & \lambda_1 & \eta_2 & 0 & \cdots \\ 0 & \eta_2 & \lambda_2 & \eta_3 & \cdots \\ 0 & 0 & \eta_3 & \lambda_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where, for $n$ outside the range $0 \le n < k$, we defined $\eta_n$ and $\lambda_n$ with the aid of the equations

$$(52)\qquad \eta_n = \eta_\nu \quad\text{and}\quad \lambda_n = \lambda_\nu \text{ for } n \equiv \nu \bmod k, \qquad 0 \le \nu < k.$$

This spectrum is characterized by

**Theorem 13.** *Let $k > 1$ be an integer, let $\eta_n > 0$ and let $\lambda_n$ be real for all $n \ge 0$, and assume (52). Then, except for a finite number of additional elements, the spectrum of the matrix $\mathbf{D}$ given in (51), considered as an operator on $l^2$, is the set of reals $\lambda$ for which*

$$(53)\qquad |D_{0,k-1}(\lambda) - \eta_0^2 D_{1,k-2}(\lambda)| < 2 \prod_{j=0}^{k-1} \eta_j,$$

*where*

(54)

$$D_{\mu,\nu}(\lambda) = \begin{vmatrix} \lambda_\mu - \lambda & \eta_{\mu+1} & 0 & \cdots & 0 & 0 \\ \eta_{\mu+1} & \lambda_{\mu+1} - \lambda & \eta_{\mu+2} & \cdots & 0 & 0 \\ 0 & \eta_{\mu+2} & \lambda_{\mu+2} - \lambda & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_{\nu-1} - \lambda & \eta_\nu \\ 0 & 0 & 0 & \cdots & \eta_\nu & \lambda_\nu - \lambda \end{vmatrix}, \quad \mu \le \nu;$$

*for $\nu < \mu$, we put $D_{\mu,\nu} = 1$.*

Since the spectrum is closed, one would expect that, instead of (53), the spectrum is described by the set of reals $\lambda$ for which

$$|D_{0,k-1}(\lambda) - \eta_0^2 D_{1,k-2}| \le 2 \prod_{j=0}^{k-1} \eta_j. \tag{55}$$

However, the set of $\lambda$'s described by this formula is not the closure of the set described by (53); the reason is that the set in (55) may contain isolated points. The $\lambda$'s that do not belong to the set described in (53) but belong to its closure will, of course, belong to the spectrum of $\mathbf{D}$, but, since there are only finitely many such points, they are accounted for by the possible exceptional elements of the spectrum of $\mathbf{D}$, mentioned in the formulation of the theorem. We do not have an example showing that there are reals $\lambda$ satisfying (55) (but not (53)) that do not belong to the spectrum of $\mathbf{D}$.

We do, however, have an example showing that the spectrum may have additional exceptional elements. These, being isolated points of the spectrum, must, of course, be eigenvalues. We are about to present such an example. In case $k = 2$, formula (53) can be written as

$$|(\lambda_0 - \lambda)(\lambda_1 - \lambda) - \eta_0^2 - \eta_1^2)| < 2\eta_0\eta_1. \tag{56}$$

Consider the case $\eta_0 = 1$, $\eta_1 = 2$, and $\lambda_0 = \lambda_1 = 0$. In this case, the matrix $\mathbf{D}$ has form

$$\mathbf{D} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

As is easily seen, 0 is an eigenvalue of this matrix; the corresponding eigenvector is

$$\left\langle 0, 1, 0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{8}, \cdots \right\rangle^*,$$

where the asterisk indicates the transpose (so that this is a column vector). This eigenvalue is not accounted for by formula (56), that is, it is one of the exceptional eigenvalues mentioned in the above theorem.

*Proof of Theorem* 13. As in the proof of Lemma 12, we will consider the operator $\mathbf{D}$ on the Hardy space $H^2$. Given $f \in H^2$, write it in the form

$$f(z) = \sum_{j=0}^{k} z^j f_j(z^k); \tag{57}$$

$z^j f_j(z^k)$ collects the terms with exponent congruent to $j \pmod{k}$ in the power series representing $f(z)$. As is easily seen, under the canonical isomorphism described in (45), the matrix $\mathbf{D}$ can be represented on $H^2$ as

$$(\mathbf{D}f)(z) = \sum_{j=0}^{k-1} (\eta_j z^{j-1} + \lambda_j z^j + \eta_{j+1} z^{j+1}) f_j(z^k) - \eta_k z^{-1} f_0(0); \tag{58}$$

notice that here $\eta_k = \eta_0$. That is, given an arbitrary complex $\lambda$, we have

(59)

$$\begin{aligned}((\mathbf{D} - \lambda\mathbf{I})f)(z) = {} & (\lambda_0 - \lambda) f_0(z^k) + \eta_1 f_1(z^k) + z^k \eta_0 f_{k-1}(z^k) \\ & + \sum_{j=1}^{k-2} z^j (\eta_j f_{j-1}(z^k) + (\lambda_j - \lambda) f_j(z^j) + \eta_{j+1} f_{j+1}(z^k)) \\ & + z^{k-1}(-z^{-k}\eta_0 f_0(0) + z^{-k}\eta_0 f_0(z^k) \\ & + \eta_{k-1} f_{k-2}(z^k) + (\lambda_{k-1} - \lambda) f_{k-1}(z^k)).\end{aligned}$$

Similarly, as in (57), write

$$g(z) = \sum_{j=0}^{k} z^j g_j(z^k); \tag{60}$$

then, by virtue of (59), the equation

$$(\mathbf{D} - \lambda\mathbf{I}) f = g \tag{61}$$

can be written in matrix form as

$$\mathbf{M}(\lambda, z)\mathbf{f}(z^k) = \mathbf{g}(z^k) + z^{-k}\eta_0 f_0(0)\mathbf{e}_{k-1}, \tag{62}$$

where $\mathbf{e}_j$ is the $k$-dimensional unit column vector with 1 for its $j$-th component, and 0 for the other components, $0 \le j \le k-1$;

$$\mathbf{f} = \langle f_0, f_1, \dots, f_{k-1}\rangle^* \quad \text{and} \quad \mathbf{g} = \langle g_0, g_1, \dots, g_{k-1}\rangle^*; \tag{63}$$

and $\mathbf{M}(\lambda, z)$ is given by

$$\mathbf{M}(\lambda, z) = \begin{pmatrix} \lambda_0 - \lambda & z^2\eta_0 + \eta_1 \\ z^{-2}\eta_0 + \eta_1 & \lambda_1 - \lambda \end{pmatrix} \quad \text{for } k = 2 \tag{64}$$

and

(65)

$$\mathbf{M}(\lambda, z) = \begin{pmatrix} \lambda_0 - \lambda & \eta_1 & 0 & \cdots & 0 & z^k\eta_0 \\ \eta_1 & \lambda_1 - \lambda & \eta_2 & \cdots & 0 & 0 \\ 0 & \eta_2 & \lambda_2 - \lambda & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_{k-2} - \lambda & \eta_{k-1} \\ z^{-k}\eta_0 & 0 & 0 & \cdots & \eta_{k-1} & \lambda_{k-1} - \lambda \end{pmatrix}$$
$$\text{for } k > 2.$$

As in the proof of Lemma 12, $\lambda$ does not belong to the spectrum of $\mathbf{D}$ if and only if equation (62) is uniquely solvable for $f$ (a solution for $\mathbf{f}$ will determine $f$) in $H^2$; note that to see this we used the Open Mapping Theorem. Now equation (62) is certainly solvable, but the solution will have certain singularities, and the question whether or not the singularities inside the closed unit disc can be compensated for by choosing $f_0(0)$ appropriately will determine whether the solution is in $H^2$. Whether the need to compensate for these solutions determines $f_0(0)$ uniquely will determine whether this solution is unique.

The singularities of the solution of the system of equations in (62) arise from the fact that the determinant $\det(\mathbf{M}(\lambda, z))$ of this system has certain zeros. However, given $g_j \in H^2$, if we find a unique solution of (62) with $f_0 \in H^2$, then we can also find $f_j \in H^2$ for $1 \le j \le k-1$ for all but finitely many values of $\lambda$. This is because we can determine these $f_j$'s by substituting $f_0$ into the system of equations given in (62) and deleting the first equation. Since the minor of $\mathbf{M}(\lambda, z)$ obtained by deleting the first row and first column is constant and nonsingular for all but finitely many values of $\lambda$, the solutions so obtained will belong to $H^2$ (notice that the term involving $z^{-k}$ on the right-hand side of the last equation in (62) will not give rise to singularities, since this term will be canceled out by a similar term on the left-hand side). The finitely many exceptional values of $\lambda$ may give rise to eigenvalues of $\mathbf{D}$. These will be among the elements of the spectrum not accounted for by (53) in the theorem to be proved.

As is easily seen, we have

(66)
$$\det(\mathbf{M}(\lambda,z)) = (-1)^{k-1}(z^k+z^{-k})\prod_{j=0}^{k-1}\eta_j - \eta_0^2 D_{1,k-2}(\lambda) + D_{0,k-1}(\lambda);$$

this equation is true for every $k \geq 2$, even though, for $k = 2$, the matrix $\mathbf{M}(\lambda, z)$ has a special form. Let $\lambda$ be fixed. It is clear from (66) that if $\lambda$ is real and (53) holds, then the equation

$$\det(\mathbf{M}(\lambda,z)) = 0 \tag{67}$$

has two roots for $z^k$ (i.e., 2k roots for $z$) on the circumference of the unit circle. On the other hand, if $\lambda$ is real and (55) is not satisfied, then this equation has two roots for $z^k$, one inside and one outside the unit circle (the case of nonreal $\lambda$ is of no interest since the operator $\mathbf{D}$ being self-adjoint implies that all elements of its spectrum are real).

Suppose now that $\lambda$ is such that (55) is not satisfied, and consider the solution for $f_0$ of the system of equations given in (62). This solution can be written as

$$f_0(z) = \det(\mathbf{M}_0(\lambda,z))/\det(\mathbf{M}(\lambda,z)), \tag{68}$$

where $\mathbf{M}_0(\lambda, z)$ is the matrix obtained by replacing the first column of the matrix $\mathbf{M}(\lambda, z)$ by the column vector

$$\langle g_0(z^k), g_1(z^k), \ldots, g_{k-1}(z^k) + z^{-k}\eta_0 f_0(0)\rangle^*. \tag{69}$$

It is easy to see that

$$\begin{aligned}\det(\mathbf{M}_0(\lambda,z)) = f_0(0)\Big((-1)^{k-1}z^{-k}\prod_{j=0}^{k-1}\eta_j - \eta_0^2 D_{1,k-2}(\lambda)\Big)\\ + g_0(z^k)D_{1,k-1} + L(g_1(z^k),\ldots,g_{k-1}(z^k)),\end{aligned} \tag{70}$$

where $L(g_1, \dots, g_{k-1})$ is a linear combination of $g_0, \dots, g_{k-1}$ the coefficients of which are polynomials in $z^k$. Let $\zeta_1^k$ be a root for $z^k$ of the equation in (67) inside the unit circle. Now it is clear that, unless

$$(-1)^{k-1}\zeta_1^{-k}\prod_{j=0}^{k-1}\eta_j - \eta_0^2 D_{1,k-2}(\lambda) = 0, \tag{71}$$

we can choose $f_0(0)$ in a unique way such that $\zeta_1^k$ is a zero for $z^k$ of $\det(\mathbf{M}_0(\lambda, z))$ as well. In this case, $f_0$ defined by (68) belongs to $H^2$. As we mentioned above, in this case we can find $f_1, \dots, f_{k-1}$ in $H^2$ such that equation (62) is satisfied, unless $\lambda$ assumes finitely many exceptional values, showing that $\lambda$ does not belong to the spectrum of $\mathbf{D}$ unless it is one of these exceptional values.

Next, we show that (71) can be satisfied for at most finitely many values of $\lambda$ provided that $\zeta_1$ in this equation is a root of equation (67). Indeed, equations (67) (with $z = \zeta_1$) and (71) imply

$$(-1)^{k-1}\zeta_1^{k}\prod_{j=0}^{k-1}\eta_j + D_{0,k-1}(\lambda) = 0,$$

and this with (71) implies

$$\left(\prod_{j=0}^{k-1}\eta_j\right)^2 = -\eta_0^2 D_{1,k-2}D_{0,k-1}. \tag{72}$$

This is a polynomial equation for $\lambda$ with a nonzero leading coefficient (since $\eta_0 \neq 0$); hence, it is satisfied only for finitely many values of $\lambda$. These values may be among the possible exceptional eigenvalues of $\mathbf{D}$.

Assume now that $\lambda$ is such that (53) is satisfied. We want to prove that $\lambda$ belongs to the spectrum of $\mathbf{D}$. Since the spectrum is closed and the set described by (53) is open, we can disregard a finite number of values of $\lambda$. Hence, we may assume that $D_{1,k-1} \neq 0$. Let $\zeta_1^k$ and $\zeta_2^k$ with $|\zeta_1^k| = |\zeta_2^k| = 1$ be the two roots for $z^k$ of equation (67) ($\zeta_1^k \neq \zeta_2^k$) in view of (53) and (66). In trying to solve equation (62), assume that $g_0$ is a polynomial in $z$ and $g_j \equiv 0$ for $1 \leq j \leq k-1$. It is clear from (70) that it is impossible to choose $f_0(0)$ in such a way that the function

$f_0(z)$ given by (68) belongs to $H^2$ for every choice of $g_0$. Indeed, the expression

$$f_0(0)\biggl((-1)^{k-1}z^{-k}\prod_{j=0}^{k-1}\eta_j - \eta_0^2 D_{1,k-2}(\lambda)\biggr) + g_0(z^k)D_{1,k-1}(\lambda)$$

would have to have zeros at $z^k = \zeta_1^k$ and $z^k = \zeta_2^k$, ($\zeta_1^k \neq \zeta_2^k$, as noted above). It is obviously possible to choose $g_0$ in such a way that this condition will not be fulfilled for any choice of $f_0(0)$ (it is enough to make sure that $g(\zeta_1^k) \neq g(\zeta_2^k)$), showing that equation (62) is not solvable in $H^2$. Therefore, the $\lambda$ in question belongs to the spectrum of $\mathbf{D}$. The proof is complete. □

Similarly to the proof of Theorem 7 presented right after Weyl's Theorem 11, we can use Weyl's theorem to derive the following consequence of Theorem 13.

**Theorem 14.** *Let $k > 1$ be an integer, and let $\alpha$ be a measure on the real line. Assume the recurrence coefficients $a_n = a_n(d\alpha)$ and $b_n = b_n(d\alpha)$ satisfy* (50). *Then the support of $\alpha$ consists of the closure $E$ of the set of reals $\lambda$ satisfying* (53) *plus a bounded countable set $F$. All limit points of the set $F$ belong to $E$.*

The proof is straightforward from what was said above. This result was essentially found by Ya. L. Geronimus [**8**; Theorem III, p. 537], but it was stated in terms of continued fractions. The usual proof of this theorem involves continued fractions and is very different from the proof given above. See also [**9, 10**], [**11**, formulas (VI.5)–(VI.7), pp. 70–71], and [**7**].

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