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PADÉ-TYPE APPROXIMANTS FOR FUNCTIONS OF MARKOV-STIELTJES TYPE

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ABSTRACT. The denominator of Padé approximants of functions of Markov-Stieltjes type are closely connected to orthogonal polynomials which leads to control of the location of the poles and to convergence of the approximants. We investigate to which extent the convergence also holds for Padé-type approximants where the location of the poles is changed.

0. Introduction. Let f be a function of the following type (Markov-Stieltjes type)

$$f(z) = \int_{-1}^{1} \frac{d\alpha(t)}{1+zt} \tag{0.1}$$

where $z \in \mathbf{C}$, the complex plane, and α is a finite positive measure whose support is an infinite subset of $[-1,1]$. Let P_{n-1}/Q_n be the $(n-1,n)$ Padé approximant of f, i.e., P_{n-1} and Q_n, $Q_n \not\equiv 0$, are polynomials of degree at most $n-1$ and n, respectively, satisfying the following interpolation condition at zero

$$(fQ_n - P_{n-1})(z) = O(z^{2n}) \quad \text{as } z \to 0. \tag{0.2}$$

Then $q_n(z) := z^n Q_n(-1/z)$ is the n-th degree orthogonal polynomial for α (see Section 2.1). This means that the zeros of $Q_n(z)$ are simple and located on $]-\infty,-1[\,\cup\,]1,\infty[$, and from this it can be proved that $P_{n-1}/Q_n \to f$ locally uniformly in $\mathbf{C}\backslash(]-\infty,-1]\cup[1,\infty[)$ (Markov's theorem [**8**]) with geometric degree of convergence (Gragg; see, for instance, [**6**]). Furthermore, if α is absolutely continuous and $\alpha'(x) > 0$ almost everywhere in $[-1,1]$, the zeros of $q_n(z)$ are distributed asymptotically according to the *arcsine* distribution, i.e., according to the equilibrium distribution of $[-1,1]$ for the logarithmic potential

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(Erdös-Turan's theorem; see, for instance, [**7**, p. 247]). Generalizations to complex measures α have been treated by Nuttall, Stahl, Al. Magnus and others; see the surveys [**7**, p. 269] and [**9**, p. 42] for discussion and references. The analogue of Markov's theorem for the case when α is supported by a half-axis was treated by Stieltjes and Carleman (see [**6**]).

In the Padé approximant all poles and zeros are free in the sense that they are determined by the interpolation condition (0.2). The problem which we want to discuss here is the effect on the convergence of the approximants of (0.1) by a perturbation of the zeros of Q_n. More precisely, how is the convergence affected if the Padé approximant is replaced by the so-called Padé-type approximant (PTA, Section 1) where we choose the location of some, or all, of the poles of the approximant in advance and determine the rest of the poles and the zeros of the approximant by a modified interpolation condition of the type (0.2)? In Section 2 we treat the case when *all* the poles are preassigned and α in (0.1) is replaced by a general complex measure with compact support. In this case some basic facts (see Theorem 2.2) follow from investigations by Walsh [**14**, Chapter 8] and Bagby [**2**] on interpolation by rational functions with preassigned poles. The fact that we here treat functions of the form (0.1) gives more explicit and detailed results. The method in Section 2 is based on a direct application of Cauchy's integral formula. In Section 3 we treat the case when some of the poles are preassigned. The method here is based on numerical quadrature and is inspired by the corresponding treatment in the Padé approximation case. It is connected to results by Jacobi, Stieltjes and M. Riesz. This paper is an extension of [**5**] which was part of the author's thesis.

1. Padé-type approximants. Let f be any formal power series in one complex variable z, let n and m be nonnegative integers, and let $v(z)$, $v(z) \not\equiv 0$, be an arbitrary polynomial of degree k where $0 \le k \le m$. We choose polynomials $P(z)$ and $w(z)$, $w(z) \not\equiv 0$, of degree at most n and $m-k$, respectively, so that we get the following interpolation condition at zero:

$$f(z)v(z)w(z) - P(z) = O(z^{n+m-k+1}) \tag{1.1}$$

where the right-hand member denotes a power series in z with lowest order term of degree $n+m-k+1$. We define $P/(vw)$ to be the (n,m)

Padé-type approximant (PTA) of f with preassigned poles at the zeros of v.

By (1.1), P/w is the $(n, m-k)$ *Padé approximant* (PA) of fv. Since the PAs exist and are unique we see that the (n, m) PTA exists and is uniquely determined by f, n, m and v. Note that, since P may be zero at a zero of v, the zeros of v need not, as a matter of fact, be true poles of the PTA. We have, in a natural way, three different types of PTAs:

1^0 No pole is preassigned $(k = 0)$; we may then take $v \equiv 1$ and P/w is the classical PA of f.

2^0 Some, but not all, poles are preassigned $(0 < k < m)$; this is the in-between case treated in Section 3.

3^0 All poles are preassigned $(k = m)$; this gives the Walsh-Bagby interpolation problem treated in Section 2.

For a general f it is difficult to get information on the location of the poles of the PA. This has been a main reason to study PTAs. For some further information on PTAs, we refer to Brezinski [**3**].

2. All poles are preassigned. In order to get simpler formulas we formulate, in this section, our problem in a different but equivalent form. We interpolate at infinity instead of at zero as in (1.1) and, instead of functions written in the form (0.1), we assume that f has the form

$$f(z) = \int_E \frac{d\alpha(t)}{z-t} \tag{2.1}$$

where $E \subset \mathbf{C}$ is compact and α is a complex measure supported by E, $\operatorname{supp}\alpha \subset E$. We consider rational functions P_{n-1}/Q_n of type $(n-1, n)$, i.e., P_{n-1} and Q_n, $Q_n \not\equiv 0$, are polynomials of degree at most $n-1$ and n, respectively.

2.1. Let us first briefly treat the PA case. If f is any function analytic and zero at infinity the $(n-1, n)$ PA of f (with interpolation at infinity) is the unique rational function P_{n-1}/Q_n of type $(n-1, n)$ such that

$$(fQ_n - P_{n-1})(z) = O(z^{-n-1}) \quad \text{as } z \to \infty. \tag{2.2}$$

We multiply (2.2) by z^j and integrate over a contour surrounding E and the origin, insert f given by (2.1) and use Fubini's and Cauchy's theorems. This gives

$$\int_E Q_n(t)t^j \, d\alpha(t) = 0, \quad \text{for } 0 \le j \le n-1,$$

i.e., the PA denominator Q_n is the n-th degree orthogonal polynomial for α as discussed in the introduction. From this, Markov's theorem then follows by using basic facts on orthogonal polynomials.

2.2. In the rest of Section 2, let P_{n-1}/Q_n be a rational function of type $(n-1, n)$ where Q_n is any polynomial of degree n chosen in advance and P_{n-1} is determined by maximal interpolation at infinity,

$$(fQ_n - P_{n-1})(z) = O(z^{-1}) \quad \text{as } z \to \infty. \tag{2.3}$$

This works for any f which is analytic and zero at infinity and means that P_{n-1}/Q_n is the $(n-1, n)$ PTA of f with prescribed poles at the zeros of Q_n (and interpolation at infinity). If Γ is any simple closed, positively oriented contour such that f is analytic outside and, on Γ, we obtain by Cauchy's theorems,

$$(fQ_n - P_{n-1})(z) = -\frac{1}{2\pi i}\int_\Gamma Q_n(t)\frac{f(t)}{t-z}\, dt, \quad z \text{ outside } \Gamma,$$

then, by using (2.1), Fubini's and Cauchy's theorems, and dividing by $Q_n(z)$,

$$(f - P_{n-1}/Q_n)(z) = \int_E \frac{Q_n(t)}{Q_n(z)}\frac{d\alpha(t)}{(z-t)}, \qquad z \in \hat{\mathbf{C}}\backslash E, \tag{2.4}$$

where $\hat{\mathbf{C}}$ is the extended complex plane. We let Q_n be monic and denote the zeros (not necessarily distinct) of Q_n by β_{jn}, $1 \le j \le n$, i.e.,

$$Q_n(z) = \prod_{j=1}^{n}(z - \beta_{jn}), \qquad \beta_{jn} \in \mathbf{C}. \tag{2.5}$$

2.3. Let us first use (2.4) to note that we cannot expect local uniform convergence in $\hat{\mathbf{C}}\backslash E$ of the PTAs if the zeros β_{jn} of Q_n may cluster in $\hat{\mathbf{C}}\backslash E$. Let α be the Dirac measure at a point $t_0 \in E$, $t_0 \ne \beta_{jn}$ for all j

and n. By (2.1), $f(z) = 1/(z - t_0)$ and, since $Q_n(t_0) \neq 0$, we see from (2.4) that P_{n-1}/Q_n has a pole at every $\beta_{jn} \in \hat{\mathbf{C}} \backslash E$. We sum up:

Theorem 2.1. *Assume that E is nondenumerable and that the set $\{\beta_{jn},\ 1 \leq j \leq n,\ n = 1, 2, \dots\}$ has a limit point in $\hat{\mathbf{C}} \backslash E$. Then there exists a function f of the form (2.1) such that the $(n-1, n)$ PTA of f with prescribed poles at β_{jn}, $1 \leq j \leq n$, does not converge locally uniformly in $\hat{\mathbf{C}} \backslash E$.*

2.4. Now we introduce the *associated measure* μ_n to the zeros β_{jn}, $1 \leq j \leq n$, of Q_n as the probability measure with point mass $1/n$ at β_{jn}, for each j, $1 \leq j \leq n$. The logarithmic potential of a measure μ is denoted by $u(z; \mu)$, i.e.,

$$u(z; \mu) = \int \log \frac{1}{|z - t|} \, d\mu(t),$$

and the logarithmic capacity of E by $\operatorname{cap} E$. We assume that E is *regular* in the sense that $\operatorname{cap} E > 0$ and that there exists a unique probability measure τ on E, the *equilibrium distribution* of E, such that $u(z; \tau)$ is identically constant on E; equivalently, the unbounded component of $\mathbf{C} \backslash E$ has a classical Green function with pole at infinity.

Theorem 2.2. *Let E be a compact regular set in $\mathbf{C}$ with connected complement and let f be given by (2.1) with a complex measure α. Let P_{n-1}/Q_n be the $(n-1, n)$ PTA of f with poles at the preassigned zeros β_{jn}, $1 \leq j \leq n$, of Q_n, and assume that the set β_{jn}, $1 \leq j \leq n$, $n = 1, 2, \dots$, has no limit point outside E. Let μ_n be the associated measure to β_{jn}, $1 \leq j \leq n$, and let τ be the equilibrium distribution of E. Then we have*

1^0 *If $u(z; \mu_n) \to u(z; \tau)$, as $n \to \infty$, for all $z \in \mathbf{C} \backslash E$, then*

$$\limsup_{n \to \infty} \max_K |f - P_{n-1} Q_n|^{1/n} < 1,$$

for every compact set $K \subset \hat{\mathbf{C}} \backslash E$. In particular, we have locally uniform convergence of P_{n-1}/Q_n to f in $\hat{\mathbf{C}} \backslash E$.

2^0 *If there exists at least one point $z \in \mathbf{C} \backslash E$ where $u(z; \mu_n)$ does not converge to $u(z; \tau)$, then there exists a positive measure α in (2.1) and a point in $\mathbf{C} \backslash E$ where P_{n-1}/Q_n does not converge to f.*

In the main, this theorem follows from Walsh [**14**] and Bagby [**2**], and we give only the key steps in the proof and refer to [**12**, Section 2.2] and [**13**] for details.

Proof. 1^0. The key to the proof is to use (2.4) and the identity

$$\frac{Q_n(t)}{Q_n(z)} = \exp\{-n(u(t;\mu_n) - u(z;\mu_n))\}. \tag{2.6}$$

From the assumption and basic facts on potentials, it follows, for compact sets $K \subset \hat{\mathbf{C}}\backslash E$, that $u(z;\mu_n) \to u(z;\tau)$ uniformly on K and that

$$\max_{z\in K} u(z;\tau) < \log(1/\operatorname{cap} E).$$

Also, for $\varepsilon > 0$ and $n > n(\varepsilon)$,

$$\min_{t\in E} u(t;\mu_n) > \log(1/\operatorname{cap} E) - \varepsilon.$$

These estimates combined with (2.6) and (2.4) give 1^0.

2^0. By choosing a subsequence, if necessary, we may assume that μ_n converges in the weak* sense to a probability measure μ on E and that $u(z;\mu) \not\equiv u(z;\tau)$ on $\mathbf{C}\backslash E$. Potential theory gives (see Section 3 of [**13**]) that

There exists $z_0 \in \mathbf{C}\backslash E$ such that $u(z_0;\mu) > \log(1/\operatorname{cap} E)$, and there exists $E_0 \subset E$ such that $\operatorname{cap} E_0 > 0$ and

$$u(t;\mu) \le \log(1/\operatorname{cap} E) \quad \text{and} \quad \liminf u(t;\mu_n) = u(t;\mu), \quad \text{for } t \in E_0.$$

These properties and (2.6) give

$$\limsup \left|\frac{Q_n(t)}{Q_n(z_0)}\right| = \infty \quad \text{for all } t \in E_0.$$

By choosing, for instance, α as the Dirac measure at a fixed $t \in E_0$, we get divergence at z_0 which proves 2^0. □

2.5. In Theorem 2.2, 2^0 we do not know where in the complement of E that the divergence point z_0 is located and we do not get good information on the class of measures α and poles $\{\beta_{jn}\}$ for which we get divergence. We shall investigate the case $E = [-1, 1]$, α positive, and use the geometry of $[-1, 1]$ and the positivity of α to obtain more precise information. Consequently, we assume that α is a nontrivial positive measure on $[-1, 1]$ and that

$$f(z) = \int_{-1}^{1} \frac{d\alpha(t)}{z - t}. \tag{2.7}$$

Let z_0 be any fixed point satisfying $|z_0 - 1| < 2$ and $\operatorname{Im} z_0 \neq 0$. We consider two disjoint, nonempty closed subintervals I and J of $[-1, 1]$ where I lies to the left of J; typically, the left-hand endpoint of I is -1 and the right-hand endpoint of J is 1. We assume that α is supported by I and that $\beta_{jn} \in J$ for all j and n. Finally, we assume that I and J are chosen so that, for a fixed $\alpha > 1$,

$$\operatorname{dist}(I, y)/|z_0 - y| \geq a > 1, \quad \text{for all } y \in J. \tag{2.8}$$

Here $\operatorname{dist}(I, y)$ is the distance from I to y and, since $|z_0 - 1| < 2$, it is possible to choose I and J in this way. We recall that P_{n-1}/Q_n is the $(n-1, n)$ PTA of f with poles at the preassigned zeros β_{jn} of Q_n. We introduce k_n and $h_n(t)$ so that

$$\frac{Q_n(t)}{Q_n(z_0)} = \prod_{j=1}^{n} \frac{t - \beta_{jn}}{z_0 - \beta_{jn}} = k_n h_n(t),$$

$|k_n| = 1$ and $h_n(t) = |Q_n(t)/Q_n(z_0)|$. Observe that k_n is independent of t, for $t \in I$, and that, by (2.8), $h_n(t) \geq a^n$ for $t \in I$. This, (2.4) and the fact that $\operatorname{Im}(1/(z_0 - t))$ is either positive for all $t \in I$ or negative for all $t \in I$, give

$$\begin{aligned} |(f - P_{n-1}/Q_n)(z_0)| &= \left| \int_{-1}^{1} k_n h_n(t) \frac{d\alpha(t)}{t - z_0} \right| = \left| \int_{-1}^{1} h_n(t) \frac{d\alpha(t)}{t - z_0} \right| \\ &\geq \left| \int_{-1}^{1} h_n(t) \operatorname{Im}\left(\frac{1}{t - z_0} \right) d\alpha(t) \right| \geq a^n \left| \int_{-1}^{1} \operatorname{Im}\left(\frac{1}{t - z_0} \right) d\alpha(t) \right| \to \infty. \end{aligned}$$

Hence, P_{n-1}/Q_n *diverges to infinity at* z_0 *for every positive, nontrivial* α *supported by* I *for all* $\beta_{jn} \in J$, $1 \leq j \leq n$, $n = 1, 2, \ldots$.

By picking z_0 close to the left-hand endpoint of J we realize that we may choose J large by taking I small (in a small neighborhood of -1). As long as (2.8) holds, we get divergence at z_0. In particular, if, for some $\varepsilon > 0$, $J = [-1+\varepsilon, 1]$, there exists α supported by $[-1, 1]$ giving divergence at certain points outside $[-1, 1]$ for all β_{jn} satisfying $\beta_{jn} \in J$, for all j and n.

On the other hand, by picking z_0 close to 1 we may instead choose I large by taking J small. By a variation of this idea we can prove

Theorem 2.3. *Let f be given by* (2.7) *where α is a positive, nontrivial measure on* $[-1, 1]$ *with* $\operatorname{supp}\alpha \neq [-1, 1]$. *Then the* $(n-1, n)$ *PTA P_{n-1}/Q_n of f with poles at the preassigned zeros β_{jn} of Q_n diverges at certain points outside* $[-1, 1]$ *for certain* $\beta_{jn} \in [-1, 1]$, *for all j and n.*

Proof. Pick a small interval J, $J \subset [-1, 1]\backslash \operatorname{supp}\alpha$ and z_0 close to J, $\operatorname{Im} z_0 \neq 0$, so that (2.8) holds with I replaced by $\operatorname{supp}\alpha$. Now $t - y$, $t \in \operatorname{supp}\alpha$, $y \in J$, may be positive or negative. By taking n to be even, we may proceed as above to get divergence at z_0. □

2.6. We have here treated the PTA with interpolation at one point (infinity). It would be natural to investigate the multipoint PTA where we interpolate at different points. A formula analogous to (2.4) holds also in this case (see [**14**, Sections 8.1 and 8.4]) and the convergence now depends on the interplay between the poles $\{\beta_{jn}\}$ and the interpolation points. This new difficulty was analyzed by Walsh and Bagby.

3. Some poles are preassigned. In this section we assume that f is given by (0.1) which means that f is a series of Stieltjes $f(z) = \sum c_j z^j$ where $c_j = (-1)^j \int_{-1}^{1} t^j \, d\alpha(t)$ for some finite positive measure α whose support is an infinite subset of $[-1, 1]$. Let $S_n(\alpha; g) = \sum_{j=1}^{n} \lambda_{jn} g(\alpha_{jn}) \approx \int_{-1}^{1} g \, d\alpha$ be an n-point quadrature formula with simple real nodes α_{jn}. We assume that $\alpha_{jn} \in [-1, 1]$ and that the PTA is defined as in Section 1. We can now formulate the following theorem which describes the connection between quadrature formulas and PTAs to a series of Stieltjes.

Theorem 3.1. *Let P_{n-1}/Q_n be a PTA to a series of Stieltjes f. Let us assume that we have chosen k, $0 \le k \le n$, simple poles β_{jn}, $1 \le j \le k$, $\beta_{jn} \in\ ]-\infty, -1] \cup [1, \infty[$, of P_{n-1}/Q_n in advance. Assume that $S_n(\alpha; g) = \sum_{j=1}^{n} \lambda_{jn} g(\alpha_{jn})$ is exact for polynomials of degree $\le 2n - k - 1$ and that $\beta_{jn} = -1/\alpha_{jn}$, $1 \le j \le k$. Then $P_{n-1}(z)/Q_n(z) = S_n(\alpha; 1/(1+zt))$.*

The proof of this theorem is omitted because it is very similar to the proof of the corresponding result for PAs (the case $k = 0$). We refer to [**6**, Section 4] for details.

It follows from Theorem 3.1 that we can use results and methods from the theory of numerical quadrature to prove convergence results for PTAs. von Sydow [**10**] has obtained estimates of the error committed when using the Gauss-Jacobi quadrature formula to approximate integrals. If we use his method and Theorem 3.1, we can prove

Theorem 3.2. *Let P_{n-1}/Q_n be a PTA to a series of Stieltjes f. Suppose that we have chosen k, $0 \le k \le n$, simple poles β_{jn}, $1 \le j \le k$, $\beta_{jn} \in\ ]-\infty, -1] \cup [1, \infty[$, of P_{n-1}/Q_n in advance. Assume that there exists a quadrature formula $S_n(\alpha; g) = \sum_{j=1}^{n} \lambda_{jn} g(\alpha_{jn})$ with the properties (a) and (b):*

(a) *S_n is exact for polynomials of degree $\le 2n - 1 - k$.*

(b) *$\alpha_{jn} = -1/\beta_{jn}$, $1 \le j \le k$, and $\alpha_{jn} \in [-1, 1]$, $1 \le j \le n$.*

Let $K \subset \mathbf{C} \backslash (]-\infty, -1] \cup [1, \infty[)$ be a compact set. Then there exist constants $R > 1$ and M such that

$$\left| \int_{-1}^{1} \frac{1}{1+zt}\, d\alpha(t) - P_{n-1}(z)/Q_n(z) \right| \le M \left(\alpha([-1,1]) + \sum_{j=1}^{n} |\lambda_{jn}| \right) R^{-2n+k},$$

for all $z \in K$.

Proof. We give only the key steps in the proof and refer to [**10**] for details. We want to estimate the error

$$E_n(g) := \int_{-1}^{1} g(t)\, d\alpha(t) - S_n(\alpha; g).$$

Let C_R be the ellipse with foci at ± 1 and semiaxes $(R \pm R^{-1})/2$, and let ε_R be the bounded component of the complement of C_R. If g is holomorphic in $\bar{\varepsilon}_R$, we can derive an alternative expression for E_n. For this purpose, we use Cauchy's integral formula. This gives

$$E_n(g) = \frac{1}{2\pi i} \int_{C_R} g(z) E_n\left(\frac{1}{z-t}\right) dz.$$

It now follows that

$$|E_n(g)| \le \frac{1}{2\pi} \int_{C_R} \left| E_n\left(\frac{1}{z-t}\right)\right| \cdot |dz| \cdot \sup_{z \in C_R} |g(z)|. \tag{3.1}$$

We now estimate the right-hand side of (3.1). Let R_n be a polynomial of degree $\le n$ and introduce the notation $p_n(g) = \inf \sup_{|x|\le 1} |g(x) - T_n(x)|$ where the infimum is taken over all polynomials T_n of degree $\le n$. We now get

$$\begin{aligned} |E_n(g)| &= \left| \int_{-1}^{1} g(t)\, d\alpha(t) - \int_{-1}^{1} R_{2n-1-k}(t)\, d\alpha(t) \right. \\ &\qquad \left. + S_n(\alpha; R_{2n-1-k}) - S_n(\alpha; g) \right| \\ &\le \left(\alpha([-1,1]) + \sum_{j=1}^{n} |\lambda_{jn}| \right) p_{2n-1-k}(g). \end{aligned} \tag{3.2}$$

To complete the proof we want an estimate for $p_{2n-1-k}((z-t)^{-1})$. But it is known that

$$p_{2n-1-k}((z-t)^{-1}) \le \frac{2|z - \sqrt{z^2-1}|^{2n-k}}{|\sqrt{z^2-1}| \cdot (1 - |z - \sqrt{z^2-1}|^2)} \tag{3.3}$$

(see Achieser [**1**, p. 62] and von Sydow [**10**, p. 61]). If we use (3.1), (3.2) and (3.3) we get

$$|E_n(g)| \le 2\left(\alpha([-1,1]) + \sum_{j=1}^{n} |\lambda_{jn}| \right) \frac{R^{-2n+k}}{1 - R^{-2}} \sup_{z \in C_R} |g(z)|.$$

We now use this estimate with $g(t) = 1/(1+zt)$ and the proof is complete by Theorem 3.1. □

Remark 1. In particular, Theorem 3.2 shows that if $\sum|\lambda_{jn}|=O(R^n)$ for all $R>1$, then P_{n-1}/Q_n converges uniformly to f on compact subsets of $\mathbf{C}\backslash(]-\infty,-1]\cup[1,\infty[)$. The special case when $k=n$ and $\sum|\lambda_{jn}|=O(1)$ was given by Brezinski [**3**, Theorem 1.13].

Remark 2. For PAs (the case $k=0$) the weights λ_{jn} are positive and $\sum\lambda_{jn}=\alpha([-1,1])$. Combined with Theorem 3.1, this leads to Markov's theorem and geometric degree of convergence as mentioned in the introduction.

We now treat the case $k=1$ where one of the poles is chosen in advance. In this case, as in the PA case, we can use results from the theory of numerical quadrature to obtain more detailed results. We have the following theorem.

Theorem 3.3. *Let P_{n-1}/Q_n be a PTA to a series of Stieltjes f. Suppose that we have chosen one simple pole $\beta_{1n}=1$ in advance. Then*

$$\limsup_{n\to\infty}\max_K\left|\int_{-1}^{1}\frac{1}{1+zt}\,d\alpha(t)-P_{n-1}(z)/Q_n(z)\right|^{\frac{1}{n}}<1$$

for every compact set $K\subset\mathbf{C}\backslash(]-\infty,-1]\cup[1,\infty[)$.

Proof. We refer to [**4**, pp. 78–79 and 80–81] or [**11**] for some basic facts which we use below on quadrature formulas. We first want to prove that, for each n, there exists a quadrature formula $S_n(\alpha;g)=\sum_{j=1}^{n}\lambda_{jn}g(\alpha_{jn})$, α_{jn} real, simple, such that S_n is exact for polynomials of degree $\leq 2n-2$, $\alpha_{1n}=-1/\beta_{1n}=-1$ and $\alpha_{jn}\in[-1,1]$ for $1\leq j\leq n$, $n\in\mathbf{Z}^+$. Let $\{R_n\}$ be the unique sequence of monic polynomials, degree $R_n=n$, which are orthogonal with respect to α, that is,

$$\int_{-1}^{1}R_n(t)R_m(t)\,d\alpha(t)=0,\qquad n\neq m.$$

We can now define sets of polynomials

$$\theta_n:=\{P:P=R_n+\beta R_{n-1},\beta\text{ real}\}.$$

Jacobi has proved that a quadrature formula S_n which is exact for polynomials of degree $\leq 2n-2$ exists if and only if the node polynomial

$M_n(z) = \prod_{j=1}^n (z-\alpha_{jn}) \in \theta_n$ and M_n has real, simple zeros. But $R_{n-1}(\alpha_{1n}) = R_{n-1}(-1) \neq 0$ (the zeros of R_{n-1} are located in $]-1,1[$) and we can choose β such that $P(-1) = R_n(-1) + \beta R_{n-1}(-1) = 0$. We see that it is possible to choose α_{jn} with $\alpha_{1n} = -1$ so that $\prod_{j=1}^n (z-\alpha_{jn}) \in \theta_n$. It is also known (M. Riesz) that all polynomials in the sets θ_n have real, simple zeros (see [**11**, p. 482]). It now follows from Jacobi's result that there exists a quadrature formula $S_n(\alpha; g) = \sum_{j=1}^n \lambda_{jn} g(\alpha_{jn})$, $\alpha_{1n} = -1$, which is exact for polynomials of degree $\leq 2n-2$. It remains to prove that $\alpha_{jn} \in [-1,1]$ for $1 \leq j \leq n$. This follows from the fact that at least $n-1$ of the zeros of a polynomial in θ_n are in $]-1,1[$ and $\alpha_{1n} = -1$. To complete the proof, we want an estimate of $\sum_{j=1}^n |\lambda_{jn}|$. It is known (Stieltjes) that $\lambda_{jn} > 0$, $1 \leq j \leq n$, if S_n is exact for polynomials of degree $\leq 2n-2$ (see [**11**, p. 479]) and this means that $\sum_{j=1}^n |\lambda_{jn}| = \sum_{j=1}^n \lambda_{jn} = \int_{-1}^1 d\alpha(t)$. Theorem 3.3 now follows from Theorem 3.2. $\square$

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