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MODEL REDUCTION AND STABILITY OF TWO-DIMENSIONAL RECURSIVE SYSTEMS

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ABSTRACT. Many approaches can be found in the literature for the realization of two-dimensional recursive digital filters using both ordinary [**12, 17, 18**] and branched continued fraction expansions [**3, 4**]. In this paper we introduce a new type of branched continued fraction expansion which has specific advantages when considering the model reduction problem. The form of the new BCF is such that, when the expansion is constructed for the transfer function of a stable system, convergents of the BCF expansion automatically satisfy one part of the Huang stability theorem [**6, 9**]. In Section 1 we shall first briefly review the one-dimensional case, and in Section 2 we shall give the algorithm for the new BCF expansion and indicate how the simplification of the stability test for the "reduced" systems follows in a natural way. We conclude Section 2 with an example.

1. One-dimensional recursive systems. Consider a one-dimensional linear shift-invariant (LSI) recursive system T satisfying a finite difference-equation of the form

$$y_n = \sum_{k=0}^{N} a_k x_{n-k} - \sum_{k=1}^{M} b_k y_{n-k}. \tag{1}$$

Then it is well known [**15**] that the transfer function $H(z)$ of the system is a rational function given by

$$H(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{\sum_{k=0}^{M} b_k z^{-k}}, \qquad b_0 = 1. \tag{2a}$$

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If we let $w = z^{-1}$, also in the sequel of the text, we can write

$$H(z) = H(w^{-1}) = \frac{\sum_{k=0}^{N} a_k w^k}{\sum_{k=0}^{M} b_k w^k} = \frac{A(w)}{B(w)}. \tag{2b}$$

A system is said to be bounded input-bounded output (BIBO) stable if and only if, for any bounded input sequence, the output sequence is bounded:

$$\forall n,\ |x(n)| \le B \quad \Longrightarrow \quad \exists\, B',\ \forall n,\ |y(n)| \le B'.$$

Theorem 1 gives a necessary and sufficient condition for BIBO stability in case one is dealing with systems described by (1).

Theorem 1 [**15**]. *Let T be an LSI recursive system whose transfer function $H(z)$ is given by* (2). *Then the following statements are equivalent*:

(i) *T is BIBO stable.*

(ii) *$B(w) \neq 0$ for $|w| \le 1$.*

In the sequel of the text, when referring to a stable system, we mean that it satisfies the condition for BIBO stability. The next theorem gives a relation between stable LSI recursive systems and continued fractions. First, we introduce the notation $R_m(w,t)$ to denote a Schur continued fraction [**10**]:

$$R_m(w,t) := \gamma_0 + \frac{(1-|\gamma_0|^2)w|}{| \quad \bar{\gamma}_0 w} + \sum_{k=1}^{m-1} \left(\frac{1|}{|\gamma_k} + \frac{(1-|\gamma_k|^2)w|}{| \quad \bar{\gamma}_k w} \right) + \frac{1|}{|t}.$$

If the Schur coefficients γ_k satisfy $|\gamma_k| < 1$, for $k = 0, 1, \dots,$ and if $\gamma_0 \in \mathbf{R}$, then the continued fraction $R_m(w,t)$ is called a positive Schur fraction.

Theorem 2 [**11**]. (i) *Let T be an LSI recursive system whose transfer function $H(z)$ is given by*

$$H(z) = \alpha R_m(z^{-1}, \gamma_m),$$

with $|\gamma_k| < 1$*, for* $k = 0, 1, \ldots, m-1$*,* $|\gamma_m| = 1$ *and* α *an arbitrary complex number. Then* T *is a stable system.*

(ii) *Let* T *be a recursive and stable system, and let* $H(z)$ *denote its transfer function.*

Then one of the following statements holds: Either there exists uniquely a finite sequence $\gamma_0, \ldots, \gamma_m$ *with* $|\gamma_k| < 1$*, for* $k = 0, \ldots, m-1$*,* $|\gamma_m| = 1$ *and a positive constant* β *such that*

$$H(z) = \beta R_m(z^{-1}, \gamma_m), \tag{3}$$

or there exists uniquely an infinite sequence $\{\gamma_k\}$ *with* $|\gamma_k| < 1$*, for* $k = 0, 1, \ldots$ *and a positive constant* β *such that*

$$H(z) = \beta \lim_{k \to \infty} R_k(z^{-1}, 1) \quad \text{for } |z| > 1. \tag{4}$$

For each r *with* $0 < r < 1$*, the convergence is uniform on* $|z| \geq 1/r$*.*

Remark. From the proof given in [**11**], it is easy to verify that we can choose the constant β by

$$\beta = \begin{cases} 1, & \text{if } \max_{|w| \leq 1} |H(w^{-1})| \leq 1, \\ \max_{|w| \leq 1} |H(w^{-1})|, & \text{if } \max_{|w| \leq 1} |H(w^{-1})| > 1. \end{cases} \tag{5}$$

Although Theorem 2 is not of real practical use if one wants to test the stability of the system, it turns out to be useful if one is interested in the model reduction problem defined in [**2**] as: given some information about a rational transfer function (of a high degree), find a system with a lower degree rational transfer function that in some sense approximates the original system. Moreover, the reduced system should as much as possible satisfy the same properties as the original system.

Let us assume from now on that the original system T is an LSI recursive and stable system. Then we know from Theorem 2 that its transfer function $H(z)$ can be written in the form (3) or (4). If we construct the approximate system in such a way that its transfer function $\tilde{H}(z)$ is a modified convergent of (3) or (4), i.e.,

$$\tilde{H}(z) = \beta R_n(z^{-1}, 1), \tag{6}$$

where $n < m$ if $H(z)$ is given by (3), then the following theorem summarizes the properties of the approximate system.

Theorem 3. *Let $H(z)$ be the transfer function of a stable LSI recursive system. If $\tilde{H}(z)$ is given by* (6), *then*:

(i) *$\tilde{H}(z)$ is a rational function of degree $[n/n]$*;

(ii) $H(z) - \tilde{H}(z) = O(z^{-n})$;

(iii) *$\tilde{H}(z)$ is the transfer function of a stable system.*

Proof. Statement (i) can very easily be verified (see also [**10**]). In order to prove the correspondence result (ii), we introduce the notation C_k/D_k, $k = 0, 1, \dots,$ for successive convergents of the continued fraction (4) and $\tilde{C}_k/\tilde{D}_k$, $k = 0, 1, \dots, 2n$, for successive convergents of $\tilde{H}(z)$ given by (6). In this way, $\tilde{H}(z) = \tilde{C}_{2n}/\tilde{D}_{2n}$. It is easy to see that

$$\begin{aligned} C_k &= \tilde{C}_k, \quad D_k = \tilde{D}_k, \qquad k = 0, \dots, 2n-1, \\ C_{2n} &= \gamma_n C_{2n-1} + C_{2n-2}, \\ D_{2n} &= \gamma_n D_{2n-1} + D_{2n-2} \end{aligned}$$

and

$$\begin{aligned} \tilde{C}_{2n} &= C_{2n-1} + C_{2n-2} = (1-\gamma_n) C_{2n-1} + C_{2n}, \\ \tilde{D}_{2n} &= D_{2n-1} + D_{2n-2} = (1-\gamma_n) D_{2n-1} + D_{2n}. \end{aligned}$$

We now have

$$\begin{aligned} H(z) - \tilde{H}(z) & \\ &= \left(H(z) - \frac{C_{2n}}{D_{2n}} \right) - \left(\frac{\tilde{C}_{2n}}{\tilde{D}_{2n}} - \frac{C_{2n}}{D_{2n}} \right) \\ &= H(z) - \frac{C_{2n}}{D_{2n}} - (1-\gamma_n) \frac{C_{2n-1} D_{2n} - C_{2n} D_{2n-1}}{D_{2n} \tilde{D}_{2n}} \\ &= H(z) - \frac{C_{2n}}{D_{2n}} - (1-\gamma_n) \frac{(1-|\gamma_0|^2) \cdots (1-|\gamma_{n-1}|^2) z^{-n}}{D_{2n} \tilde{D}_{2n}}. \end{aligned} \tag{7}$$

From [**10**] we know that the order of correspondence of the even convergents C_{2n}/D_{2n} to $H(z)$ is

$$H(z) - \frac{C_{2n}}{D_{2n}} = O(z^{-(n+1)}), \tag{8}$$

while it is easily verified that

$$D_{2n}\tilde{D}_{2n} = 1 + \cdots + \bar{\gamma}_0^2 \gamma_n z^{-2n}.$$

This, together with (7) and (8), proves statement (ii). For part (ii) of the theorem, we point out that, since $H(z)$ is the transfer function of a stable system, the coefficients γ_k in (6) satisfy $|\gamma_k| < 1$ for $k = 0, \ldots, n-1$. Statement (iii) now immediately follows from Theorem 2. □

The above theorem states that if the original system is stable, choosing the approximate transfer function $\tilde{H}(z)$ to be a modified convergent of the Schur continued fraction expansion of the original transfer function, guarantees that the approximate system is stable. However, the degree of correspondence to the original transfer function is lower than the degree one would obtain by constructing a Padé approximant with the same numerator and denominator degree for $H(z)$. The drawback of using Padé approximants is that there is no guarantee that the constructed Padé approximant will realize a stable system. The way to compute the Schur continued fraction for $H(z)$, and, hence, the approximate transfer function $\tilde{H}(z)$, depends on the information given. If the rational transfer function $H(z)$ is given explicitly, then the Schur coefficients γ_k can be computed by means of the algorithm as given in [**10**]:

$$\begin{aligned} f_0(w) &= \frac{H(w^{-1})}{\beta}, \\ f_{k+1}(w) &= \frac{1}{w}\frac{f_k(w) - \gamma_k}{1 - \bar{\gamma}_k f_k(w)}, \quad \gamma_k = f_k(0),\ k \geq 0. \end{aligned}$$

If, on the other hand, $H(w^{-1})$ is given by its Taylor series, another way to compute the coefficients γ_k is to start a Viscovatov-type algorithm. Let

$$\frac{H(w^{-1})}{\beta} = f(w) = \sum_{i=0}^{\infty} c_i w^i,$$

and put

$$\begin{aligned} \gamma_0 &= c_0, \\ f_0(w) &= f(w) - \gamma_0, \\ f_1(w) &= (1 - |\gamma_0|^2)w - \bar{\gamma}_0 w f_0(w). \end{aligned} \tag{9a}$$

In this way, we have

$$f_0(w) = w\sum_{i=0}^{\infty} c_i^{(0)} w^i,$$
$$f_1(w) = w\sum_{i=0}^{\infty} c_i^{(1)} w^i,$$

with

$$\begin{aligned} c_i^{(0)} &= c_{i+1}, \qquad i \geq 0,\\ c_0^{(1)} &= 1-|\gamma_0|^2,\\ c_i^{(1)} &= -\bar{\gamma}_0 c_{i-1}^{(0)}, \qquad i\geq 1. \end{aligned} \tag{9b}$$

In a similar way, choose, for $k \geq 1$,

$$\begin{aligned} \gamma_k &= \frac{c_0^{(2k-2)}}{c_0^{(2k-1)}},\\ f_{2k}(w) &= f_{2k-2}(w) - \gamma_k f_{2k-1}(w),\\ f_{2k+1}(w) &= (1-|\gamma_k|^2) w f_{2k-1}(w) - \bar{\gamma}_k w f_{2k}(w). \end{aligned} \tag{9c}$$

Then

$$f_{2k}(w) = w^{k+1}\sum_{i=0}^{\infty} c_i^{(2k)} w^i,$$
$$f_{2k+1}(w) = w^{k+1}\sum_{i=0}^{\infty} c_i^{(2k+1)} w^i,$$

with

$$\begin{aligned} c_i^{(2k)} &= c_{i+1}^{(2k-2)} - \gamma_k c_{i+1}^{(2k-1)}, \qquad i\geq 0,\\ c_0^{(2k+1)} &= (1-|\gamma_k|^2) c_0^{(2k-1)},\\ c_i^{(2k+1)} &= (1-|\gamma_k|^2) c_i^{(2k-1)} - \bar{\gamma}_k c_{i-1}^{(2k)}, \qquad i\geq 1. \end{aligned} \tag{9d}$$

Finally we obtain, if $c_0^{(2k-1)} \neq 0$ in (9c), the Schur continued fraction of the form

$$\gamma_0 + \frac{(1-|\gamma_0|^2)w|}{|\quad \bar{\gamma}_0 w} + \sum_{k=1}^{\infty}\left(\frac{1|}{|\gamma_k} + \frac{(1-|\gamma_k|^2)w|}{|\quad \bar{\gamma}_k w}\right). \tag{10}$$

Denoting the successive convergents of the continued fraction (10) by C_k/D_k, it is easy to see that [**13**]

$$\frac{H(w^{-1})}{\beta}D_k - C_k = f(w)D_k - C_k = (-1)^k f_k(w).$$

It is easily verified (see also [**10**]) that, for $k = 0, 1, \ldots,$

$$\begin{aligned} D_{2k} &= 1 + \cdots + \bar{\gamma}_0 \gamma_k w^k, \\ D_{2k+1} &= \bar{\gamma}_k w + \cdots + \bar{\gamma}_0 w^{k+1}. \end{aligned}$$

Hence, (10) corresponds to $H(w^{-1})/\beta$ in the sense that

$$\frac{H(w^{-1})}{\beta} - \frac{C_{2k}}{D_{2k}} = O(w^{k+1}),$$

and, when $|\gamma_k| \neq 0$ for $k \geq 0$,

$$\frac{H(w^{-1})}{\beta} - \frac{C_{2k+1}}{D_{2k+1}} = O(w^k).$$

Using this correspondence result it is also easy to prove that if (10) exists, it is unique. In order to see that the formulas (9) can always be applied if $H(z)$ is the transfer function of a stable system, it is sufficient to note from (9) that

$$\begin{aligned} c_0^{(1)} &= 1 - |\gamma_0|^2, \\ c_0^{(2k+1)} &= (1 - |\gamma_k|^2) c_0^{(2k-1)} \end{aligned}$$

and to make use of the results of Theorem 2: either $\exists\, m$ with $|\gamma_m| = 1$, in which case the Schur fraction for $H(z)$ is finite and there is no need to compute γ_{m+1} from (9c), or, $\forall k$, $|\gamma_k| < 1$, and, hence, $c_0^{(2k-1)} \neq 0$ for all k. It is the Viscovatov-type algorithm (9) that we will generalize to the two-dimensional case.

2. Two-dimensional systems. As in the one-dimensional case, we will consider two-dimensional first-quadrant LSI recursive systems satisfying a finite difference equation of the form

$$
\begin{aligned}
y(n_1,n_2) = &\sum_{\substack{(k_1,k_2)\in N\\ N\subset \mathbf{N}^2}} a(k_1,k_2)x(n_1-k_1,n_2-k_2)\\
&- \sum_{\substack{(k_1,k_2)\in M\\ M\subset \mathbf{N}^2\setminus\{(0,0)\}}} b(k_1,k_2)y(n_1-k_1,n_2-k_2).
\end{aligned}
\tag{11}
$$

The transfer function of the system (11) is then given by [**7**]:

$$
\begin{aligned}
H(z_1,z_2) &= \frac{\sum_{(k_1,k_2)\in N} a(k_1,k_2)z_1^{-k_1}z_2^{-k_2}}{1+\sum_{(k_1,k_2)\in M} b(k_1,k_2)z_1^{-k_1}z_2^{-k_2}}\\
&= \frac{\sum_{(k_1,k_2)\in N} a(k_1,k_2)z_1^{-k_1}z_2^{-k_2}}{\sum_{(k_1,k_2)\in M\cup\{(0,0)\}} b(k_1,k_2)z_1^{-k_1}z_2^{-k_2}}.
\end{aligned}
$$

where we have set $b(0,0) = 1$. Using the notation $w_1 = z_1^{-1}$ and $w_2 = z_2^{-1}$, one can write

$$
\begin{aligned}
H(z_1,z_2) &= H(w_1^{-1},w_2^{-1})\\
&= \frac{\sum_{(k_1,k_2)\in N} a(k_1,k_2)w_1^{k_1}w_2^{k_2}}{\sum_{(k_1,k_2)\in M\cup\{(0,0)\}} b(k_1,k_2)w_1^{k_1}w_2^{k_2}}\\
&= \frac{A(w_1,w_2)}{B(w_1,w_2)}.
\end{aligned}
\tag{12}
$$

A two-dimensional system is said to be BIBO stable if the output signal corresponding to a bounded input signal is bounded. As in the one-dimensional case, this definition of stability can be reformulated in terms of the transfer function of the system. The following theorem summarizes some of the existing results.

Theorem 4 [**7, 14**]. *Let T be a two-dimensional first-quadrant LSI system with a rational transfer function given by* (12) *and having no nonessential singularities of the second kind on the unit bicircle. Then the system is stable if and only if*

(i) $B(w_1,w_2)\neq 0$ *for* $|w_1|\le 1$, $|w_2|\le 1$ *if and only if* (ii)(a) $B(w_1, w_2) \neq 0$ *for* $|w_1| \leq 1$, $|w_2| = 1$

(b) $B(w_1, w_2) \neq 0$ *for* $|w_1| = 1$, $|w_2| \leq 1$ *if and only if*

(iii)(a) $B(w_1, w_2) \neq 0$ *for* $|w_1| = 1$, $|w_2| \leq 1$,

(b) $B(w_1, 0) \neq 0$ *for* $|w_1| \leq 1$ *if and only if*

(iv)(a) $B(w_1, w_2) \neq 0$ *for* $|w_1| \leq 1$, $|w_2| = 1$,

(b) $B(0, w_2) \neq 0$ *for* $|w_2| \leq 1$.

Proof. For the proof of the theorem, we refer to [**6, 9, 16**]. □

We remark that conditions (iii–iv) consist on one hand of a one-dimensional condition ((iii)(b),(iv)(b)), while the other part is two-dimensional in nature. In [**1, 8**] it was shown how the two-dimensional condition (iii)(a) (or (iv)(a)) can be tested, while condition (iii)(b) (respectively, (iv)(b)) states that the projected system with transfer function $H_1(w^{-1}) = A(w,0)/B(w,0)$ (respectively, $H_2(w^{-1}) = A(0,w)/B(0,w)$) should be stable. With this theorem and the results of Section 1 in mind, we propose to construct a BCF expansion for a bivariate function $f(w_1, w_2)$ as follows. In view of simplifying conditions (iii), the proposed BCF will be of the form

$$\begin{aligned} &\left(\gamma_0 + \sum_{i=1}^{\infty} \frac{\delta_{i0}^{(0)} w_2 \,|}{|\quad 1}\right) + \frac{(1-|\gamma_0|^2) w_1 |}{|\quad \bar{\gamma}_0 w_1} \\ &\quad + \sum_{k=1}^{\infty} \left(\frac{1 \quad\;\; |}{\left| \gamma_k + \sum_{i=1}^{\infty} \frac{\delta_{i0}^{(k)} w_2 |}{| \quad 1} \right.} + \frac{(1-|\gamma_k|^2) w_1 \Big|}{\Big| \quad \bar{\gamma}_k w_1} \right) \end{aligned} \tag{13a}$$

such that, after projection ($w_2 = 0$), the BCF reduces to a Schur fraction in w_1. When considering conditions (iv) the roles of w_1 and

w_2 are interchanged and the BCF will be of the form

$$\left(\gamma_0 + \sum_{i=1}^{\infty} \frac{\delta_{i0}^{(0)} w_1 \,|}{|\quad 1}\right) + \frac{(1-|\gamma_0|^2)w_2|}{|\quad \bar{\gamma}_0 w_2} + \sum_{k=1}^{\infty} \left(\frac{1 \quad |}{\left| \gamma_k + \sum_{i=1}^{\infty} \frac{\delta_{i0}^{(k)} w_1 |}{| \quad 1} \right.} + \frac{(1-|\gamma_k|^2)w_2 \big|}{\big| \quad \bar{\gamma}_k w_2} \right). \tag{13b}$$

In both BCF forms the even partial denominators are themselves continued fractions in w_2, respectively w_1. In the sequel of the text, we shall restrict ourselves to the form (13a). Results for the other form can be obtained in a completely analogous way. The coefficients in the BCF (13a) can be computed from the Taylor series expansion for $f(w_1, w_2)$ as follows. Let

$$\begin{aligned}
f(w_1, w_2) &= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{ij} w_2^j \right) w_1^i, \\
\gamma_0 &= c_{00}, \\
h_0(w_2) &= \sum_{j=0}^{\infty} c_{0j} w_2^j = \gamma_0 + \sum_{j=1}^{\infty} c_{0j} w_2^j, \\
f_0(w_1, w_2) &= f(w_1, w_2) - h_0(w_2), \\
f_1(w_1, w_2) &= (1 - |\gamma_0|^2) w_1 - \bar{\gamma}_0 w_1 f_0(w_1, w_2).
\end{aligned} \tag{14a}$$

In this way

$$\begin{aligned}
f_0(w_1, w_2) &= w_1 \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{ij}^{(0)} w_2^j \right) w_1^i, \\
f_1(w_1, w_2) &= w_1 \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{ij}^{(1)} w_2^j \right) w_1^i,
\end{aligned}$$

with

$$
\begin{aligned}
c_{ij}^{(0)} &= c_{i+1,j}, \qquad i \geq 0,\ j \geq 0,\\
c_{00}^{(1)} &= 1 - |\gamma_0|^2,\\
c_{0j}^{(1)} &= 0, \qquad j \geq 1,\\
c_{ij}^{(1)} &= -\bar{\gamma}_0 c_{i-1,j}^{(0)}, \qquad i \geq 1,\ j \geq 0.
\end{aligned} \tag{14b}
$$

In a similar way choose, for $k \geq 1$,

$$
\begin{aligned}
h_k(w_2) &= \sum_{j=0}^{\infty} d_j^{(k)} w_2^j = \gamma_k + \sum_{j=1}^{\infty} d_j^{(k)} w_2^j,\\
f_{2k}(w_1, w_2) &= f_{2k-2}(w_1, w_2) - h_k(w_2) f_{2k-1}(w_1, w_2),\\
f_{2k+1}(w_1, w_2) &= (1 - |\gamma_k|^2) w_1 f_{2k-1}(w_1, w_2) - \bar{\gamma}_k w_1 f_{2k}(w_1, w_2),
\end{aligned}
$$

where the $d_j^{(k)}$ are obtained by equating coefficients of equal powers of w_2 in

$$
\sum_{j=0}^{\infty} c_{0j}^{(2k-2)} w_2^j - \sum_{j=0}^{\infty} d_j^{(k)} w_2^j \sum_{j=0}^{\infty} c_{0j}^{(2k-1)} w_2^j = 0
$$

or, explicitly,

$$
\begin{aligned}
d_0^{(k)} &= \gamma_k = \frac{c_{00}^{(2k-2)}}{c_{00}^{(2k-1)}},\\
d_j^{(k)} &= \frac{1}{c_{00}^{(2k-1)}} \left(c_{0j}^{(2k-2)} - \sum_{i=0}^{j-1} c_{0,j-i}^{(2k-1)} d_i^{(k)} \right), \quad j > 0.
\end{aligned} \tag{14c}
$$

Then

$$
\begin{aligned}
f_{2k}(w_1, w_2) &= w_1^{k+1} \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{ij}^{(2k)} w_2^j \right) w_1^i,\\
f_{2k+1}(w_1, w_2) &= w_1^{k+1} \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{ij}^{(2k+1)} w_2^j \right) w_1^i,
\end{aligned}
$$

with

$$
\begin{aligned}
c_{ij}^{(2k)} &= c_{i+1,j}^{(2k-2)} - \sum_{l=0}^{j} d_l^{(k)} c_{i+1,j-l}^{(2k-1)}, \qquad i \geq 0,\ j \geq 0,\\
c_{0j}^{(2k+1)} &= (1 - |\gamma_k|^2) c_{0j}^{(2k-1)}, \qquad j \geq 0,\\
c_{ij}^{(2k+1)} &= (1 - |\gamma_k|^2) c_{ij}^{(2k-1)} - \bar{\gamma}_k c_{i-1,j}^{(2k)}, \qquad i \geq 1,\ j \geq 0.
\end{aligned} \tag{14d}
$$

For uniformity of notation we set $c_{0j} = d_j^{(0)}$ in $h_0(w_2)$ so that, for each $k \geq 0$, the coefficients $d_j^{(k)}$ are associated with $h_k(w_2)$ and always $d_0^{(k)} = \gamma_k$.

We remark that if we set $w_2 = 0$ in algorithm (14) it reduces to the one-dimensional algorithm (9) applied to $f(w_1, 0)$ and the BCF (13a) reduces to the Schur continued fraction (10). Hence, the coefficients γ_k in (14) are equal to the Schur coefficients γ_k for $f(w_1, 0)$.

Assuming that all $c_{00}^{(2k-1)} \neq 0$ in (14c), we can use the formulas (14) to construct a BCF of the form

$$h_0(w_2) + \frac{(1-|\gamma_0|^2)w_1|}{|\quad \bar{\gamma}_0 w_1} + \sum_{k=1}^{\infty}\left(\frac{1 \quad|}{|h_k(w_2)} + \frac{(1-|\gamma_k|^2)w_1|}{|\quad \bar{\gamma}_k w_1}\right). \tag{15}$$

If we denote by C_k/D_k the k^{th} convergent of (15), where the even partial denominators are the infinite expressions $h_k(w_2)$, then, in analogy with the univariate case,

$$f(w_1, w_2) - \frac{C_{2k}}{D_{2k}} = \frac{f_{2k}(w_1, w_2)}{D_{2k}} = O(w_1^{k+1} w_2^l, l \geq 0), \tag{16}$$

where $O(w_1^{k+1} w_2^l, l \geq 0)$ means that the only terms occurring are of the form $w_1^{k+1+i} w_2^j$ with $i \geq 0$ and $j \geq 0$. We shall now show that if the BCF (15) for $f(w_1, w_2)$ exists, it is unique. Assume we can write

$$\begin{aligned} f(w_1, w_2) = h_0(w_2) &+ \frac{(1-|\gamma_0|^2)w_1|}{|\quad \bar{\gamma}_0 w_2} \\ &+ \sum_{k=1}^{\infty}\left(\frac{1 \quad|}{|h_k(w_2)} + \frac{(1-|\gamma_k|^2)w_1|}{|\quad \bar{\gamma}_k w_1}\right) \end{aligned} \tag{17a}$$

and

$$\begin{aligned} f(w_1, w_2) = g_0(w_2) &+ \frac{(1-|\eta_0|^2)w_1|}{|\quad \bar{\eta}_0 w_1} \\ &+ \sum_{k=1}^{\infty}\left(\frac{1 \quad|}{|g_k(w_2)} + \frac{(1-|\eta_k|^2)w_1|}{|\quad \bar{\eta}_k w_1}\right). \end{aligned} \tag{17b}$$

We use the notation $\tilde{C}_k/\tilde{D}_k$ for the k^{th} convergent of (17b), where the even partial denominators are the infinite expressions $g_k(w_2)$. Setting $w_1 = 0$ in (17a)–(17b) it follows immediately that $g_0(w_2) = h_0(w_2)$. Now, assuming that $h_k(w_2) = g_k(w_2)$, and hence also $\gamma_k = \eta_k$ for $k = 0, \ldots, m-1$, we shall show that this also holds for $k = m$. By induction,

$$\begin{aligned}\frac{C_{2m}}{D_{2m}} - \frac{\tilde{C}_{2m}}{\tilde{D}_{2m}} &= (h_m(w_2) - g_m(w_2))\frac{(C_{2m-1}D_{2m-2} - C_{2m-2}D_{2m-1})}{D_{2m}\tilde{D}_{2m}} \\ &= (h_m(w_2) - g_m(w_2))\frac{(1-|\gamma_0|^2)\cdots(1-|\gamma_{m-1}|^2)w_1^m}{D_{2m}\tilde{D}_{2m}}.\end{aligned}$$

On the other hand, we know from (16) that

$$\frac{C_{2m}}{D_{2m}} - \frac{\tilde{C}_{2m}}{\tilde{D}_{2m}} = \frac{O(w_1^{m+1}w_2^l, l \geq 0)}{D_{2m}\tilde{D}_{2m}}.$$

This proves the uniqueness of the BCF (15). We still need to indicate how the coefficients $\delta_{j0}^{(k)}$ in (13a) can be computed from the coefficients $d_j^{(k)}$ in the Taylor series of $h_k(w_2)$. This can be done by means of the well-known univariate Viscovatov scheme [**5**]:

$$\begin{aligned}&\delta_{00}^{(k)} = 1, \\ &\delta_{0i}^{(k)} = 0, \qquad i \geq 1, \\ &\delta_{1i}^{(k)} = d_{i+1}^{(k)}, \qquad i \geq 0, \\ &\delta_{ji}^{(k)} = \delta_{j-2,i+1}^{(k)} - \frac{\delta_{j-2,0}^{(k)}}{\delta_{j-1,0}^{(k)}}\delta_{j-1,i+1}^{(k)}, \qquad i \geq 0,\ j \geq 2.\end{aligned} \tag{18}$$

From (14) and the Viscovatov algorithm (18) it is clear that a sufficient condition for the existence of the BCF (13a) is that, for $k = 0, 1, \ldots$,

$$\begin{aligned}&c_{00}^{(2k+1)} \neq 0, \\ &\delta_{j0}^{(k)} \neq 0, \qquad j = 1, 2, \ldots.\end{aligned}$$

We now go back to the model reduction problem. Let $H(z_1, z_2)$ be the transfer function of a stable system for which we want to find a reduced model. We shall indicate how modified convergents of the BCF (13a) for $H(z_1, z_2)$ all satisfy condition (iii)(b) of the stability Theorem 4. Hence, checking the stability of two-dimensional systems represented by modified convergents of (13a) is reduced to checking condition (iii)(a), and this can be done as described in [**1, 8**]. We shall also indicate the degree of correspondence of the modified convergents of (13a) to $H(z_1, z_2)$.

Let $H(z_1, z_2)$, given by (12), represent a stable digital filter. As before, we assume that $H(z_1, z_2)$ has no nonessential singularities of the second kind on the unit bicircle. As in the univariate case, we shall construct a BCF (13a) for $H(z_1, z_2)/\beta$ instead of for $H(z_1, z_2)$ itself, where it will become clear in a moment how the positive constant β should be chosen. In order to construct the BCF (13a) for $H(z_1, z_2)/\beta$, we first compute the Taylor series expansion

$$\frac{H(z_1, z_2)}{\beta} = \frac{H(w_1^{-1}, w_2^{-1})}{\beta} = \frac{1}{\beta}\frac{A(w_1, w_2)}{B(w_1, w_2)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} w_1^i w_2^j$$

and then apply the Viscovatov-type algorithms (14) and (18). We have mentioned above that the coefficients γ_k in (14) when applied to

$$\frac{1}{\beta}\frac{A(w_1, w_2)}{B(w_1, w_2)}$$

are equal to the Schur coefficients for $A(w_1, 0)/(\beta B(w_1, 0))$. Now, since $H(z_1, z_2)$ represents a stable two-dimensional system, we know from Theorem 4 that $A(w_1, 0)/B(w_1, 0)$ is the transfer function of a stable one-dimensional system. Let us choose β according to (5) by

$$(19) \qquad B = \begin{cases} 1, & \text{if } \max_{|w_1|\le 1} \left|\frac{A(w_1,0)}{B(w_1,0)}\right| \le 1, \\ \max_{|w_1|\le 1} \left|\frac{A(w_1,0)}{B(w_1,0)}\right|, & \text{if } \max_{|w_1|\le 1} \left|\frac{A(w_1,0)}{B(w_1,0)}\right| > 1. \end{cases}$$

Then, using Theorem 2 we know that the Schur coefficients γ_k for $A(w_1, 0)/(\beta B(w_1, 0))$ either satisfy $\exists\, l$ with $|\gamma_k| < 1$, for $k = 0, \dots, l-1$ and $|\gamma_l| = 1$, or, $\forall k$, $|\gamma_k| < 1$. In this last case we set $l = \infty$. In any case, we have $|\gamma_k| < 1$ for $k < l$. Hence, it is now clear from formulas (14b)–(14d) that we can apply the Viscovatov-type algorithm (14) to

$A(w_1, w_2)/(\beta B(w_1, w_2))$, with β given by (19), to construct

$$R_{n,m}(w_1, w_2, t) = \left(\gamma_0 + \sum_{i=1}^{m} \frac{\delta_{i0}^{(0)} w_2 \,|}{|\quad 1}\right) + \frac{(1-|\gamma_0|^2)w_1|}{|\quad \bar{\gamma}_0 w_1} + \sum_{k=1}^{n-1}\left(\frac{1 \qquad |}{\left|\gamma_k + \sum_{i=1}^{m} \frac{\delta_{i0}^{(k)} w_2}{|\quad 1}\right.} + \frac{(1-|\gamma_k|^2)w_1\Big|}{\Big|\quad \bar{\gamma}_k w_1}\right) + \frac{1|}{|t},$$

where, clearly, $n \leq l$ and where we have assumed that the one-dimensional Viscovatov algorithm can be applied to each of the functions $h_k(w_2)$, for $k = 0, \dots, n-1$. The rational functions $R_{n,m}(w_1, w_2, t)$ are modified convergents of the BCF expansion (13a) for $H(w_1^{-1}, w_2^{-1})/\beta$. If we choose the reduced system to have a transfer function $\tilde{H}(z_1, z_2)$ given by

$$\tilde{H}(z_1, z_2) = \beta R_{n,m}(z_1^{-1}, z_2^{-1}, 1), \tag{20}$$

then the following holds. For simplicity, but without loss of generality, we set $\beta = 1$ in the sequel of the text.

Theorem 5. *Let $H(z_1, z_2)$ represent a stable two-dimensional first-quadrant LSI recursive system having no nonessential singularities of the second kind on the unit bicircle. The following statements hold for*

$$\tilde{H}(z_1, z_2) = R_{n,m}(z_1^{-1}, z_2^{-1}, 1).$$

(i) *$\tilde{H}(z_1, z_2)$ is a rational function of the form*

$$\tilde{H}(z_1, z_2) = \tilde{H}(w_1^{-1}, w_2^{-1}) = \frac{\tilde{A}(w_1, w_2)}{\tilde{B}(w_1, w_2)},$$

where

$$\tilde{A}(w_1, w_2) = \sum_{i=0}^{n} \sum_{j=0}^{n\lfloor \frac{m+1}{2} \rfloor} \tilde{a}(i,j) w_1^i w_2^j,$$

$$\tilde{B}(w_1, w_2) = \sum_{i=0}^{n} \sum_{j=0}^{(n-1)\lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor} \tilde{b}(i,j) w_1^i w_2^j,$$

with $\lfloor \cdot \rfloor$ denoting the integer part of its argument.

(ii) $H(z_1,z_2)-\tilde{H}(z_1,z_2)=O(z_1^{-n}z_2^{-j},z_1^{-j}z_2^{-(m+1)},j\geq 0)$.

(iii) *Given that* $\tilde{H}(z_1,z_2)$ *has no nonessential singularities of the second kind on the unit bicircle,* $\tilde{H}(z_1,z_2)$ *is the transfer function of a stable system if and only if*

$$\tilde{B}(w_1,w_2)\neq 0,\ \text{for } |w_1|=1,\ |w_2|\leq 1.$$

Proof. The proof of (i) is by induction on n. For $n=1$, it is clear that

$$\left(\gamma_0+\sum_{i=1}^{m}\frac{\delta_{i0}^{(0)}w_2\,|}{|\quad 1}\right)+\frac{(1-|\gamma_0|^2)w_1|}{|\quad \bar{\gamma}_0 w_1}+\frac{1|}{|1}=\frac{\sum_{i=0}^{1}\sum_{j=0}^{\lfloor\frac{m+1}{2}\rfloor}\tilde{a}(i,j)w_1^iw_2^j}{\sum_{i=0}^{1}\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\tilde{b}(i,j)w_1^iw_2^j}.$$

Now assume that

$$\begin{aligned}\left(\gamma_1+\sum_{i=1}^{m}\frac{\delta_{i0}^{(1)}w_2\,|}{|\quad 1}\right)&+\frac{(1-|\gamma_1|^2)w_1|}{|\quad \bar{\gamma}_1 w_1}\\ &+\sum_{k=2}^{n-2}\left(\frac{1\qquad|}{\left|\gamma_k+\sum_{i=1}^{m}\frac{\delta_{i0}^{(k)}w_2|}{|\quad 1}\right.}+\frac{(1-|\gamma_k|^2)w_1|}{|\quad \bar{\gamma}_k w_1}\right)+\frac{1|}{|1}\\ &=\frac{\sum_{i=0}^{n-1}\sum_{j=0}^{(n-1)\lfloor\frac{m+1}{2}\rfloor}c(i,j)w_1^iw_2^j}{\sum_{i=0}^{n-1}\sum_{j=0}^{(n-2)\lfloor\frac{m+1}{2}\rfloor+\lfloor\frac{m}{2}\rfloor}d(i,j)w_1^iw_2^j}.\end{aligned}$$

Then, clearly,

$$\begin{aligned}
\tilde{H}(w_1^{-1}, w_2^{-1}) &= \left(\gamma_0 + \sum_{i=1}^{m} \frac{\delta_{i0}^{(0)} w_2|}{| \quad 1}\right) + \frac{(1-|\gamma_0|^2)w_1|}{| \quad \bar{\gamma}_0 w_1} \\
&\quad + \frac{\displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{(n-2)\lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor} d(i,j) w_1^i w_2^j}{\displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{(n-1)\lfloor \frac{m+1}{2} \rfloor} c(i,j) w_1^i w_2^j} \\
&= \frac{\displaystyle\sum_{i=0}^{n} \sum_{j=0}^{n\lfloor \frac{m+1}{2} \rfloor} \tilde{a}(i,j) w_1^i w_2^j}{\displaystyle\sum_{i=0}^{n} \sum_{j=0}^{(n-1)\lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor} \tilde{b}(i,j) w_1^i w_2^j}
\end{aligned}$$

which proves statement (i). From (16) it is easy to see that

$$\begin{aligned}
H(z_1, z_2) - R_{n,m}\left(z_1^{-1}, z_2^{-1}, \gamma_n + \sum_{i=1}^{m} \frac{\delta_{i0}^{(n)} w_2|}{| \quad 1}\right) \\
= O(z_1^{-(n+1)} z_2^{-j}, z_1^{-j} z_2^{-(m+1)}, j \geq 0).
\end{aligned}$$

Using this result, the proof of the correspondence result (ii) is completely analogous to the one given in Theorem 3. To prove (iii) it is sufficient to show, according to Theorem 4, that $\tilde{A}(w_1, 0)/\tilde{B}(w_1, 0)$ represents a stable one-dimensional system. We have

$$\frac{\tilde{A}(w_1, 0)}{\tilde{B}(w_1, 0)} = \gamma_0 + \frac{(1-|\gamma_0|^2 w_1|}{| \quad \bar{\gamma}_0 w_1} + \sum_{k=1}^{n-1} \left(\frac{1|}{|\gamma_k} + \frac{(1-|\gamma_k|^2)w_1|}{| \quad \bar{\gamma}_k w_1}\right) + \frac{1|}{|1}.$$

We have already pointed out that, since $H(z_1, z_2)$ represents a stable two-dimensional system, $|\gamma_k| < 1$ for $k = 0, \ldots, l-1$. This, together with Theorem 2, guarantees the stability of $\tilde{A}(w_1, 0)/\tilde{B}(w_1, 0)$. □

We shall conclude this section with an example, but first we recall the definition of root map which, as indicated in [**7**], can be quite useful to investigate the stability of a two-dimensional LSI system. We shall write $B[w_1](w_2)$ to indicate that we interpret the two-dimensional polynomial $B(w_1, w_2)$ as a one-dimensional polynomial of the variable w_2 with coefficients which are themselves one-dimensional polynomials of the variable w_1. The root map of $B(w_1, w_2)$ consists of two root images—one root image shows the loci of the roots of $B[w_1](w_2)$ as the parameter w_1 traverses the unit circle $w_1 = e^{i\varphi}$ for $-\pi \leq \varphi_1 \leq \pi$. The other root image shows the loci of the roots of $B[w_2](w_1)$ as the parameter w_2 traverses the unit circle $w_2 = e^{i\varphi_2}$. We remark that the conditions (ii) of Theorem 4 will be satisfied if both root images of $B(w_1, w_2)$ lie outside the respective unit circles.

Now consider the LSI system given by the transfer function

$$H(z_1, z_2) = \frac{1}{B(z_1^{-1}, z_2^{-1})}, \tag{21a}$$

where
(21b)

$$\begin{aligned} B(w_1, w_2) = {} & (w_2 - w_1 + 3)(w_2 - w_1 - 3)(w_2 + w_1 + 3)(w_2 + w_1 - 3) \\ & \times (w_2 - w_1 + 4)(w_2 - w_1 - 4)(w_2 + w_1 + 4)(w_2 + w_1 - 4) \\ & \times (w_2 - w_1 + 5)(w_2 - w_1 - 5)(w_2 + w_1 + 5)(w_2 + w_1 - 5). \end{aligned}$$

The root image of $B[w_2](w_1)$ is given in Figure 1. Note that since $B(w_1, w_2) = B(w_2, w_1)$, the root image of $B[w_1](w_2)$ is identical to the one given in Figure 1. From the root map given in Figure 1 and Theorem 4(ii) it immediately follows that (21) represents a stable system.

In order to construct a reduced model for (21), we compute the BCF (13a) for $H(z_1, z_2)/\beta$, where β is given by (19). It is easy to check that

$$\max_{|w_1| \leq 1} \left| \frac{1}{B(w_1, 0)} \right| \leq 1,$$

and, hence, we have $\beta = 1$. The Taylor series expansion of $H(z_1, z_2) = 1/B(z_1^{-1}, z_2^{-1})$ is given by

$$\begin{aligned} \frac{1}{B(w_1, w_2)} = {} & \frac{1}{(3 \cdot 4 \cdot 5)^4} + \frac{1538}{.60^6} w_2^2 + O(w_2^4) \\ & + w_1^2 \left(\frac{1538}{60^6} + \frac{3753610}{60^8} w_2^2 + O(w_2^4) \right) + O(w_1^4 w_2^j, j \geq 0). \end{aligned}$$

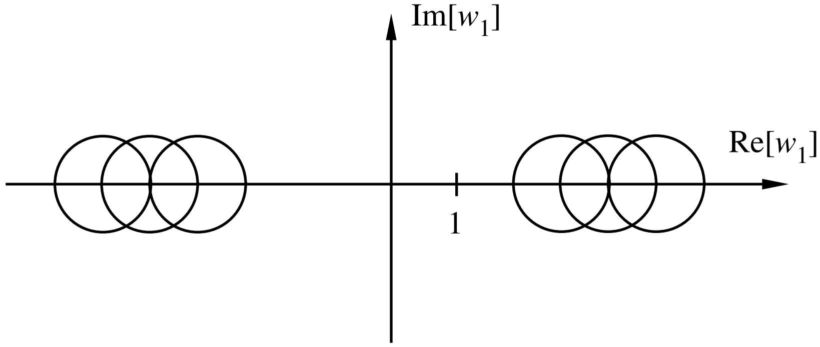

FIGURE 1.

We can now apply the formulas (14) to compute the BCF (15) for $1/B(w_1, w_2)$. This gives

$$\begin{array}{llll}
\gamma_0 = \dfrac{1}{60^4} & & & \\
d_0^{(0)} = \gamma_0 & d_1^{(0)} = 0 & d_2^{(0)} = \dfrac{1538}{60^6} & \cdots \\
c_{00}^{(0)} = 0 & c_{01}^{(0)} = 0 & c_{02}^{(0)} = 0 & \cdots \\
c_{10}^{(0)} = \dfrac{1538}{60^6} & c_{11}^{(0)} = 0 & c_{12}^{(0)} = \dfrac{3753610}{60^8} & \cdots \\
\vdots & & & \\
c_{00}^{(1)} = 1 - \dfrac{1}{60^8} & c_{01}^{(1)} = 0 & c_{02}^{(1)} = 0 & \cdots \\
c_{10}^{(1)} = 0 & c_{11}^{(1)} = 0 & c_{12}^{(1)} = 0 & \cdots \\
c_{20}^{(1)} = -\dfrac{1538}{60^{10}} & c_{21}^{(1)} = 0 & c_{22}^{(2)} = -\dfrac{3753610}{60^{12}} & \cdots \\
\vdots & & &
\end{array}.$$

For $k = 1$ in (14), we find

$$\gamma_1 = 0,$$
$$d_0^{(1)} = \gamma_1, \quad d_j^{(1)} = 0, \qquad j \geq 0,$$

and so

$$\begin{aligned} c_{ij}^{(2)} &= c_{i+1,j}^{(0)} \\ c_{ij}^{(3)} &= c_{ij}^{(1)}, \end{aligned} \qquad i \geq 0, \qquad j \geq 0,$$

while, for $k = 2$,

$$\gamma_2 = \frac{5536800}{60^8 - 1}$$
$$d_0^{(2)} = \gamma_2 \quad d_1^{(2)} = 0 \quad d_2^{(2)} = \frac{3753610}{60^8 - 1} \quad \cdots .$$

Grouping these results we find that the BCF (15) for $1/B(w_1, w_2)$ becomes

$$(22) \quad \frac{1}{60^4} + \frac{1538}{60^6} w_2^2 + \cdots + \frac{(1 - \frac{1}{60^8}) w_1 |}{| \frac{1}{60^4} w_1} + \frac{1|}{|0} + \frac{w_1|}{|\ 0}$$
$$+ \frac{1 \qquad |}{\left| \frac{5536800}{60^8 - 1} + \frac{3753610}{60^8 - 1} w_2^2 + \cdots \right.} + \frac{\left(1 - \left(\frac{5536800}{60^8 - 1}\right)^2\right) w_1 \Big|}{\Big| \quad \frac{5536800}{60^8 - 1} w_1} + \frac{1|}{|\cdots}.$$

Note that, in this case, applying the formulas (18) to $h_0(w_2)$ and $h_2(w_2)$, gives that $\delta_{10}^{(0)} = \delta_{10}^{(2)} = 0$. This implies that we have to apply the "singular rules" of the Viscovatov-scheme [**5**, p. 55] to $h_0(w_2)$ and $h_2(w_2)$. Doing this leaves (22) unchanged. In order to obtain reduced models for the system represented by (21), consider modified convergents of the BCF (22). If we take $n = 3$ and $m = 2$ in (20), we find that the transfer function of the reduced system is given by

(23)
$$\tilde{H}(z_1, z_2) = R_{3,2}(z_1^{-1}, z_2^{-1}, 1) = \frac{\tilde{A}(w_1, w_2)}{\tilde{B}(w_1, w_2)}$$
$$= \frac{1}{60^4} + \frac{1538}{60^6} w_2^2 + \left(1 - \frac{1}{60^8}\right)$$
$$\times \frac{\frac{5536800}{60^8-1} w_1^2 + w_1^3 + \frac{3753610}{60^8-1} w_1^2 w_1^2 (1 + \frac{5536800}{60^8-1} w_1)}{1 + \frac{5536800}{60^8-1} w_1 + \frac{5536800}{60^4(60^8-1)} w_1^2 + \frac{1}{60^4} w_1^3 + \frac{3753610}{60^4(60^8-1)} w_1^2 w_2^2 (1 + \frac{5536800}{60^8-1} w_1)}.$$

Since the conditions of Theorem 5 are satisfied, we can conclude the following for the reduced system (23). The order of correspondence of $\tilde{H}(z_1, z_2)$ to $H(z_1, z_2)$ is

$$H(z_1, z_2) - \tilde{H}(z_1, z_2) = O(z_1^{-3} z_2^{-j}, z_1^{-j}, z_2^{-3}, j \geq 0),$$

and, whereas the number of terms in the numerator and denominator polynomial of the original transfer function equals 92, for $\tilde{H}(z_1, z_2)$, this number amounts to 16. From the theorem it also follows that $\tilde{A}(w_1, 0)/\tilde{B}(w_1, 0)$ represents a stable one-dimensional system, and, hence, testing the stability of $\tilde{H}(z_1, z_2)$ is reduced to verifying that

$$\tilde{B}(w_1, w_2) \neq 0, \qquad |w_1| = 1, \ |w_2| \leq 1. \tag{24}$$

As mentioned above, this condition can be checked using the technique described in [**1, 8**]. We have not used this technique here but instead consider the root image of $\tilde{B}[w_1](w_2)$ as w_1 traverses the unit circle. Condition (24) will be satisfied if this root image lies outside the unit circle in the complex w_2-plane. From (23) we have that the roots of $\tilde{B}[w_1](w_2)$ as a function of w_1 are given by

$$w_2^2 = -\frac{60^4(60^8 - 1)}{3753610} \cdot \frac{1 + \frac{536800}{60^8-1} w_1 + \frac{536800}{60^4(60^8-1)} w_1^2 + \frac{1}{60^4} w_1^3}{w_1^2 (1 + \frac{536800}{60^8-1} w_1)}.$$

If we set $w_1 = e^{i\varphi_1}$ in the above expression, it can easily be verified that the corresponding values of w_2 satisfy $|w_2| > 10^6$. Hence, it is clear that the root image of $\tilde{B}[w_1](w_2)$ lies outside the unit circle and we can conclude that $\tilde{H}(z_1, z_2)$ given by (23) represents a stable reduced system for the system (21).

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