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LOCALLY INVARIANT MANIFOLDS FOR QUASILINEAR PARABOLIC EQUATIONS

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1. Introduction. The theory of invariant manifolds for differential equations is becoming more and more important since its origin [**11**] two decades ago. After it has been realized that the ideas are also valid for infinite-dimensional systems [**13**], the realm of partial differential equations was redeveloped under this new *geometrical point of view* [**10**]. In a series of papers the machinery was extended to fit for parabolic equations [**10**], damped hyperbolic problems [**4**], and finally elliptic equations in cylindrical domains [**12, 8**].

Yet, up to now almost all work was restricted to semilinear problems. That means, for a differential equation of the form

$$\dot{x} - Lx = f(x), \tag{1.1}$$

we have to assume that f is a smooth function from $D(L^\gamma)$ into $X = D(L^0)$ for some $\gamma < 1$. The quasilinear case, $\gamma = 1$, is only treated in very special cases [**6, 7, 15, 16**]. Here we give a general approach which includes arbitrary parabolic problems.

The main difficulty we have to deal with is the regularity loss through the nonlinearity f. This has to be compensated by the regularizing properties of the linear equation

$$\dot{x} - Lx = g(t), \quad x(0) = \xi. \tag{1.2}$$

In this paper we use the interpolation spaces $D_L(\theta)$, $\theta \in (0,1)$, defined in [**9**]. If L generates a holomorphic semigroup, then we have the following *maximal regularity result*: for $g \in C([0,T], D_L(\theta))$ and $L\xi \in D_L(\theta)$ the solution of (1.2) satisfies $Lx \in C([0,T], D_L(\theta))$. Using an appropriate modification of this result it is straightforward

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to generalize existing invariant manifold theorems to the quasilinear case.

In Section 2 we prove an abstract theorem for the existence of a center-unstable manifold for (1.1). As a corollary we obtain the existence of center and unstable manifolds. In Section 3 some relevant properties of the interpolation space $D_L(\theta)$ are given. If L is an $L_p(\Omega)$-realization of an elliptic operator, then $D_L(\theta)$ is in the class of *Nikolski spaces*. In the last section it is finally shown how the abstract theory applies to quasilinear parabolic equations.

2. Invariant manifolds. We consider the nonlinear evolution equation

$$\dot{x} - Lx = f(t, \lambda, x), \tag{2.1}$$

where $\lambda \in \Lambda$ is a set of real parameters and $L : D(L) \to X$ is the generator of a holomorphic semigroup $(e^{Lt})_{t\geq 0}$. For the time being we restrict our attention to the construction of a center-unstable manifold, which will contain all solutions of (2.1) sufficiently small for $t \to -\infty$.

Therefore, we assume that the Banach space X splits into closed L-invariant subspaces X_1 and X_2 such that (2.1) takes the form

$$\begin{aligned} \dot{x}_1 - Ax_1 &= f_1(t, \lambda, x_1, x_2), \\ \dot{x}_2 - Bx_2 &= f_2(t, \lambda, x_1, x_2). \end{aligned} \tag{2.2}$$

Here, $\dim X_1 < \infty$ and all eigenvalues of $A = L|_{X_1}$ have nonnegative real part. The operator $B = L|_{X_2} : D(B) = D(L) \cap X_2 \to X_2$ is closed, densely defined, and satisfies the resolvent estimate

$$||(B - z)^{-1}||_{X_2 \to X_2} \leq \frac{C}{1 + |z|}, \tag{2.3}$$

for all z with $\operatorname{Re} z \geq 0$. Then B generates a holomorphic semigroup $(e^{Bt})_{t\geq 0}$ with $||e^{Bt}|| \leq Ce^{-\alpha t}$ for $t \geq 0$.

According to [**9**] we define, for $\theta \in (0, 1)$, the interpolation spaces

$$\begin{aligned} &D_B(\theta) = \{x_2 \in X_2 \mid ||s^\theta B(B - s)^{-1} x_2|| \to 0 \text{ for } s \to \infty\}, \\ &||x_2||_\theta = \max\{||s^\theta B(B - z)^{-1} x_2|| \mid s \geq 0\}, \\ &D_B(\theta + 1) = \{x_2 \in D(B) \mid Bx_2 \in D_B(\theta)\},\ ||x_2||_{\theta+1} = ||Bx_2||_\theta. \end{aligned}$$

Lemma 2.1. *Assume $B : D(B) \to X_2$ is closed, densely defined and satisfies* (2.3). *Let $\theta \in (0,1)$. Then, the mapping $g_2 \to x_2$ with*

$$x_2(t) = \int_{-\infty}^{t} e^{B(t-s)} g_2(s)\, ds$$

is a bounded linear mapping from $C_b((-\infty,0], D_B(\theta))$ into $C_b^1((-\infty,0],$

$D_B(\theta)) \cap C_b((-\infty,0], D_B(\theta+1))$.

Here $C_b(I,Y)$ denotes the space of bounded continuous functions $u : I \to Y$. The lemma is a direct consequence of Theorem 3.1 in [**9**], so we omit the proof.

For fixed $\theta \in (0,1)$ we require the following conditions for (2.2).

A1: $\dim X_1 < \infty$, $\mathrm{Re}\,\mathrm{spec}\,(A) \geq 0$.

A2: $B : D(B) \to X_2$ is closed, $D(B)$ is dense in X_2, and (2.3) is satisfied for all z with $\mathrm{Re}\, z \geq 0$.

A3: There exist neighborhoods $U_1 \subset X_1$ and $U_2 \subset D_B(\theta+1)$ of zero and an integer $k \geq 1$ such that

$$f = (f_1, f_2) = C^k_{b,\mathrm{unif}}(\mathbf{R} \times \Lambda \times U_1 \times U_2, X_1 \times D_B(\theta)).$$

For some $\lambda_0 \in \Lambda$ and all t we have $f(t,\lambda_0,0)=0$, $(\partial/\partial x) f(t,\lambda_0,0)=0$.

Theorem 2.2. (Existence of a local center-unstable manifold). *Let* A1, A2, *and* A3 *be satisfied. Then there exist neighborhoods $U_1' \subset U_1$ and $U_2' \subset U_2$ of zero, a neighborhood $\Lambda' \subset \Lambda$ of λ_0, and a function*

$$h = h(t,\lambda,x_1) \in C_b^k(\mathbf{R} \times \Lambda' \times U_1', U_2')$$

with the following properties (*for $\lambda \in \Lambda'$ fixed*):

a) *The set $M = \{(t, x_1, h(t,\lambda,x_1)) \in \mathbf{R} \times X_1 \times D_B(\theta+1) \mid (t,x_1) \in \mathbf{R} \times U_1'\}$ is a local integral manifold of* (2.2). (*For a definition of local integral manifolds, see* [**14**].)

b) *Every solution $(x_1(\cdot), x_2(\cdot)) \in C^1((-\infty,t_0], X_1 \times D_B(\theta))$ which belongs to $U_1' \times U_2'$ for all $t \leq t_0$ lies in M, i.e., $x_2(t) = h(t,\lambda,x_1(t))$, $t \leq t_0$.*

c) *For all* $t \in \mathbf{R}$, $h(t, \lambda_0, 0) = 0$, $(\partial/\partial x)h(t, \lambda_0, 0) = 0$.

Proof. Observe that Lemma 2.1 guarantees just a gain of regularity of the order B; but this is exactly the same as the loss through the nonlinearity f. Hence, by a careful bookkeeping of the regularity we will be able to solve (2.2) via the same iteration procedure as in the semilinear case. This proof essentially goes along with the proof of the center manifold theorem in [**14**], and, therefore, we only point out the necessary changes one has to do.

We may assume, after multiplying with a suitable cut-off function, if necessary, that $f \in C^k_{b,\mathrm{unif}}(\mathbf{R} \times \Lambda \times X_1 \times D_B(\theta+1),\ X_1 \times D_B(\theta))$ whereby $\delta = \sup\left\{ \left\| \frac{\partial f}{\partial x}(t, \lambda, x) \right\| \mid (t, \lambda, x) \in \mathbf{R} \times \Lambda \times X_1 \times D_B(\theta+1) \right\}$ is sufficiently small. If there is no smooth cut-off function on $D_B(\theta+1)$ we can proceed as in [**14**], where only a cut-off on the finite-dimensional part X_1 is needed.

To find all solutions of (2.2) being bounded on $(-\infty, \tau]$ we transform (2.2) into the corresponding integral equation with the additional initial value condition $x_1(\tau) = \xi$. Let $y(t) = x(t+\tau)$ for $t \leq 0$, then y has to satisfy the fixed point problem

$$y(\cdot) = T(y, \tau, \lambda, \xi)(\cdot) \tag{2.4}$$

where

$$T_1(y, \tau, \lambda, \xi)(t) = e^{At}\xi - \int_t^0 e^{A(t-s)} f_1(s+\tau, \lambda, y(s))\, ds$$

and

$$T_2(y, \tau, \lambda, \xi)(t) = \int_{-\infty}^t e^{B(t-s)} f_2(s+\tau, \lambda, y(s))\, ds.$$

For $\nu \in \mathbf{R}$ define the weighted function spaces

$$Y_\nu = \{ y \in C((-\infty, 0], X_1 \times D_B(\theta+1)) \mid |y|_\nu < \infty \},$$
$$|y|_\nu = \sup\{ e^{\nu t}(||y_1|| + ||y_2||_{\theta+1}) \mid t \leq 0 \}.$$

Using the estimates $||e^{At}|| \leq C(1-t)^m$, $t \leq 0$, $||e^{Bt}|| \leq Ce^{-\alpha t}$, $t \geq 0$, and Lemma 2.1 we see that $T : Y_\nu \times \mathbf{R} \times \Lambda \times X_1 \to Y_\nu$ is locally Lipschitz continuous whenever $\nu \in (0, \alpha)$. For sufficiently small $\delta = \delta(\nu)$ it is even a uniform contraction in $y \in Y_\nu$. Thus, (2.4) has a unique fixed point $y = \bar{y}(\tau, \lambda, \xi)(\cdot)$ in Y_ν, depending continuously on (τ, λ, ξ). The function h is defined as $h(\tau, \lambda, \xi) = \bar{y}_2(\tau, \lambda, \xi)(0)$; hence we have $h \in C(\mathbf{R} \times \Lambda \times X_1, D_B(\theta + 1))$.

All further properties of $\bar{y}$ (resp. h) can be obtained in a similar fashion as in [**14**]. □

The flow on the center-unstable manifold M is now completely described by the *reduced equation*

$$\dot{x}_1 - Ax_1 = f_1(t, \lambda, x_1, h(t, \lambda, x_1)) \tag{2.5}$$

which is an ordinary differential equation since $\dim X_1 < \infty$. However, to this equation the classical theorems for invariant manifolds [**11, 5**] apply, and we obtain

Corollary 2.3. (Existence of local center manifold and local unstable manifold). *Let* A1, A2, *and* A3 *be satisfied by our system* (2.2). *Then there exist a local center manifold and a local unstable manifold.*

Remarks. 1. This approach does not yield the uniqueness of the unstable manifold. Yet, if we construct the unstable manifold directly (using a similar technique as for Theorem 2.1 but with $\nu < 0$) we need no cut-off function and the uniqueness can be shown.

2. The same method should apply to the construction of the infinite-dimensional stable manifold (cf. [**7**]).

3. Nikolski spaces. Parabolic equations are most conveniently viewed as abstract evolution equations in $X = L_p(\Omega)$ with $p \in (1, \infty)$. Then, $L : D(L) \to X$ is an elliptic operator of order $2m$, and $D(L)$ is some closed subspace of the Sobolev space $W_p^{2m}(\Omega)$. The corresponding interpolation spaces $D_L(\theta)$ are the Nikolski spaces $h_p^{2\theta m}(\Omega)$. Because of the limited size of this paper, we only give the definitions and some of the main properties. For further information and proofs, we refer the reader to [**9**]. Subsequently, we assume that Ω is a bounded region in $\mathbf{R}^n$ with $\partial\Omega$ of class C^{2m}.

Definition 3.1. Let $\sigma \in (0,1)$, $p \in (1,\infty)$, and $m \in N_0$. Then

$$\begin{aligned} h_p^\sigma(\mathbf{R}^n) &= \{u \in L_p(\mathbf{R}^n) \mid |y|^{-\sigma}||u(\cdot+y)-u(\cdot)||_{L_p} \to 0 \ \text{ for } y \to 0\}, \\ h_p^\sigma(\Omega) &= \{u \in L_p(\Omega) \mid \exists \tilde{u} \in h_p^\sigma(\mathbf{R}^n) : \tilde{u}|_\Omega = u\}, \\ h_p^{m+\sigma}(\Omega) &= \{u \in W_p^m(\Omega) \mid D^\beta u \in h_p^\sigma(\Omega), \quad 0 \le |\beta| \le m\}. \end{aligned}$$

It is known [**9**] that the interpolation space $D_L(\theta)$ depends only on $D(L)$ and X but not on the operator L itself. Therefore, we are able to introduce the interpolation spaces $(Y,X)_\theta$ for every Banach space Y continuously embedded in X. Then we have $D_L(\theta) = (D(L),X)_{1-\theta}$.

Lemma 3.2. *Let $m \in \mathbf{N}$, $p \in (1,\infty)$, and $\theta \in (0,1)$ such that $m\theta \notin \mathbf{N}$.*

a) *Then, $(W_p^m(\Omega), L_p(\Omega))_{1-\theta} = h_p^{\theta m}(\Omega)$.*

b) *Let $\overset{\circ}{W}{}_p^m(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W_p^m(\Omega)$, and let Y be any closed subspace of $W_p^m(\Omega)$ such that $\overset{\circ}{W}{}_p^m(\Omega) \subset Y \subset W_p^m(\Omega)$. If additionally $\theta m < 1/p$, then $(Y, L_p(\Omega))_{1-\theta} = h_p^{\theta m}(\Omega)$.*

Lemma 3.3. a) *For $s > n/p$, $s \notin \mathbf{N}$, the space $h_p^s(\Omega)$ is continuously embedded in $C^0(\overline{\Omega})$ and, thus, forms an algebra.*

b) *For $s = m+\sigma > n/p$, $m \in \mathbf{N}_0$ and $\sigma \in (0,1)$, and $f \in C^{m+k}(\mathbf{R}^l, \mathbf{R})$, then evaluation mapping $(u_1(\cdot),\dots,u_l(\cdot)) \in (h_p^s(\Omega))^l \to f(u_1(\cdot),\dots,u_l(\cdot)) \in h_p^s(\Omega)$ is k-times continuously differentiable.*

4. Quasilinear parabolic systems. For a vector-valued function $u = (u_1,\dots,u_r)$ we consider a quasilinear equation on the bounded domain $\Omega \in R^n$.

$$\begin{aligned} \dot{u} - \tilde{L}u &= F(u) \qquad && \text{for } x \in \Omega, \\ B_1 u = \cdots = B_m u &= 0 && \text{for } x \in \partial\Omega, \end{aligned} \tag{4.1}$$

where $\tilde{L}u = \sum_{|\alpha|\le 2m} A_\alpha(x) D^\alpha u$ is an elliptic operator of order $2m$ and $B_1,\dots,B_m$ is a set of boundary operators of degree $0 \le \beta_1 < \cdots < \beta_m < 2m$, respectively. Let $p > \max\{1, n/2\}$ and assume that $L : D(L) \to L_p(\Omega)$; $u \to Lu$ is the generator of a holomorphic semigroup $(e^{Lt})_{t\geq 0}$ (sufficient conditions for this are given in [**1,2**]). Here $D(L)$ is defined as $\{u \mid u_i \in W_p^{2m}(\Omega),\ i = 1, \ldots, r;\ B_j u = 0,\ j = 1, \ldots, m, \text{ on } \partial\Omega\}$. The function $F = F(u)$ may be a classical function of all the derivatives $D^\alpha u$ up to order $2m$; yet the highest derivatives $D^\alpha u$, $|\alpha| = 2m$, may appear linearly only. Moreover, we assume $F(u) = \mathcal{O}(||u||^2 + ||Lu||^2)$ for $u \to 0$.

We choose $\theta > 0$ such that $1/p > 2\theta m > n/p - 1$. Then, because of Lemma 3.2b, we know $D_L(\theta) = h_p^{2\theta m}(\Omega)$ and, because of Lemma 3.3a, $h_p^{2\theta m+1}(\Omega)$ is continuously imbedded in $C^0(\overline{\Omega})$. The quasilinearity and Lemma 3.3b now yield that the mapping $F = F(u) \in C^k(h_p^{2(\theta+1)m}(\Omega), h_p^{2\theta m}(\Omega))$. On the other hand, the resolvent $(L-s)^{-1}$ exists for sufficiently large $s > 0$ and is compact since Ω is bounded. Hence, the spectrum of L is discrete and the L-invariant eigenspace X_1, corresponding to the spectral points z with $\operatorname{Re} z \geq 0$, is finite dimensional.

Altogether we have now shown that the assumptions A1, A2, and A3 are satisfied for our system (4.1). Thus, Theorems 2.2 and 2.3 give the existence of center-unstable, center, and unstable manifolds.

Remark. Here we have restricted ourselves to linear homogeneous boundary conditions. However, using the method described in [**3**], it is possible to treat nonlinear boundary conditions also.

REFERENCES

1. S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

2. ———, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions* II, Comm. Pure Appl. Math. **17** (1964), 35–92.

3. H. Amann, *Parabolic evolution equations with nonlinear boundary conditions*, Proc. Sympos. Pure Math. **45** (1986), 17–27.

4. J. Carr, *Applications of center manifold theory*, Appl. Math. Sci. **35**, Springer-Verlag, Heidelberg, Berlin, New York, 1981.

5. S.-N. Chow, J.K. Hale, *Methods of bifurcation theory*, Springer-Verlag, Heidelberg, Berlin, New York, 1982.

6. G. Da Prato, A Lunardi, *Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach space*, Arch. Rational Mech. Anal. **101** (1988), 115–141.

7. C.L. Epstein, M.I. Weinstein, *A stable manifold theorem for the curve shortening equation*, Comm. Pure Appl. Math. **40** (1987), 119.

8. G. Fischer, *Zentrumsmannigfaltigkeiten bei elliptischen Differentialgleichungen*, Math. Nachr. **115** (1984), 137–157.

9. P. Grisvard, G. da Prato, *Equations d'évolutions abstraites non linéaires de type parabolique*, Ann. Mat. Pura Appl. (4) **120** (1979), 329–396.

10. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. **840**, Springer-Verlag, 1981.

11. A. Kelley, *The stable, center-stable, center, center-unstable, unstable manifolds*, J. Differential Equations **3** (1967), 547–570.

12. K. Kirchgässner, *Wave solutions of reversible systems and applications*, J. Differential Equations **45** (1982), 113–127.

13. J.E. Marsden, M.F. McCracken, *The Hopf bifurcation and its applications*, Springer-Verlag, Berlin, New York, 1976.

14. A. Mielke, *A reduction principle for nonautonomous systems in infinite-dimensional spaces*, J. Differential Equations **65** (1986), 68–88.

15. ———, *Reduction of quasilinear elliptic equations in cylindrical domains with applications*, Math. Methods Appl. Sci. **10** (1988), 51–66.

16. ———, *Normal hyperbolicity of center manifolds and Saint-Venant's principle*, Arch. Rational Mech. Anal. **110** (1990), 353–372.

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