ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 3, Summer 1991

A METHOD FOR THE NUMERICAL SOLUTION OF A CLASS OF NONLINEAR DIFFUSION EQUATIONS

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ABSTRACT. We propose a finite difference scheme for the diffusion equation, $(*)$ $u_t = d(u)\Delta u + f(u)$, on a general spatial domain of $\mathbf{R}^m$, $m \geq 1$, $d(u)$ is a bounded positive smooth function.

For the numerical solution of $(*)$ one usually uses a finite difference method based on the well-known θ-method, which requires a factorization of a matrix at each time step.

Here we propose a numerical scheme in which we need a single factorization of a matrix for each time level.

We prove that if W is an invariant region for $(*)$, it is also invariant for the proposed method. Comparisons between our scheme and the explicit/implicit Euler method are made.

We give an error bound which implies the first order convergence of the method and shows that the error does not exceed diam(W) for $t \to +\infty$. Finally, we show a numerical application.

1. Introduction. In recent years much interest has been shown in the numerical solution of the nonlinear diffusion equation

$$u_t = d(u)\Delta u + f(u), \qquad \Omega \times (0, +\infty) \tag{1.1}$$

with the addition of certain initial and boundary conditions. In (1.1), Ω is a bounded domain of $\mathbf{R}^m$, $m \geq 1$ with smooth boundary $\partial\Omega$; Δ denotes the Laplace operator in the spatial variable x and $d(\cdot)$ is a positive smooth function. Let $W = [0, b]$ be the u-space interval and let us suppose that $d(u)$ is bounded on W, that is, two positive constants α and β exist such that

$$0 < \alpha < d(u) < \beta \quad \text{for any } u \in W. \tag{1.2}$$

Received by the editors on June 20, 1986.
Work performed under the auspices of the "Centro Interuniversitario di Analisi Numerica e Matematica Computazionale."

Moreover, we assume that $f(\cdot)$ is a smooth function such that a constant c exists for which

$$|f'(u)| \leq c \quad \text{for any } u \in W \tag{1.3}$$

and

$$f(u) \cdot n(u) \leq 0, \qquad u \in \partial W, \tag{1.4}$$

where $n(u)$ is the outer normal on the boundary ∂W of W. We suppose that (1.1) has a unique sufficiently smooth solution $u(x,t)$ and that W is an invariant region for (1.1): that means that if the initial function $u(x,0)$ belongs to W, for any $x \in \Omega$, then $u(x,t)$ belongs to W, for any $(x,t) \in \Omega \times (0,+\infty)$ (see [1]).

Example. As a practical example of diffusion equation (1.1) we can consider the temperature controlled problem in the one-dimensional domain $(0,1)$:

$$\begin{gathered} v_t = \mathbf{A}(v)v_{xx}, \qquad 0 < x < 1, 0 < t < T, \\ v(x,0) = 0, \qquad 0 < x < 1, \\ v(0,t) = f_0(t), \qquad v(1,t) = 0, \qquad 0 < t < T, \end{gathered}$$

which has been studied by Duchateau in [**4**].

Models of the form (1.1) can also be found in biology. In this case u denotes the density of the species being considered, while $d(u)$ takes into account that the diffusion process depends on the density of the individuals of the species.

In recent years several schemes have been proposed for the numerical solution of nonlinear diffusion equations of the form (1.1) (see, for example, [**2,3,5**]).

In [**5**] Hoff studied a family of finite difference schemes based on the well-known θ-method:

$$[I - \theta L(\mathbf{V}^n)]\mathbf{V}^{n+1} = [I + \phi L(\mathbf{V}^n)]\mathbf{V}^n + \Delta t F(\mathbf{V}^n), \tag{1.5}$$

where $\mathbf{V}^n$ is the numerical solution at the time t_n and $L(\mathbf{V}^n)$ is a matrix which depends on the diffusion.

He proved that, under certain conditions on the mesh, the numerical solution given by (1.5) is stable, in the sense that W is also invariant for the numerical solution ($\mathbf{V}^\circ \in W$ implies $\mathbf{V}^n \in W$ for any $n > 0$).

We now observe that, if $0 < \theta < 1$, (1.5) requires one factorization of a matrix at each time step. On the other hand, if $\theta = 0$, (1.5) becomes an explicit scheme; no factorization is needed, but in order to have stability we must greatly restrict the time step.

Hence a method which avoids all the factorizations of (1.5) and preserves the stability, under reasonable restrictions on the mesh, would be suitable.

In this paper we propose a scheme which requires a single factorization for each time level.

The method is suggested by the fact that we can write $d(u)$ as

$$d(u) = d(u^*) + \tilde{d}(u) \quad \text{for any } u \in W, \tag{1.6}$$

where u^* is the point in which the minimum value of $d(u)$ occurs and $\tilde{d}(u)$ is the positive function given by $\tilde{d}(u) = d(u) - d(u^*)$.

Then, the matrix $L(\mathbf{V}^n)$ in (1.5) can be written as

$$L(\mathbf{V}^n) = L(\mathbf{V}^*) + \tilde{L}(\mathbf{V}^n), \tag{1.7}$$

where $\mathbf{V}^* = (u^*, u^*, \ldots, u^*)^t$ and $\tilde{L}(\mathbf{V}^n)$ is a matrix defined by $\tilde{d}(u)$. Considering (1.7), from (1.5) we can obtain the scheme

$$[\mathbf{I} - \theta L(\mathbf{V}^*)]\mathbf{V}^{n+1} = [\mathbf{I} + \phi L(\mathbf{V}^n) + \theta L(\mathbf{V}^n)]\mathbf{V}^n + \Delta t F(\mathbf{V}^n) \tag{1.8}$$

for $n \geq 0$.

The computational advantage of (1.8) with $\theta > 0$ on (1.5), with $\theta > 0$, is more evident when the spatial dimension m is greater than one and Ω is a general domain of $\mathbf{R}^m$. In fact, in this case the matrix on the left of (1.5) is a full matrix and a factorization of such a matrix at any time is very expensive (see [**7**]).

Moreover, the advantage of (1.8), with $\theta > 0$, over (1.5) with $\theta = 0$ (that is, the Euler explicit scheme) concerns the stability conditions.

In fact, in the one dimensional case, we shall prove that the invariance condition, for the numerical solution obtained by (1.8), is given by

$$\Delta t < \frac{1}{c + 2(\beta - \theta\alpha)/\Delta x^2}; \tag{1.9}$$

while, for the Euler explicit scheme, we have

$$\Delta t < \frac{1}{c + 2\beta/\Delta x^2}. \tag{1.10}$$

Then, comparing (1.9) and (1.10), we can say that (1.9) requires a less restrictive bound on Δt than the one required by (1.10), because $\beta > \beta - \alpha$. Of course, the more that $(\beta - \alpha)/\beta$ is less than 1, the more the bound on Δt given by (1.9), is less restrictive than the one given by (1.10).

The paper is organized as follows: in section 2 we recall the θ-method for nonlinear diffusion equations and introduce the new scheme in the one spatial dimensional case; in section 3 we show that the proposed scheme preserves some invariant properties of the θ-method; in section 4 we give a bound for the error which tends to diam W as $t \to +\infty$ and which is $O(\Delta t + \Delta x^2)$ for fixed t; in section 5 we extend the method to a more general spatial domain $\Omega \in \mathbf{R}^2$ and show a numerical application.

2. Description of the method. For the sake of simplicity, we now propose the numerical method in the one spatial dimensional case. In the case $m = 2$ the method will be easily derived in section 5.

Hence, consider

$$u_t = d(u)u_{xx} + f(u), \qquad x \in (0,1), t \in (0, +\infty) \tag{2.1a}$$

subject to the initial condition

$$u(x, 0) = u_\circ(x), \qquad x \in (0,1). \tag{2.1b}$$

To simplify the exposition we assume homogeneous boundary conditions

$$u_x(0,t) = u_x(1,t) = 0, \qquad t \in (0, +\infty) \tag{2.1c}$$

but observe that the main results of this section can also be found for more general boundary conditions. Let Δx be the step in the spatial variable and Δt the step in the time variable. Let $x_k = (k-1)\Delta x$ be a point of the spatial interval $[0,1]$, for any $k = 1, \ldots, K$ with K as a positive integer, such that $\Delta x(K-1) = 1$. We also discretize the time interval by $t_n = n\Delta t$ for $n > 0$, and denote by u_k^n and v_k^n the exact and approximate solutions of (1.1) evaluated at the point (x_k, t_n). We now consider the known operator

$$\delta^2 w(x_k) = \frac{w(x_{k+1}) - 2w(x_k) + w(x_{k-1})}{\Delta x^2},$$

where $w : [0,1] \to \mathbf{R}$. Then (1.1) can be replaced by the following system of difference equations

$$\begin{aligned} \frac{v_k^{n+1} - v_k^n}{\Delta t} &= d(v_k^n)\delta^2(\theta v_k^{n+1} + \phi v_k^n) + f(v_k^n) \\ v_k^\circ &= u_\circ(x_k), \end{aligned} \tag{2.2}$$

for $k = 1, \ldots, K$, $n > 0$. (2.2) results from the application of the θ-method to (1.1), where we compute f at the previous time level (see Richmeyer & Morton [**8**]). In (2.2) θ and ϕ are two positive constants, the sum of which is one.

Denote, by $\mathbf{V}^n$ and $F(\mathbf{V}^n)$, the two vectors

$$\mathbf{V}^n = (v_1^n, \ldots, v_k^n)^t, \qquad F(\mathbf{V}^n) = (f(v_1^n), \ldots, f(v_k^n))^t$$

and, by $L(\mathbf{V}^n)$, the $K \times K$ tridiagonal matrix

$$L(\mathbf{V}^n) = r \begin{bmatrix} -2d(v_1^n) & 2d(v_1^n) & & 0 \\ d(v_2^n) & -2d(v_2^n) & d(v_2^n) & \\ \ddots & \ddots & \ddots & \\ 0 & & 2d(v_K^n) & -2d(v_K^n) \end{bmatrix}$$

with $r = \Delta t/\Delta x^2$.

Then (2.2) can be written in compact form as

$$\begin{aligned} [I - \theta L(\mathbf{V}^n)]\mathbf{V}^{n+1} &= [I + \phi L(\mathbf{V}^n)]\mathbf{V}^n + \Delta t F(\mathbf{V}^n), \qquad n > 0, \\ \mathbf{V}^\circ &= \{\mathbf{V}_k^\circ\}_{k=1}^K. \end{aligned} \tag{2.3}$$

Now observe that, if $0 < \theta < 1$, the numerical efforts in (2.3) are dominated by the factorization of a time-dependent matrix, and hence it would be very advantageous to reduce the number of factorizations required by (2.3). For this purpose, we write the diffusion function $d(u)$ as

$$d(u) = d(u^*) + \tilde{d}(u), \quad \text{for any } u \in W, \tag{2.4}$$

where $\tilde{d}(\cdot)$ is the positive function $\tilde{d}(u) = d(u) - d(u^*)$ and u^* is the minimum point of $d(u)$ on W.

Then, we introduce the tridiagonal matrix $\tilde{L}(\mathbf{V}^n)$ given by

$$\tilde{L}(\mathbf{V}^n) = r \begin{bmatrix} -2\tilde{d}(v_1^n) & 2\tilde{d}(v_1^n) & & 0 \\ \tilde{d}(v_2^n) & -2\tilde{d}(v_2^n) & \tilde{d}(v_2^n) & \\ \ddots & \ddots & \ddots & \\ 0 & & 2\tilde{d}(v_K^n) & -2\tilde{d}(v_K^n) \end{bmatrix}.$$

Thus $L(\mathbf{V}^n)$ can be written as

$$L(\mathbf{V}^n) = L(\mathbf{V}^*) + \tilde{L}(\mathbf{V}^n), \quad \text{for any } n \geq 0, \tag{2.5}$$

where $\mathbf{V}^*$ is the K-vector given by $\mathbf{V}^* = (u^*, u^*, \dots, u^*)^t$.

Now, putting (2.5) on the left hand side of (2.3) and computing the term $L(\mathbf{V}^n)\mathbf{V}^{n+1}$ at the previous time level, we have

$$[\mathbf{I} - \theta L(\mathbf{V}^*)]\mathbf{V}^{n+1} = [\mathbf{I} + \phi L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n)]\mathbf{V}^n + \Delta t F(\mathbf{V}^n), \tag{2.6}$$

which requires a single factorization of the matrix $\mathbf{I} - \theta L(\mathbf{V}^*)$. We now show the stability of (2.6) in the sense of the invariance of W for the numerical solution.

3. Invariance properties of the method. In this section we shall deal with square nonnegative matrices, i.e., matrices $\mathbf{A} = (A_{ks})$, with $A_{ks} > 0$ for all k, s, which we shall denote by $\mathbf{A} > 0$. We now recall some known definitions and results:

Definition 3.1. A square matrix $\mathbf{A}$ with $A_{ks} < 0$, for $K \neq s$, is called an M-matrix if it is nonsingular and $\mathbf{A}^{-1} > 0$. Moreover, it is called a singular M-matrix if it is singular and if, for all scalar $\varepsilon > 0$, $\mathbf{A} + \varepsilon \mathbf{I}$ is an M-matrix.

Theorem 3.1. *If* $\mathbf{A}$*, with* $A_{ks} < 0$ *for all* $k \neq s$*, and* $\mathbf{A}_{kk} > 0$ *for all* k*, is an irreducible diagonally dominant matrix, then* $\mathbf{A}^{-1} > 0$.

We now show the following results:

Lemma 3.1. *If we suppose that the condition*

$$1 > -\phi L_{kk} - \theta \tilde{L}_{kk}, \qquad k = 1, \ldots, K, \tag{3.1}$$

is satisfied, then we have

(a) $[I - \theta L(V^*)]^{-1} \geq 0$,

(b) $||[I - \theta L(V^*)]^{-1}||_\infty \leq 1$,

(d) $||I + \phi L(V^n) + \theta \tilde{L}(V^n)||_\infty \leq 1$.

Proof. Observe that

$$L_{ks} \geq 0, \quad \text{for } k \neq s, \qquad L_{kk} \leq 0; \qquad \sum_s L_{ks} = 0. \tag{3.2}$$

Then, from the properties of the matrix L, $\mathbf{I} - \theta L(\mathbf{V}^*)$ is an irreducible M-matrix, and Theorem 2.1 ensures that the relation (a) is satisfied.

From (3.1) and (3.2) it follows that

$$\mathbf{I} + \theta L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n) \geq 0.$$

Consequently, one can easily prove (b) and (c). □

Now we can give the stability theorem:

Theorem 3.2. *If we fix the step* Δx *and if the time step* Δt *is such that*

$$1 - c\Delta t > -\phi L_{kk} - \theta \tilde{L}_{kk}, \quad \textit{for any } k = 1, \ldots, K, \tag{3.3}$$

where c *is the constant in* (1.3)*, then* $W = [0, b]$ *is invariant for the numerical solution given by* (2.6).

Proof. The proof will follow by induction. Let $\mathbf{B}$ be the K-vector given by $\mathbf{B} = (b, b, \ldots, b)^t$, and if we suppose that $\mathbf{V}^n - \mathbf{B} \leq 0$, that is $\mathbf{V}^n \leq \mathbf{B}$, then we can show that $\mathbf{V}^{n+1} - \mathbf{B} \leq 0$.

From (2.6) and $\tilde{L}(\mathbf{V}^n) = L(\mathbf{V}^n) - L(\mathbf{V}^*)$,

$$\begin{aligned}\mathbf{V}^{n+1} - \mathbf{B} = [\mathbf{I} - \theta L(\mathbf{V}^*)]^{-1}\{[\mathbf{I} + \phi L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n)](\mathbf{V}^n - \mathbf{B}) \\ + L(\mathbf{V}^n)\mathbf{B} + \Delta t F(\mathbf{V}^n)\}.\end{aligned} \tag{3.4}$$

From (3.1) we have

$$\{L(\mathbf{V}^n)B\}_k = \sum_{r=1}^{K} L_{kr} b = 0. \tag{3.5}$$

Furthermore,

$$F(\mathbf{V}^n) \leq -c(\mathbf{V}^n - B), \tag{3.6}$$

which follows in fact from

$$\{F(\mathbf{V}^n)\}_k = f(v_k^n) = f(b) + f'(\xi)(v_k^n - b), \qquad \xi \in (v_k^n, b),$$

(1.3) and (1.4).

Considering (3.3), (3.5) and (3.6) obtains

$$\begin{aligned}\{[\mathbf{I} + \phi L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n)](\mathbf{V}^n - B) + L(\mathbf{V}^n)B + \Delta t F(\mathbf{V}^n)\} \\ \leq [(1 - c\Delta t)\mathbf{I} + \phi L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n)](\mathbf{V}^n - \mathbf{B}) \leq 0,\end{aligned} \tag{3.7}$$

and, since (3.3) implies (3.1), $[\mathbf{I} - \theta L(\mathbf{V}^*)]^{-1}$ is a nonnegative matrix. Then, from the relations (3.3) and (3.7) we get $\mathbf{V}^{n+1} \leq \mathbf{B}$, i.e., $\mathbf{V}^{n+1} - \mathbf{B} < 0$. In the same way, we can show that, if $\mathbf{V}^n \geq 0$, it follows that $\mathbf{V}^{n+1} \geq 0$. Hence, we conclude that, when $\mathbf{V}^\circ$ belongs to W, $\mathbf{V}^{n+1}$ remains in W for all $n \geq 0$. □

Remark 3.1. The above result gives the conditions that we have to assume in order to have a positive numerical solution, which is most important in many applications.

Remark 3.2. In order to explain the stability condition (3.3) better, we observe that, if we require

$$1 - c\Delta t > 2\phi r d(u) + 2\theta r \tilde{d}(u), \quad \text{for any } u \in W,$$

or,

$$1 - c\Delta t > 2\phi r\beta + 2\theta r(\beta - \alpha),$$

then the stability condition (3.3) is verified. The last restriction can be replaced by the more practical condition on Δt,

$$\Delta t < \frac{1}{c + 2(\beta - \theta\alpha)/\Delta x^2} \tag{3.8}$$

which, Δx fixed, gives a bound on Δt in terms of α, β, c. From (3.8), because $\beta > \beta - \alpha$, it follows that the numerical scheme (2.6), with $\theta = 1$, requires a bound on Δt which is less restrictive than the one that we have to use if we employ (2.6) with $\theta = 0$.

The advantage becomes more effective the more the quotient $(\beta - \alpha)/\beta$ is less than 1. This is the case of diffusion function $d(u)$ with a weak dependence on u, that is, $d(u)$ almost constant on W.

4. An error bound. We now derive a bound for the error of the numerical scheme (2.6), given by

$$\mathbf{E}^n = \mathbf{U}^n - \mathbf{V}^n, \qquad n \geq 0,$$

where $\mathbf{U}^n$ denotes the exact solution at the spatial points of the interval $[0, 1]$:

$$\mathbf{U}^n = (u_1^n, \ldots, u_K^n)^t.$$

We define the local truncation error of (2.6) by

$$\tau_{n+1} = [\mathbf{I} - \theta L(\mathbf{V}^*)]\mathbf{U}^{n+1} - [\mathbf{I} + \Phi L(\mathbf{U}^n) + \theta \tilde{L}(\mathbf{U}^n)]\mathbf{U}^n + \Delta t F(\mathbf{U}^n), \tag{4.1}$$

for $n \geq 0$.

From the smoothness of the exact solution $u(x, t)$ of (1.1) and of $d(u)$ and $f(u)$, we can prove that

$$||\tau_{n+1}||_\infty \leq S_1 \Delta t h, \tag{4.2}$$

where S_1 is a positive constant independent of Δt, and Δx but dependent on u_{tt}, u_{xxxx} and $d(u)$, $f(u)$, with $h = \Delta t + \Delta x^2$.

Then we can prove the following theorem showing that, for a fixed h, the error does not exceed diam $W (= b)$ as time increases.

Theorem 4.1. *Under the hypotheses of Theorem* 3.2 *there exists a positive function* $S(t)$ *and a positive constant* p *such that*

$$(4.3)\quad ||E^{n+1}||_\infty \le \frac{e^{-pt_{n+1}}}{h + e^{-(p+S)t_{n+1}}}[||E^\circ||_\infty + Sh] + \frac{h}{h + e^{-(p+S)t_{n+1}}} b.$$

Proof. From (4.1) it follows that

(4.4)

$$[\mathbf{I} - \theta L(\mathbf{V}^*)]\mathbf{U}^{n+1} = [\mathbf{I} + \phi L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n)]\mathbf{U}^n + \Delta t F(\mathbf{U}^n) + \tau_{n+1} + \{\phi[L(\mathbf{U}^n) - L(\mathbf{V}^n)]\}\mathbf{U}^n + \{\theta[\tilde{L}(\mathbf{U}^n) - \tilde{L}(\mathbf{V}^n)]\}\mathbf{U}^n.$$

Thus, subtracting (2.6) by (4.4), we have

(4.5)

$$[\mathbf{I} - \theta L(\mathbf{V}^*)]\mathbf{E}^{n+1} = [\mathbf{I} + \phi L(\mathbf{V}^n) + \theta \tilde{L}(\mathbf{V}^n)]\mathbf{E}^n + \Delta t[F(\mathbf{U}^n) - F(\mathbf{V}^n)] + \tau_{n+1} + \{\phi[L(\mathbf{U}^n) - L(\mathbf{V}^n)]\}\mathbf{U}^n + \{\theta[\tilde{L}(\mathbf{U}^n) - \tilde{L}(\mathbf{V}^n)]\}\mathbf{U}^n.$$

Now consider that

$$\tilde{L}(\mathbf{U}^n) - \tilde{L}(\mathbf{V}^n) = L(\mathbf{U}^n) - L(\mathbf{V}^n).$$

Hence, if we put

$$w = \phi[L(\mathbf{U}^n) - L(\mathbf{V}^n)]\mathbf{U}^n + \theta[\tilde{L}(\mathbf{U}^n) - \tilde{L}(\mathbf{V}^n)]\mathbf{V}^n,$$

then

$$w_k = \{L(\mathbf{U}^n) - L(\mathbf{V}^n)]\mathbf{U}^n\}_k = \Delta t[d(u_k^n) - d(v_k^n)]\delta^2 u_k^n,$$

and, from the smoothness of $d(u)$,

$$(4.6)\quad ||w||_\infty \le S_2 \Delta t ||E^n||_\infty,$$

where S_2 is independent of Δt, and Δx by depending on $\max_{0\le t\le t_{n+1}} |u_{xxxx}(\cdot,t)|$. Moreover, we can see that

$$||F(\mathbf{U}^n) - F(\mathbf{V}^n)||_\infty \le S_3||\mathbf{E}_n||_\infty, \qquad n \ge 0, \tag{4.7}$$

with S_3 a positive constant.

Then, by considering that $||\mathbf{I} + \phi L(\mathbf{V}^n) + \theta\tilde{L}(\mathbf{V}^n)||_\infty \le 1$ and (4.2), (4.6) and (4.7), it follows that

$$||\mathbf{E}^{n+1}||_\infty \le (1 + S\Delta t)||\mathbf{E}^n||_\infty + S\Delta th, \tag{4.8}$$

where S depends on S_1, S_2, S_3.

From (4.8), by using the discrete Gronwall's lemma,

$$||\mathbf{E}^{n+1}||_\infty \le e^{St_{n+1}}(||\mathbf{E}^\circ||_\infty + Sh), \qquad n \ge 0. \tag{4.9}$$

Moreover, since $\mathbf{U}^n$ and $\mathbf{V}^n$ belong to W, then

$$||\mathbf{E}^n||_\infty \le b, \quad \text{for any } n \ge 0. \tag{4.10}$$

Now, if we consider $\sigma = h/(h + e^{-(p+S)t}n + 1)$ with $p > 0$ and multiply (4.10) by σ and (4.9) by $1 - \sigma$, then the result (4.3) follows. □

Remark 4.1. (4.3) shows the first order convergence of the numerical scheme (2.6), for a fixed time, and that the error is bounded by $\operatorname{diam} W = b$, as time increases and h is fixed.

5. Bidimensional case: m = 2. We now suppose that Ω is a spatial irregular domain of $\mathbf{R}^2$. Let Δx^j be the increment in the spatial variables x^j for $j = 1, 2$. Let $x_k = (k_1\Delta x^1, k_2\Delta x^2)$ be a point on the grid of Ω for k_1, k_2 integers and $k \in \mathcal{T}$, where $\mathcal{T}$ is an appropriate index set of $\mathbf{N}$. If $w : \Omega \to \mathbf{R}$, we denote by w_k the value of $w(x_k)$ for $k \in \mathcal{T}$ and indicate by δ_j^2 a discretization operator of the second derivative of w with respect to the variable x^j, $j = 1, 2$.

For example, we can approximate the second derivatives at an irregular mesh point P, as described in Figure 1 by the following formulas:

$$\delta_1^2 w(P) = \frac{h_3 w(P_1) - (h_1 + h_3)w(P) + h_1 w(P_3)}{h_1 h_3(h_1 + h_3)/2} \tag{5.1a}$$

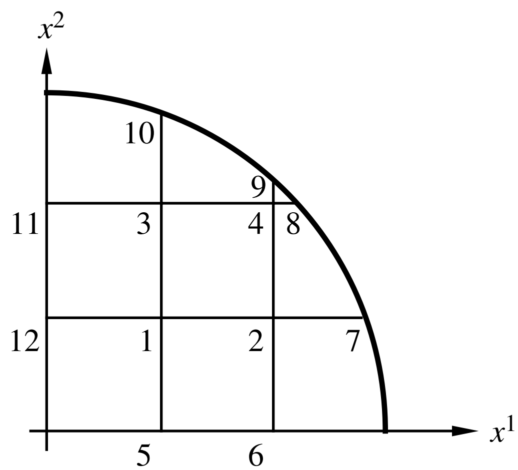

FIGURE 1.

and

$$\delta_2^2 w(P) = \frac{h_2 w(P_4) - (h_2 + h_4) w(P) + h_4 w(P_2)}{h_2 h_4 (h_2 + h_4)/2} \tag{5.1b}$$

(see [**10**]).

Instead, if P is a regular mesh point the previous formulas give the usual discretization operators for the spatial second derivatives.

Now, if we denote by v_k^n an approximation of the exact solution at x_k for any $k \in \mathcal{T}$ and at the time t_n, the θ-method can be formulated as

$$\frac{v_k^{n+1} - v_k^n}{\Delta t} = d(v_k^n) \sum_{j=1}^{2} \delta_j^2 (\theta v_k^{n+1} + \phi v_k^n) + f(v_k^n), \tag{5.2}$$

for $k \in \mathcal{T}$, $n \geq 0$. Then, setting $d(v_k^n) = \tilde{d}(v_k^n) + d(v^*)$ and discussing, as in section 2, from (5.2) we obtain the proposed method.

It follows that (5.2) can be written in compact form as in (2.3), where L is now a full matrix, while the new section can be written as in (2.6).

For example, if Ω is the irregular domain of $\mathbf{R}^2$ shown here

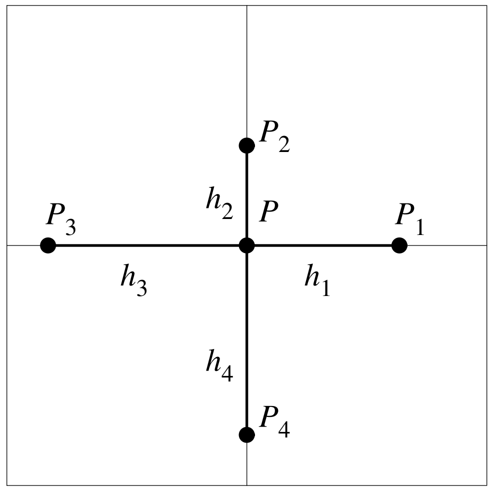

FIGURE 2. Treatment of an irregular mesh point.

and, using (5.1) to discretize the spatial derivatives, we have that the structure of $L(\mathbf{V}^n)$ will be of the form

$$L(\mathbf{V}^n) = \begin{bmatrix} l_{11} & l_{12} & l_{13} & 0 \\ l_{21} & l_{22} & 0 & l_{23} \\ l_{31} & 0 & l_{33} & l_{34} \\ 0 & l_{42} & l_{43} & l_{44} \end{bmatrix},$$

where the elements of $L(\mathbf{V}^n)$ depend on the spatial steps and on the diffusion.

In the case $m = 2$, the results of sections 3 and 4 can be derived in the same way.

6. A numerical example. In order to furnish a practical application of (1.8), we consider the one-dimensional controlled temperature problem, studied in [**4**]:

$$v_t = \mathbf{A}(v)v_{xx}, \qquad 0 < x < 1,\ 0 < t < 1,$$

subject to the initial and boundary conditions

$$\begin{aligned} &v(x,0) = 0, & & 0 < x < 1, \\ &v(0,t) = f_0(t), \qquad v(1,t) = 0, & & 0 < t < 1, \end{aligned}$$

where we assume

$$A(v) = 0.1 + v^2, \qquad 0 < v < 1$$
$$f_0(t) = t, \qquad 0 < t < 1.$$

In column III of the table which follows we show the numerical solution obtained by (1.8) for $\Delta t = 0.1$, $\Delta x = 0.05$ at the time $t = 1$ and at the spatial mesh points. It is compared with the results given by the implicit Euler method (IEM) for the same steps (column II) and with the exact solution (ES) which has been obtained by the Euler explicit scheme, for very small steps (column I).

TABLE

x	ES	IEM	(1.8)	ES-IEM	ES-(1.9)
.05	.89E+00	.05E+00	.87E+00	.40E+00	.23E-01
.10	.79E+00	.43E+00	.76E+00	.36E+00	.32E-01
.15	.69E+00	.36E+00	.65E+00	.33E+00	.41E-01
.20	.59E+00	.31E+00	.55E+00	.29E+00	.48E-01
.25	.50E+00	.25E+00	.45E+00	.25E+00	.52E-01
.30	.42E+00	.21E+00	.37E+00	.21E+00	.54E-01
.35	.34E+00	.17E+00	.27E+00	.17E+00	.52E-01
.40	.27E+00	.14E+00	.23E+00	.14E+00	.46E-01
.45	.21E+00	.11E+00	.18E+00	.11E+00	.38E-01
.50	.16E+00	.86E-01	.14E+00	.79E-01	.30E-01
.55	.12E+00	.67E+00	.10E+00	.57E-01	.22E-01
.60	.93E-01	.53E-01	.78E-01	.41E-01	.15E-01
.65	.69E-01	.41E-01	.59E-01	.28E-01	.98E-02
.70	.50E-01	.31E-01	.44E-01	.19E-01	.61E-02
.75	.36E-01	.23E-01	.33E-01	.13E-01	.33E-02
.80	.25E-01	.17E-01	.23E-01	.82E-02	.19E-02
.85	.17E-01	.12E-01	.16E-01	.51E-02	.88E-03
.90	.10E-01	.75E-02	.10E-01	.29E-02	.35E-03
.95	.49E-02	.36E-02	.48E-02	.13E-02	.11E-03

These numerical results show that (1.8) is advantageous on the IEM because the first method requires only one factorization of a matrix for all time levels, while the IEM needs a factorization at each time step. The columns IV, V show a better accuracy of (1.8) on IEM, with respect to the exact solution in column I. Moreover, (1.8) permits a larger time step than that of the explicit Euler method. In fact, in this case, in order to have the stability, we have to assume $\Delta t \cong 10^{-3}$.

Acknowledgment. Thanks are given to Professor Vincenzo Casulli of the University of Trento for helpful discussions on the subject.

REFERENCES

1. K.N. Chueh, C.C. Conley and J.A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana J. Math. **26** (1977), 373–392.

2. J. Douglas, *Effective time-stepping methods for the numerical solution of nonlinear parabolic problems*, in *The Mathematics of finite elements and applications* III (J.R. Whiteman ed.), Academic Press, 1978, 289–304.

3. ———, T. Dupont and P. Percell, *A time-stepping method for Galerkin approximations for nonlinear parabolic equations*, Lecture Notes in Math. **630** Springer, Berlin, 1978, 67–75.

4. P. Duchateau, *Boundary behavior and monotonicity estimates for the solutions to nonlinear diffusion equations*, Rocky Mountain J. Math. **15** (1985), 769–786.

5. D. Hoff, *Stability and convergence of finite difference methods for systems of nonlinear reaction-diffusion equations*, SIAM J. Num. Anal. **15** (1978), 1161–1177.

6. L. Lopez, *Stability and asymptotic behaviour for the numerical solution of a reaction-diffusion model for a deterministic diffusive epidemic*, IMA J. Num. Anal. **13** (1983), 341–351.

7. C. Mastroserio, M. Montrone, *Stability of invariant regions for an alternating direction method for systems of nonlinear reaction-diffusion equations*, CALCOLO V. XIX, (1982), 253–268.

8. R.D. Richtmyer and K.W. Morton, *Difference methods for initial value problems*, Interscience, New York, 1967.

9. R.S. Varga, *Matrix iterative analysis*, Prentice-Hall, New York, 1962.

10. D.M. Young and R.T. Gregory, *A survey of numerical mathematics*, Vol. II, Addison-Wesley Publishing Company, Reading, MA, 1972.

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