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ON ITERATED TORSION PRODUCTS OF ABELIAN p-GROUPS

PATRICK KEEF

ABSTRACT. The question of when $\mathrm{Tor}\,(A_1, \ldots, A_n)$ is a dsc for abelian groups $A_1, \ldots, A_n$ is discussed. The proofs involve inductions on both the number and cardinality of the groups. When $A_1, \ldots, A_n$ have countable length and cardinality $\aleph_n$, necessary and sufficient conditions are given using invariants from set theory.

0. Introduction. In this paper the term "group" will be used to mean an abelian p-group for some fixed prime p (except for a momentary consideration of cotorsion groups in Theorem 12). In [**12**] Nunke investigates the question of when $\mathrm{Tor}\,(A, B)$ is a direct sum of countable groups (dsc). He arrives at an answer (cf ., our Theorem 1) in the case where A and B have different lengths. The case where the lengths are the same was left unresolved. In that paper Nunke also looks at iterated torsion products with an eye toward constructing $\aleph_n$-*cyclic* groups (groups whose subgroups of cardinality strictly less than $\aleph_n$ are direct sums of cyclics). A special case of Nunke's question is considered by Hill [**5**]. He arrives at separate necessary and sufficient conditions for $\mathrm{Tor}\,(A, B)$ to be a direct sum of cyclic groups which essentially induct on the cardinality of the groups involved. These conditions are generalized in Keef [**7**] which considers when $\mathrm{Tor}\,(A, B)$ is a dsc whose length is a limit ordinal.

The purpose of the present paper is to generalize the above results in two ways. First, instead of simply considering when $\mathrm{Tor}\,(A, B)$ is a dsc, we look at the question of when $\mathrm{Tor}\,(A_1, \ldots, A_n)$ is a dsc. This allows us not only to induct on the cardinality of the various groups, but also on the number of groups involved. We also discuss the case where the length of the groups involved is an isolated countable ordinal. Perhaps the most interesting result of the present paper is a determination of when $\mathrm{Tor}\,(A_1, \ldots, A_n)$ is a dsc of countable length and cardinality $\aleph_n$ (Theorem 10).

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One problem of the inductive approaches of [**5**] and [**7**] is that the answers given seem to depend upon selecting ascending chains of subgroups with certain required properties. In the present paper we borrow ideas from set theory and the study of κ-*free* groups to present results which do not employ particular chains of subgroups. Specifically, we introduce an invariant ΓA which has values in a certain quotient boolean algebra which is a simple generalization of a construction in this other area.

To quickly summarize the contents of this work, the first section is concerned with preliminaries and definitions. In particular, we present a generalization of the notion of a C_λ group to the case where λ is an isolated countable ordinal. The second section is concerned with applying these notions to iterated torsion products. Some of these results apply when the groups have different cardinalities and others when the cardinalities are the same. The third section is devoted to extending some of these results to the case where λ is uncountable. By a result of [**7**] we are led to a consideration of the *balanced projective dimension* (b.p.d.) of the groups involved. A generalization of the notion of a κ-free group is given involving the b.p.d. of a group. A version of Shelah's singular compactness theorem is presented. These notions allow us to generalize some of the results of the previous section to larger cardinalities, provided the groups involved are "almost" of b.p.d. at most 1.

Briefly reviewing some standard notation and ideas, we identify a cardinal with the first ordinal of that cardinality. The cardinality of a group A will be denoted by $|A|$. If α is an ordinal, let $A(\alpha) = \{a \in A : ht(a) \geq \alpha\}$ and $l(A)$ be the *length* of A, i.e., the smallest ordinal such that $A(\alpha) = A(\alpha + 1)$. By an α-*high* subgroup we mean a subgroup maximal with respect to trivial intersection with $A(\alpha)$. We assume familiarity with the standard facts on the Tor functor (see [**12**]) as well as those on α-*pure* exact sequences (see [**10**] and [**11**]), e.g., an α-high subgroup is $\alpha + 1$-pure.

1. In [**8**], for a limit ordinal $\lambda \leq \Omega$, Megibben calls G a C_λ group if for every $\alpha < \lambda$ the group $G/G(\alpha)$ is a dsc. We wish first to extend this definition to isolated countable ordinals. If $\lambda < \Omega$ is isolated it is easy to see that as it stands the above merely requires that $G/G(\lambda)$ is a dsc. An alternative, which will be more appropriate for our purposes, is to define a C_λ group to be a group which has, for every ordinal $\alpha < \lambda$, an α-high subgroup which is a dsc. By [**7**, Theorem 8], when λ is a limit ordinal these two definitions agree. That they say different things when λ is isolated can be seen by observing that if G is a group with an ω-high subgroup which is a direct sum of cyclic groups, then G is a $C_{\omega+1}$ group under the second definition. It is easy to construct such a G with the further property that $G/G(\omega)$ is not a direct sum of cyclics and such a G fails to satisfy the first definition. In this paper we will use this second extension of Megibben's definition.

Throughout this paper we will let λ denote an ordinal not exceeding Ω. Observe that if λ is an isolated ordinal, then a group G is a C_λ if and only if it has a $(\lambda - 1)$-high subgroup which is a dsc. To see this, note that, for any $\alpha < \lambda$, an α-high subgroup can be extended to a $(\lambda - 1)$-high subgroup, and, by a result of Nunke [**12**], if one $(\lambda - 1)$-high subgroup is a dsc then they all are. The result then follows by a well-known result of Hill's on isotype subgroups of dsc's [**3**]. Observe that this easily implies that when λ is isolated, a C_λ group G with $G(\lambda) = 0$ is *summable* (i.e., its socle is free as a valuated vector space). This follows since K is a $(\lambda - 1)$-high subgroup of G, then $G[p] \cong K[p] \oplus G(\lambda - 1)[p]$ and since K is a dsc, $K[p]$ is summable. This is in marked contrast to the situation at limit ordinals (cf. [**8**]).

The results in this paper will often be stated for all $\lambda \leq \Omega$, but we will occasionally ignore the case where λ is finite. We leave it to the reader to fill in the usually trivial details of this case. The advantage of infinite values of λ is that λ-pure sequences are pure in the usual sense of the word and the Tor functor is exact on the category of pure exact sequences. The triviality of the finite case is due to the fact that bounded groups are direct sums of cyclics.

We note in passing the following simple fact.

Proposition 1. *If K is an isotype subgroup of the C_λ group G, then K is also a C_λ group.*

Proof. If $\alpha < \lambda$ and L is an α-high subgroup of K, then L is contained as an isotype subgroup of an α-high subgroup of G. Since α is countable, Hill's theorem on isotype subgroups of dsc's implies the result. □

One nice feature of our definition is that it allows us to recast a result of Nunke's [**12**].

Theorem 1. *Let A and B be reduced groups with $\lambda = l(A) < l(B)$. Then* $\mathrm{Tor}\,(A,B)$ *is a* dsc *if and only if A is a* dsc *and B is a C_λ group.*

The following characterization of C_λ groups is a slight variation on Theorem 1.

Theorem 2. *Let G be a group and H a reduced* dsc *of length λ. Then G is a C_λ if and only if* $\mathrm{Tor}\,(G,H)$ *is a* dsc.

Proof. When λ is a limit ordinal, the result follows from induction since H is the direct sum of dsc's of shorter length. If λ is isolated and K is a $(\lambda-1)$-high subgroup of G, then

$$0 \to \mathrm{Tor}\,(K,H) \to \mathrm{Tor}\,(G,H) \to \mathrm{Tor}\,(G/K,H)(\cong \oplus H) \to 0$$

is λ-pure exact and hence splitting. So $\mathrm{Tor}\,(G,H)$ is a dsc if and only if $\mathrm{Tor}\,(K,H)$ is a dsc and the result follows from Theorem 1. □

The last theorem can be used to give the following necessary and sufficient condition for a C_λ group of length λ to be a dsc. We say a C_λ group G is *normal* if $G(\lambda) = 0$.

Proposition 2. *If G is a normal C_λ group, then G is a* dsc *if and only if G is a λ-pure projective.*

Proof. Necessity being clear, we suppose G is a λ-pure projective. If λ is a limit ordinal, the result is clear using Megibben's definition of C_λ and Nunke's homological characterization of dsc's (see [**11**]). If λ is isolated, let L be a $(\lambda-1)$-high subgroup of a dsc group H of length λ. Since G is a λ-pure projective, the sequence

$$0 \to \mathrm{Tor}\,(L,G) \to \mathrm{Tor}\,(H,G) \to \oplus G \to 0$$

splits because it is λ-pure. So since, by Theorem 2, $\mathrm{Tor}\,(H,G)$ is a dsc, so is G. □

The following lemma will be crucial in ensuring that we can construct chains of subgroups which are “well-behaved.” We let H_λ denote the “generalized Prüfer group” of length λ.

Lemma 1. *Suppose K is a pure subgroup of G. Then* $\operatorname{Tor}(K, H_\lambda)$ *is a summand of* $\operatorname{Tor}(G, H_\lambda)$ *if and only if K is a λ-pure subgroup of G. If, in addition, G is a C_λ group, then so is G/K.*

Proof. If K is a λ-pure subgroup, then

$$0 \to \operatorname{Tor}(K, H_\lambda) \to \operatorname{Tor}(G, H_\lambda) \to \operatorname{Tor}(G/K, H_\lambda) \to 0$$

is also λ-pure, and since the last group is λ-pure projective it must split.

For the converse, recall that for a group A there is a (natural) map $\partial_A : \operatorname{Tor}(A, H_\lambda) \to A$ and a short exact sequence $X \to Y \xrightarrow{\alpha} A$ is λ-pure exact if and only if ∂_A factors through α (see [**10**]). Suppose $\pi : G \to G/K$ is the natural projection. If

$$\phi : \operatorname{Tor}(G/K, H_\lambda) \to \operatorname{Tor}(G, H_\lambda)$$

is a right inverse for $\operatorname{Tor}(\pi, 1_{H_\lambda})$, then we have

$$\pi \circ (\partial_G \circ \phi) = \partial_{G/K} \circ \operatorname{Tor}(\pi, 1_{H_\lambda}) \circ \phi = \partial_{G/K}$$

and so K is a λ-pure subgroup of G.

The last statement follows from Theorem 2. □

We come now to a definition which is fundamental for the results of this paper. Let G be an infinite C_λ group. By a λ-*development* of G we mean an ascending chain of subgroups $\{G_i\}_{i \in I}$ of G, where $I = [0, \alpha]$ for some limit ordinal α such that we have

(a) for all $i < \alpha$, G_i is a λ-pure subgroup of G;

(b) $G_0 = 0$, $G_\alpha = G$;

(c) if $i \le \alpha$ is a limit ordinal, $\cup_{j<i} G_j = G_i$ (i.e., the chain is *smooth*).

If, in addition to the above, we have

(d) $|G_i| < |G|$ for all $i < \alpha$,

we will call the λ-development *proper*. Finally, if in addition to (a)–(d) we have

(e) $(G_{i+1}/G_i)(\lambda) = 0$ if and only if $(G/G_i)(\lambda) = 0$, for all $i < \alpha$,

we will call the λ-development *normal*. Observe one consequence of Lemma 1: if $\{G_i\}$ is a λ-development of G, then for each i the groups G_{i+1}/G_i and G/G_i are C_λ groups.

Theorem 3. *Suppose A and B are normal C_λ groups with $|\lambda| \leq |A| < |B|$. Then* $\mathrm{Tor}\,(A,B)$ *is a* dsc *if and only if B has a λ-development $\{B_i\}$ such that, for each i,* $\mathrm{Tor}\,(A, B_{i+1}/B_i)$ *is a* dsc. *Furthermore, if* $\mathrm{Tor}\,(A,B)$ *is a* dsc, *we may choose the λ-development to be normal.*

Proof. If B has a λ-development as stated, then each sequence

$$0 \to \mathrm{Tor}\,(A, B_i) \to \mathrm{Tor}\,(A, B_{i+1}) \to \mathrm{Tor}\,(A, B_{i+1}/B_i) \to 0$$

must split because it is λ-pure and the last group is a λ-pure projective. So $\mathrm{Tor}\,(A,B)$ is clearly isomorphic to the direct sum of the groups $\mathrm{Tor}\,(A, B_{i+1}/B_i)$, which proves this part.

Conversely, if $\mathrm{Tor}\,(A,B)$ is given to be a dsc, we construct the required λ-development as follows. Suppose $\{x_i\}_{i<|B|}$ is a well-ordering of B. Observe that by Theorem 2, $\mathrm{Tor}\,(H_\lambda, B)$ is a dsc. Fix decompositions of $\mathrm{Tor}\,(A,B)$ and $\mathrm{Tor}\,(H_\lambda, B)$. Let $B_0 = 0$. Suppose, for some $j \leq |B|$ we have constructed a smooth ascending chain $\{B_i\}_{i<j}$ satisfying

(a) B_i is pure,

(b) $\mathrm{Tor}\,(A, B_i)$ and $\mathrm{Tor}\,(H_\lambda, B_i)$ are the direct sums of certain terms in the decompositions of $\mathrm{Tor}\,(A,B)$ and $\mathrm{Tor}\,(H_\lambda, B)$, respectively,

(c) $|B_i| \leq |A||i|$,

(d) if $i+1 < j$, then $x_i \in B_{i+1}$,

(e) if $i+1 < j$, then $(B_{i+1}/B_i)(\lambda) = 0$ if and only if $(B/B_i)(\lambda) = 0$.

If j is a limit ordinal, then smoothness dictates our choice of B_j. If $j = i+1$ is isolated, we begin by constructing an extension K_0 of B_i

which contains x_i so that $(K_0/B_i)(\lambda) \neq 0$ if and only if $(B/B_i)(\lambda) \neq 0$. Using a "back-and-forth" technique, we can construct extensions K_n which are alternately pure subgroups and subgroups for which $\operatorname{Tor}(A, K_n)$ and $\operatorname{Tor}(H_\lambda, K_n)$ are the direct sum of certain terms in fixed decompositions of $\operatorname{Tor}(A, B)$ and $\operatorname{Tor}(H_\lambda, B)$, respectively. This can clearly be done so that $|K_n| \leq |A||i|$. Let $B_j = \cup_n K_n$. Observe that conditions (a) and (b) together with Lemma 1 imply that B_j is a λ-pure subgroup of B (cf. case II of [**5**, Theorem 2]). $\square$

The last result is a generalization of [**7**, Theorem 17]. The present proof, instead of employing the notions of λ-*isotype* and λ-*nice* subgroups in the successive building of extensions, uses Lemma 1. This not only has the advantage of simplicity, but it also applies equally well to the case where λ is isolated. That this is a true generalization follows from the fact that when λ is a limit ordinal and A is a C_λ group, a sequence $X \to Y \to A$ is λ-pure if and only if it is λ-*balanced* (see [**11**, Theorem 2.9]). We will use this technique frequently in our proofs to guarantee that we are, in fact, creating λ-pure subgroups.

Corollary 1. *If B is a C_λ group with $|B| > |\lambda|$, then B has a normal λ-development.*

Proof. Use the last result with $A = H_\lambda$. $\square$

If α is an uncountable cardinal and S_1, S_2 are subsets of $[0, \alpha)$, define $S_1 \sim S_2$ if $S_1 \cap C = S_2 \cap C$ for some *cub* (closed and unbounded) $C \subseteq [0, \alpha)$. This can easily be seen to be an equivalence relation and the usual set operations can be performed on the various equivalence classes. Another way to view this construction is to define I to be the collection of all $S \subseteq [0, \alpha)$ such that $S \cap C = \varnothing$ for some cub C. It can be seen that I is an ideal in $\mathcal{P}([0, \alpha))$ and the equivalence classes mentioned are simply cosets in the quotient boolean algebra. Denote this quotient boolean algebra by ρ_α. Let $0 = [\varnothing]$ and $1 = [[0, \alpha)]$.

Recall that a cardinal α is called *regular* if the cofinality of α is α. Otherwise, it is called *singular*. Suppose $\alpha > |\lambda|$ is a regular cardinal and A is a C_λ group of cardinality α. If $\{B_i\}_{i<\alpha}$ is a normal

λ-development of A, let

$$\Gamma'\{B_i\} = \{i < \alpha : (A_{i+1}/A_i)(\lambda) \neq 0\}$$

and $\Gamma A = [\Gamma'\{B_i\}]$ in ρ_α.

Lemma 2. *With the above notation,* ΓA *does not depend on the particular normal* λ*-development chosen.*

Proof. If $\{B_i'\}$ is another normal λ-development, then by the regularity of α,

$$C = \{i < \alpha : B_i = B_i'\}$$

is a cub, and by the normality condition,

$$\Gamma'\{B_i\} \cap C = \Gamma'\{B_i'\} \cap C,$$

hence $[\Gamma'\{B_i\}] = [\Gamma'\{B_i'\}]$ in ρ_α. □

If we are given a normal λ-development of a C_λ group and we select from it those terms corresponding to some cub, the result is another normal λ-development (which we will call a *subdevelopment*). So if $\{B_i\}$ is a normal λ-development for A (where $|A| > \lambda$ is regular) and $\Gamma A = 0$, then by passing to a subdevelopment, we may assume $\Gamma'\{B_i\} = \varnothing$.

When there are several cardinals being discussed we may use the notation $\Gamma_\alpha A$ for emphasis. We may also use $\Gamma_n A$ for $\Gamma_{\aleph_n} A$.

Observe that if A is a C_λ group with $|A| \leq |\lambda|$, then A may not have a proper λ-development. In this case we define $\Gamma_{|A|} A$ to be $0 = [\varnothing]$ if A is a dsc and $1 = [[0, |A|)]$ otherwise. This is consistent with the following observations:

Proposition 3. *Suppose* A *is a normal* C_λ *group.*

(a) *If* $|A| = \aleph_1$*, then* $\Gamma A = 0$ *if and only if* A *is a* dsc.

(b) *If* $|A| = \aleph_2$ *and* $\lambda = \Omega$*, then* $\Gamma A = 0$ *if and only if the b.p.d. of* A *is at most* 1.

Proof. In (a) we may clearly consider only the case where λ is countable. If A is a dsc, then letting the B_i be the direct sum of certain terms in a decomposition of A, we clearly have $\Gamma'\{B_i\} = \varnothing$. Conversely, if $\Gamma'\{B_i\} = \varnothing$, then for each $i < \omega_1$, the short exact sequence $B_i \to B_{i+1} \to B_{i+1}/B_i$ must split, giving a decomposition of A, as required, into countable groups.

(b) is essentially a restatement of [**7**, Theorem 22]. ◻

2. Recall that the iterated torsion product is defined inductively by

$$\mathrm{Tor}\,(A_1, \dots, A_n) = \mathrm{Tor}\,(\mathrm{Tor}\,(A_1, \dots, A_{n-1}), A_n).$$

It is consistent with the above to define $\mathrm{Tor}\,(A) = A$. Because of standard commutativity and associativity relations, the iterated torsion product can be "built up" in many different ways from products involving fewer terms.

Proposition 4. (a) *If $A_1, \dots, A_n$ are C_λ groups, then* $\mathrm{Tor}\,(A_1, \dots,$

$A_n)$ *is a C_λ group.*

(b) *If* $\mathrm{Tor}\,(A_1, \dots, A_n)$ *is a C_λ group of length at least λ, then each A_j is a C_λ group.*

Proof. By induction we may assume $n = 2$. Let H be a dsc of length λ. In (a) we may assume $l(A_j) \geq \lambda$ for each j (otherwise replace A_j by $A_j \oplus H$). Considering the isomorphism

$$\mathrm{Tor}\,(\mathrm{Tor}\,(A_1, A_2), H) \cong \mathrm{Tor}\,(A_1, \mathrm{Tor}\,(A_2, H)),$$

Theorem 2 now implies this part. As to (b), let $\alpha < \lambda$ and K be an α-high subgroup of A_1. Then $\mathrm{Tor}\,(K, A_2)$ is an isotype subgroup of $\mathrm{Tor}\,(A_1, A_2)$ of length at most α, so it must be a dsc. So by Theorem 1, K must be a dsc. Therefore, A_1, and similarly A_2, is a C_λ group. ◻

Suppose we wish to decide when $\mathrm{Tor}\,(A_1, \dots, A_n)$ is a dsc of length λ for reduced groups $A_1, \dots, A_n$. If we renumber the groups so that $l(A_j) = \lambda$ for $j \leq k$ and $A_j(\lambda) \neq 0$ for $j > k$, then by Theorem 1 and our last proposition,

$$\mathrm{Tor}\,(A_1, \dots, A_n) \cong \mathrm{Tor}\,(\mathrm{Tor}\,(A_1, \dots, A_k), \mathrm{Tor}\,(A_{k+1}, \dots, A_n))$$

is a dsc if and only if $\operatorname{Tor}(A_1, \ldots, A_k)$ is a dsc. So in deciding when $\operatorname{Tor}(A_1, \ldots, A_n)$ is a dsc, there is no loss in generality in restricting to the case where each A_j is a normal C_λ group.

Theorem 4. *Suppose $A_1, \ldots, A_n$ are normal C_λ groups with λ-developments $\{B_{1,i}\}, \ldots, \{B_{n,i}\}$. If for each i and j, the group*

$$X_{j,i} = \operatorname{Tor}(B_{1,i}, \ldots, B_{j-1,i}, B_{j,i+1}/B_{j,i}, B_{j+1,i+1}, \ldots, B_{n,i+1})$$

is a dsc*, then* $\operatorname{Tor}(A_1, \ldots, A_n)$ *is a* dsc.

Proof. For each i and j there a is a λ-pure short exact sequence

$$\operatorname{Tor}(B_{1,i}, \ldots, B_{j,i}, B_{j+1,i+1}, \ldots, B_{n,i+1}) \\ \to \operatorname{Tor}(B_{1,i}, \ldots, B_{j-1,i}, B_{j,i+1}, \ldots, B_{n,i+1}) \to X_{j,i}$$

which clearly must split. Therefore, $\operatorname{Tor}(A_1, \ldots, A_n)$ is isomorphic to the direct sum of the $X_{j,i}$. □

In the remainder of this paper, the notation $X_{j,i}$ will be reserved for the above iterated torsion product. The above condition can be used to prove

Theorem 5. *Suppose $A_1, \ldots, A_n$ are normal C_λ groups of cardinality at most $\aleph_{n-1}$. Then* $\operatorname{Tor}(A_1, \ldots, A_n)$ *is a* dsc *group.*

Proof. We may clearly assume $|A_j| = \aleph_{n-1}$ for each j (otherwise replace A_j by the direct sum of $\aleph_{n-1}$ copies of A_j). The result is clear for $n = 1$. When $n = 2$ and $\lambda = \Omega$ the result is [**6**, Theorem 6]. So assume the result is true for $n - 1 \geq 1$ and $\aleph_{n-1} > |\lambda|$. By Theorem 3 there are proper λ-developments $\{B_{1,i}\}, \ldots, \{B_{n,i}\}$ of $A_1, \ldots, A_n$. By induction and Theorem 2, each of the groups

$$X_{j,i} \cong \operatorname{Tor}(\operatorname{Tor}(B_{1,i}, \ldots, B_{j-1,i}, B_{j+1,i+1}, \ldots, B_{n,i+1}) B_{j,i+1}/B_{j,i})$$

is a dsc, so the result follows from the last theorem. □

Corollary 2. *If $A_1, \ldots, A_n$ are a normal C_λ group, and λ is countable, then any isotype subgroup of* $\operatorname{Tor}(A_1, \ldots, A_n)$ *of cardinality at most $\aleph_{n-1}$ is a* dsc *group.*

Proof. Any isotype subgroup of cardinality at most $\aleph_{n-1}$ is contained in a subgroup of the form $\mathrm{Tor}(K_1, \ldots, K_n)$, where K_j is an isotype subgroup of A_j of cardinality at most $\aleph_{n-1}$. Since isotype subgroups of dsc groups of countable length are dsc groups, the result follows from the preceding theorem and Proposition 1. □

Corollary 3. *Suppose $A_1, \ldots, A_n$ are normal C_λ groups of cardinality $\aleph_n$. If $\Gamma A_1, \ldots, \Gamma A_n$ are all* 0*, then* $\mathrm{Tor}(A_1, \ldots, A_n)$ *is a* dsc.

Proof. Passing to subdevelopments, if necessary, suppose $\{B_{1,i}\}, \ldots, \{B_{n,i}\}$ are normal λ-developments such that for each i and j, $(B_{j,i+1}/B_{j,i})(\lambda) = 0$. By Theorem 5 each $X_{j,i}$ is a dsc, so by Theorem 4 we are done. □

We will have use for the following simple observation.

Lemma 3. *Suppose $A_1, \ldots, A_n$ are C_λ groups of cardinality at most $\aleph_{n-1}$. If $A_1, \ldots, A_{n-1}$ are normal, then* $\mathrm{Tor}(A_1, \ldots, A_n)$ *is a* dsc *if and only if either* $\mathrm{Tor}(A_1, \ldots, A_{n-1})$ *is a* dsc *or A_n is normal.*

Proof. If A_n is normal, this follows from Theorem 5. If $A_n(\lambda) \neq 0$, it follows from Theorem 1. □

Theorem 6. *Suppose $A_1, \ldots, A_n$ are normal C_λ groups with $|\lambda| \leq |A_1| < \cdots < |A_n|$, where for each j, $|A_j|$ is regular. If* $\mathrm{Tor}(A_1, \ldots, A_n)$ *is a* dsc*, then, for some j, $\Gamma_{|A_j|} A_j = 0$.*

Proof. If A_1 is a dsc, we are clearly done, so suppose it is not. Choose j such that $\mathrm{Tor}(A_1, \ldots, A_{j-1})$ is not a dsc, but

$$\mathrm{Tor}(A_1, \ldots, A_j) \cong \mathrm{Tor}(\mathrm{Tor}(A_1, \ldots, A_{j-1}), A_j)$$

is a dsc. By Theorem 3, A_j has a normal λ-development $\{B_i\}$ such that for all i,

$$\mathrm{Tor}(\mathrm{Tor}(A_1, \ldots, A_{j-1}), B_{i+1}/B_i)$$

is a dsc. By the choice of j and Theorem 1, we must have $(B_{i+1}/B_i)(\lambda) = 0$ for all i, and we are done. □

The following observation can be viewed as a generalization of Nunke's construction of $\aleph_n$ -cyclic groups (see [**12**]).

Corollary 4. *If λ is countable and $A_1, \ldots, A_n$ are normal, infinite C_λ groups of strictly increasing, regular cardinality, with $\Gamma_{|A_j|}A_j \neq 0$ for each j, then* $\operatorname{Tor}(A_1, \ldots, A_n)$ *is not a* dsc*, but every isotype subgroup of cardinality at most $\aleph_{n-1}$ is a* dsc.

Proof. This is a result of Corollary 2 and Theorem 6. □

The following indicates that for sufficiently small cardinals, the converse of Theorem 6 is valid.

Theorem 7. *Suppose for $j = 1, \ldots, n$, C_j is a normal C_λ of cardinality $\aleph_j$. Then* $\operatorname{Tor}(C_1, \ldots, C_n)$ *is a* dsc *if and only if there is a j such that $\Gamma_j C_j = 0$.*

Proof. Necessity being Theorem 6, suppose $\{B_i\}$ is a normal λ-development of C_j with $(B_{i+1}/B_i)(\lambda) = 0$ for all i. By Theorem 5, $\operatorname{Tor}(C_1, \ldots, C_{j-1}, B_{i+1}/B_i)$ is a dsc for each i, so by Theorem 3, $\operatorname{Tor}(C_1, \ldots, C_j)$, and hence $\operatorname{Tor}(C_1, \ldots, C_n)$ is a dsc. □

The remainder of this section we will be primarily concerned with the case where λ is countable. In the next section we will present versions of these results for $\lambda = \Omega$.

Theorem 8. *Suppose λ is countable, $A_1, \ldots, A_n$ are normal C_λ groups of equal uncountable cardinality. If* $\operatorname{Tor}(A_1, \ldots, A_n)$ *is a* dsc*, then there are normal λ-developments $\{B_{1,i}\}, \ldots, \{B_{n,i}\}$ such that for all j and i,*

$$Y_{j,i} = \operatorname{Tor}(B_{1,i}, \ldots, B_{j-1,i}, B_{j,i+1}/B_{j,i}, B_{j+1,i}, \ldots, B_{n,i})$$

is a dsc.

Proof. Using a "back-and-forth" argument as in Theorem 3, we can construct normal λ-balanced towers $\{B_{j,i}\}$, such that, for all i,

$\mathrm{Tor}\,(B_{1,i}, \dots, B_{n,i})$ is a summand of $\mathrm{Tor}\,(A_1, \dots, A_n)$, and hence also of

$$\mathrm{Tor}\,(B_{1,i}, \dots, B_{j-1,i}, B_{j,i+1}, B_{j+1,i}, \dots, B_{n,i}).$$

This group is a dsc by Hill's theorem, and so the quotient, which is isomorphic to $Y_{j,i}$, is also a dsc. □

We introduce now some additional notation. If $\{B_i\}$ is a normal λ-development for the normal C_λ group A, and $|A|$ is regular, let

$$\Lambda'\{B_i\} = \{i : B_i \text{ is not a dsc}\}$$

and $\Lambda A = [\Lambda'\{B_i\}]$ in $\rho_{|A|}$. As in Lemma 2, ΛA is independent of the particular λ-development employed. Observe that if λ is countable, then by Hill's theorem ΛA is either 0 or 1. Also, if $A_1, \dots, A_n$ are normal C_λ groups of the same regular cardinality, define

$$\tau(A_1, \dots, A_n) = \cup_j \{\Gamma A_j \cap \Lambda \mathrm{Tor}\,(A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_n)\}.$$

We use this new notation in the following theorem.

Theorem 9. *Suppose λ is countable, $n \geq 2$, and $A_1, \dots, A_n$ are normal C_λ groups of cardinality $\aleph_n$. Then* $\mathrm{Tor}\,(A_1, \dots, A_n)$ *is a* dsc *if and only if $\tau(A_1, \dots, A_n) = 0$ in $\rho_{\aleph_n}$.*

Proof. If $\mathrm{Tor}\,(A_1, \dots, A_n)$ is a dsc, then by Theorem 8 there are λ-developments such that

$$Y_{j,i} \cong \mathrm{Tor}\,(\mathrm{Tor}\, B_{1,i}, \dots, B_{j-1,i}, B_{j+1,i}, \dots, B_{n,i}), B_{j,i+1}/B_{j,i})$$

is a dsc. By Lemma 3, $Y_{j,i}$ is a dsc if and only if $(B_{j,i+1}/B_{j,i})(\lambda) = 0$ or

$$\mathrm{Tor}\,(B_{1,i}, \dots, B_{j-1,i}, B_{j+1,i}, \dots, B_{n,i})$$

is a dsc, which implies that the complement of $\tau(A_1, \dots, A_n)$ in $\rho_{\aleph_n}$ is 1, and this part follows.

Conversely, if $\tau(A_1, \dots, A_n) = 0$ choose λ-developments $\{B_{1,i}\}, \dots, \{B_{n,i}\}$ of $A_1, \dots, A_n$ so that for all j, $\Gamma'\{B_{j,i}\} = \varnothing$ whenever $\Gamma A_j = 0$. Choosing subdevelopments, if necessary, we may assume that

$$\mathrm{Tor}\,(B_{1,i}, \dots, B_{j-1,i}, B_{j+1,i}, \dots, B_{n,i})$$

is either always or never a dsc. Observe that this second restriction also guarantees that

$$\operatorname{Tor}(B_{1,i},\dots,B_{j-1,i},B_{j+1,i+1},\dots,B_{n,i+1})$$

is always or never a dsc. Now using these λ-developments, if $\tau(A_1,\dots,$

$A_n)=0$, then each $X_{j,i}$ in Theorem 4 will be a dsc, and hence so will $\operatorname{Tor}(A_1,\dots,A_n)$. $\square$

The following generalizes Theorem 7 to groups of arbitrary cardinality.

Theorem 10. *Suppose λ is countable and $A_1,\dots,A_n$ are normal C_λ groups. Then* $\operatorname{Tor}(A_1,\dots,A_n)$ *has an isotype subgroup of cardinality $\aleph_n$ which is not a* dsc *if and only if after possibly reordering, there exist isotype subgroups $C_1,\dots,C_n$ of $A_1,\dots,A_n$ respectively such that for each j, $|C_j|=\aleph_j$ and $\Gamma_j C_j\neq 0$.*

Proof. If the given C_j exist, then $\operatorname{Tor}(C_1,\dots,C_n)$ is an isotype subgroup of $\operatorname{Tor}(A_1,\dots,A_n)$ which is not a dsc, by Theorem 7.

As to necessity, we may clearly assume that $|A_j|\leq\aleph_n$ for each j (cf. Corollary 2), and $\operatorname{Tor}(A_1,\dots,A_n)$ fails to be a dsc. Renumber the A_j so that $|A_j|<\aleph_n$ for $j\leq k$ and $|A_j|=\aleph_n$ for $j>k$. By Theorem 5, $k<n$. For $j\leq k$, let $\overline{A}_j=\oplus_{\aleph_n}A_j$. Clearly, $\Gamma_n\overline{A}_j=0$. By Theorem 9, after possibly interchanging some A_j for $j>k$ with A_n, we have $\Gamma_n A_n\neq 0$ and

$$\Lambda\operatorname{Tor}(\overline{A}_1,\dots,\overline{A}_k,A_{k+1},\dots,A_{n-1})=1.$$

Let $C_n=A_n$. Clearly, the above condition implies that there are isotype subgroups $A'_1,\dots,A'_{n-1}$ of $\overline{A}_1,\dots,\overline{A}_k,A_{k+1},\dots,A_{n-1}$ of cardinality $\aleph_{n-1}$ such that $\operatorname{Tor}(A'_1,\dots,A'_{n-1})$ fails to be a dsc. Clearly this also implies that

$$\operatorname{Tor}(A_1,\dots,A_k,A'_{k+1},\dots,A'_{n-1})$$

also fails to be a dsc (for $\operatorname{Tor}(A'_1,\dots,A'_n)$ is contained as an isotype subgroup in the direct sum of a collection of copies of this). Induction, using the groups $A_1,\dots,A_k,A'_{k+1},\dots,A'_{n-1}$, implies the result. $\square$

It can easily be seen that in the last result we can equally well use λ-pure subgroups instead of isotype ones.

Corollary 5. *Suppose λ is countable and $A_1, \ldots, A_n$ are normal C_λ groups of cardinality $\aleph_n$. If for each j, every isotype subgroup of A_j of cardinality $\aleph_1$ is a* dsc*, then* $\operatorname{Tor}(A_1, \ldots, A_n)$ *is a* dsc.

Proof. This follows from Theorem 10 and Proposition 3(a). □

It seems to the author that without major excursions into logic and set theory a satisfactory answer to the question of when $\operatorname{Tor}(A, B)$ is even a direct sum of cyclic groups will be impossible. As a simple illustration of the difficulties involved, consider the following. Suppose A is an unbounded torsion complete group with a countable basic subgroup.

Proposition 5. *The continuum hypothesis is equivalent to the statement that* $\operatorname{Tor}(A, A)$ *is a direct sum of cyclics.*

Proof. If the continuum hypothesis is true then $|A| = \aleph_1$, and then $\operatorname{Tor}(A, A)$ is a direct sum of cyclics by Theorem 5. If the continuum hypothesis is false, then A has pure subgroups C_1 and C_2 of cardinality $\aleph_1$ and $\aleph_2$ respectively. It is easy to see that $\Gamma_1 C_1 = 1$ and $\Gamma_2 C_2 = 1$, so by Theorem 10, $\operatorname{Tor}(A, A)$ is not a dsc.

3. In this section we concentrate on the ordinal Ω. If G is a group with $G(\Omega) = 0$, a subgroup W of G is called *separable* if for every $g \in G$,

$$\sup\{ht(g + w) : w \in W\} < \Omega.$$

Let $\mathcal{F}$ denote the class of groups G with $G(\Omega) = 0$ whose b.p.d. is at most 1. The next two results are from [**2**].

Theorem 11. *Let G be a group with $G(\Omega) = 0$. Then G is in $\mathcal{F}$ if and only if G is the union of a smooth ascending chain of subgroups $\{B_i\}$ such that for each i, B_i is a separable subgroup of G with $|B_{i+1}/B_i| \leq \aleph_1$.*

Corollary 6. *The class $\mathcal{F}$ is closed with respect to isomorphism and the formation of subgroups.*

Proof. Closure with respect to isomorphism being trivial, suppose G' is a subgroup of $G \in \mathcal{F}$. If $\{B_i\}$ satisfies Theorem 11 then, letting $B_i' = B_i \cap G'$, it can easily be seen that G' also satisfies Theorem 11. □

Recall that if G is a cotorsion group and α is an ordinal cofinal with ω, then $G/G(\alpha)$ is complete in its α-topology (see [**9**]). The following then gives a similar characterization of those C_Ω groups in $\mathcal{F}$.

Theorem 12. *A normal C_Ω group A is in $\mathcal{F}$ if and only if for every group W,*

$$Z_W = \text{Ext}\,(A, W)/\text{Ext}\,(A, W)(\Omega)$$

is complete in its Ω-topology.

Proof. Let $K \to H \to A$ be a balanced resolution of A, where H is a dsc. Note that since A is a C_Ω group, this is also an Ω-pure sequence. For any group W we have an exact sequence

$$\cdots \to \text{Hom}\,(K, W) \to \text{Ext}\,(A, W)(\Omega) \to \text{Ext}\,(H, W)(\Omega) = 0$$

which leads to an Ω-pure exact sequence

$$0 \to Z_W \to \text{Ext}\,(H, W) \to \text{Ext}\,(K, W) \to 0.$$

Note $H \cong \oplus_\alpha L_\alpha$ where the length of each L_α is strictly less than Ω. Therefore,

$$\text{Ext}\,(H, W) \cong \prod_\alpha \text{Ext}\,(L_\alpha, W)$$

is complete in the Ω-topology (each term in the product is discrete in this topology). So for all W, Z_W is complete in its Ω-topology if and only if $\text{Ext}\,(K, W)(\Omega) \equiv 0$ if and only if K is Ω-projective if and only if K is a dsc if and only if A is in $\mathcal{F}$. □

We now restate a theorem of [**7**]:

Theorem 13. *If A and B are C_Ω groups in $\mathcal{F}$, then* $\mathrm{Tor}\,(A,B)$ *is a* dsc.

If α is a cardinal, we say a group G is α-$\mathcal{F}$ if every subgroup of G of cardinality strictly less than α is in $\mathcal{F}$. It is essentially a consequence of Shelah's singular compactness theorem that we can restrict our attention to regular cardinals in the above definition. To verify this we use the following version of Shelah's result, valid in valuated vector spaces, due to Eklof (see [**1**]). Call a subspace S of a vector space V *small* if $\dim S < \dim V$.

Theorem 14. *Let V be a valuated vector space over a field F (e.g., $\mathbf{Z}_p$). If $\dim V$ is singular and every small subspace of V is contained in a free subspace, then V is free.*

This implies

Theorem 15. *Let G be a group with $G(\Omega) = 0$ and α be a singular cardinal. If G is α-$\mathcal{F}$, then G is α^+-$\mathcal{F}$.*

Proof. We may assume $|G| = \alpha$. Let $K \to H \to G$ be a balanced-projective resolution of G, where H is a dsc. Clearly, H can be chosen so that $|H| = \alpha$. We need to show that K is also a dsc. By a result of Hill [**4**], it is sufficient to show that $K[p]$ is free as a valuated vector space. If $L \subseteq K[p]$ with $\dim L < \alpha$, then by a straightforward "back-and-forth" argument we can construct subgroups $K' \subseteq K$, $H' \subseteq H$ and $G' \subseteq G$, such that

(a) $L \subseteq K'$,

(b) H' is a summand of H and hence a dsc,

(c) $K' \to H' \to G'$ is balanced short exact,

(d) $|G'| < |G|$.

Since $G' \in \mathcal{F}$, K' must be a dsc and hence $K'[p]$ is a free valuated vector space. So by Eklof's theorem we are done. □

We now apply this concept to get some generalizations of the results of the last section to the case where $\lambda = \Omega$. For their statements we let β be an ordinal. In the corresponding results of the last section we had $\beta = 2$.

Theorem 16. *Suppose $n \geq 2$ and for each $j = 1, \ldots, n$, A_j is a normal C_Ω group which is $\aleph_\beta$-$\mathcal{F}$. If $|A_j| < \aleph_{\beta+n-2}$ for all j, then* $\mathrm{Tor}\,(A_1, \ldots, A_n)$ *is a* dsc.

Proof. If $n = 2$, this is clear from Theorem 13. If the result is true for $n - 1 \geq 2$, and for each j, $\{B_{j,i}\}$ is an Ω-development of A_j with $|B_{j,i}| < \aleph_{\beta+n-3}$, then by induction each $X_{j,i}$ in Theorem 4 is a dsc and, hence, so is $\mathrm{Tor}\,(A_1, \ldots, A_n)$. $\square$

Corollary 7. *Suppose $n \geq 2$ and for each $j = 1, \ldots, n$, A_j is a normal C_Ω group which is $\aleph_\beta$-$\mathcal{F}$. If $|A_j| \leq \aleph_{\beta+n-2}$ for all j, then* $\mathrm{Tor}\,(A_1, \ldots, A_n)$ *is in $\mathcal{F}$, in fact* $\Lambda\mathrm{Tor}\,(A_1, \ldots, A_n) = 0$.

Proof. If $\{B_{j,i}\}$ is an Ω-development of A_j with $|B_{j,i}| < \aleph_{\beta+n-2}$, then $\mathrm{Tor}\,(A_1, \ldots, A_n)$ is the union of the isotype subgroups $\mathrm{Tor}\,(B_{1,i}, \ldots, B_{n,i})$ and by the last result these are all dsc's. The result then follows by [**2**, Theorem 5.2]. $\square$

Corollary 8. *Suppose $n \geq 2$ and for each $j = 1, \ldots, n$, A_j is a normal C_Ω group which is $\aleph_\beta$-$\mathcal{F}$. Then* $\mathrm{Tor}\,(A_1, \ldots, A_n)$ *is $\aleph_{\beta+n-1}$-$\mathcal{F}$.*

Proof. Any subgroup of $\mathrm{Tor}\,(A_1, \ldots, A_n)$ of cardinality $\leq \aleph_{\beta+n-2}$ is contained in a subgroup of the form $\mathrm{Tor}\,(C_1, \ldots, C_n)$, where C_j is an isotype subgroup of A_j of cardinality $\leq \aleph_{\beta+n-2}$. So the result follows from Corollaries 6 and 7. $\square$

The following is a version of three of the results of the last section valid when λ is uncountable (Theorems 8, 9 and 10).

Theorem 17. *Suppose $n \geq 2$ and for each $j = 1, \ldots, n$, A_j is a normal C_Ω group of cardinality $\aleph_{\beta+n-2}$, which is $\aleph_\beta$-$\mathcal{F}$.*

(a) *If* $\operatorname{Tor}(A_1, \dots, A_n)$ *is a* dsc, *then there are* Ω-*developments* $\{B_{1,i}\}, \dots, \{B_{n,i}\}$ *such that for all* i,

$$Y_{j,i} = \operatorname{Tor}(B_{1,i}, \dots, B_{j-1,i}, B_{j,i+1}/B_{j,i}, B_{j+1,i}, \dots, B_{n,i})$$

is a dsc.

(b) *If* $\operatorname{Tor}(A_1, \dots, A_n)$ *is a* dsc, *then* $\tau(A_1, \dots, A_n) = 0$.

(c) *If* $\operatorname{Tor}(A_1, \dots, A_n)$ *fails to be a* dsc, *then after possibly reordering, there are isotype subgroups* $C_1, \dots, C_n$ *of* $A_1, \dots, A_n$ *respectively such that for each* j, $|C_j| = \aleph_{\beta+j-2}$ *and* $\Gamma C_j \neq 0$.

Proof. The proof of (a) follows as in Theorem 8, with one change. Instead of using Hill's theorem to ensure that

$$\operatorname{Tor}(B_{1,i}, \dots, B_{j-1,i}, B_{j,i+1}, B_{j+1,i}, \dots, B_{n,i})$$

use Theorem 16. Parts (b) and (c) follow from part (a) in exactly the same manner as the corresponding parts of Theorems 9 and 10 follow from Theorem 8. □

We conclude with a pair of results which are valid for the ordinary torsion product (i.e., when $n = 2$).

Theorem 18. *Suppose* A *and* B *are normal* C_Ω *groups of equal regular cardinality. If* $\operatorname{Tor}(A, B)$ *is a* dsc, *then* $\Gamma A \cap \Gamma B = 0$.

Proof. Choose normal Ω-developments $\{K_i\}$ and $\{L_i\}$ such that for each i, $\operatorname{Tor}(K_i, L_i)$ is a summand of $\operatorname{Tor}(A, B)$. Then

$$P_i = \operatorname{Tor}(K_{i+1}, L_{i+1})/\operatorname{Tor}(K_i, L_i)$$

is a dsc. By [**7**, Theorem 1], there is an Ω-pure exact sequence

$$0 \to P_i \to \operatorname{Tor}(K_{i+1}/K_i, L_{i+1}) \oplus \operatorname{Tor}(K_{i+1}, L_{i+1}/L_i) \to \operatorname{Tor}(K_{i+1}/K_i, L_{i+1}/L_i) \to 0.$$

Observe that if G denotes the middle term of the above, then clearly $G(\Omega) = 0$. Since the dsc P_i is *absolutely separable* (i.e., separable in any group which contains it as an isotype subgroup), we must clearly have

$$\mathrm{Tor}\,(K_{i+1}/K_i, L_{i+1}/L_i)(\Omega) = 0$$

for each i and so either $K_{i+1}/K_i(\Omega)$ or $L_{i+1}/L_i(\Omega)$ vanishes and the result follows. □

The following is an analogue of Theorem 9 (or a converse of Theorem 17 (b)) for $n = 2$.

Theorem 19. *Suppose A and B are normal C_Ω-groups of cardinality $\aleph_2$. Then* $\mathrm{Tor}\,(A, B)$ *is a* dsc *if and only if*

$$\tau(A, B) = [\Gamma A \cap \Lambda B] \cup [\Lambda A \cap \Gamma B] = 0 \in \rho_{\omega_2}.$$

Proof. By [**7**, Theorem 22], $\mathrm{Tor}\,(A, B)$ is a dsc if and only if there are Ω-developments $\{K_i\}$ and $\{L_i\}$ such that for all i, $\mathrm{Tor}\,(K_i, L_{i+1}/L_i)$ and $\mathrm{Tor}\,(K_{i+1}/K_i, L_i)$ are dsc groups (the λ-developments in that result are only assumed to be proper, but they could have easily been constructed to also be normal). By Lemma 3 these are dsc's if and only if K_i is a dsc or L_{i+1}/L_i is normal, and K_{i+1}/K_i is normal or L_i is a dsc. The result is now clear. □

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Mathematics Department, Whitman College, Walla Walla, WA 99362