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CONSTRICTED SYSTEMS

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0. Introduction. Recent interest in the investigation of chaotic behavior of dynamic systems has led to a broad renewed interest in L_1 Markov operators. The recent monograph of Lasota and Mackey [**11**] gives a very readable introduction to the methods and applications of this approach to the randomness of deterministic processes. Lasota, Li, and Yorke [**10**] and Lasota and Yorke [**12**] have studied the asymptotic periodicity of an L_1 Markov operator which has a constricting set which attracts densities. For strongly constricted systems we obtain asymptotic finite dimensionality for a contraction on any B-space. Similar results are obtained for weakly compact constrictors on B-spaces for which the geometry can be exploited. Bartoszek [**1**] has extended the strong constrictor results of [**10**] to positive operators on arbitrary Banach lattices. The approach here has some advantages in that positivity is not required and the B-space can be arbitrary.

Using a clever and elementary argument Komornik [**7**] has shown that weak implies strong for a constricted L_1 Markov operator. While the deLeeuw–Glicksberg machinery we bring to bear does give some results in an effortless fashion, it does not give this result. We give an example of a positive isometry of $C(X)$ which is weakly but not strongly constricted to show that a full strength Komornik Theorem is not possible in $C(X)$.

1. Let T be a linear contraction on a B-space $\mathbf{X}$. Suppose F is a compact subset of $\mathbf{X}$ with the property that for each x in the unit ball, $B(\mathbf{X})$,

$$\operatorname{dist}(T^n x, F) \to 0.$$

We will call F a (norm or strong) *constrictor* for T. Note that we have not assumed that F is convex or invariant under the action of T. The first defect is easily fixed, for if F is a constrictor for T so is the norm closed convex hull of F. The second defect is also easily fixed—indeed

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in two different ways. The external solution is to replace F with the closure of its orbit under T,

$$F' = \text{closure } \cup \{T^n F : n \text{ in } \mathbf{Z}_+\}.$$

As F' is larger the distance condition holds for F'. It is not as obvious at this stage that the orbit closure is again compact but this will be clearer in what follows. The internal solution to the lack of invariance is next considered. It is obvious that each x in $B(\mathbf{X})$ has a compact orbit closure, so for each such x there is a nonempty ω limit set which is contained in F. We then define

$$\Omega = \cup\{\omega(x) : x \text{ in } B(\mathbf{X})\}$$

to obtain a compact invariant constrictor for T.

A construction T is called *strongly almost periodic* if each point has a compact orbit closure. For such operators there is a deLeeuw–Glicksberg decomposition. The space $\mathbf{X}$ has a T reducing decomposition

$$\mathbf{X} = \mathbf{X}_0 \oplus \mathbf{X}_1.$$

For x_0 in $\mathbf{X}_0$ we have

$$||T^n x_0|| \to 0$$

while $\mathbf{X}_1$ can be characterized as the closure of the span of the eigenvectors corresponding to unimodular eigenvalues. This description must be modified for the action on real B-spaces. All we will need to know is that on $\mathbf{X}_1$ the action of T is a surjective isometry and each point x_1 in $\mathbf{X}_1$ is recurrent in the sense that x_1 is a norm cluster point of $\{T^n x_1\}$. The applications are usually to positive operators on function spaces and in those circumstances the choice of using real or complex functions is a matter of convenience. The space $\mathbf{X}_1$ is the range of a contractive projection π which commute with T. This deLeeuw–Glicksberg decomposition (along with its weak counterpart) is one of the most useful results of algebraic analysis. A very readable exposition is to be found in [**8**, p. 103]. With the aid of the decomposition we obtain the following characterization of strongly constricted operators.

Theorem 1. *A contraction T is strongly constricted if and only if T is strongly almost periodic and* $\dim \mathbf{X}_1 < \infty$.

Proof. If T is strongly almost periodic with $\dim \mathbf{X}_1 < \infty$ we can take F to be $B(\mathbf{X}_1)$ which is compact if X_1 is finite dimensional. On the other hand, if T is strongly constricted it is strongly almost periodic. We see that if x is in $B(\mathbf{X}_1)$, we must have $\{T^n x\}$ clustering to x while at the same time $\operatorname{dist}(T^n x, F)$ goes to zero. This implies that each point x of $B(\mathbf{X}_1)$ is in F. Then as $\mathbf{X}_1$ has a norm compact ball it must be finite dimensional.

Now we can return to our earlier claim. If T is strongly almost periodic and F is any compact set, the orbit $\{T^n F : n \text{ in } Z_+\}$ is norm precompact. This follows either from the Moore–Osgood double limit theorem or is an easy consequence of sequence chasing in the orbit closure. □

Next we consider the implications of some special constricted contractions.

Theorem 2. *Suppose T is a strongly constricted contraction on a real L_1 space. Then T is periodic on $\mathbf{X}_1$ and is asymptotically periodic in the sense that*

$$||T^n f - T^n \pi f|| \to 0$$

where π is the deLeeuw–Glicksberg projection onto $\mathbf{X}_1$.

Proof. Now $\mathbf{X}_1$ is the range of an L_1 contractive projection π. The isometric structure of such spaces is well known [**3** or **9**]. $\mathbf{X}_1$ is isometric to a finite dimensional L_1 space. T on that space is a surjective isometry and so must permute the extreme points. This shows that T is periodic on $\mathbf{X}_1$. The asymptotic statement is just the characterization of $\mathbf{X}_0$ in disguise. □

Remarks. (1) The same sort of result is available in L_p, $p \neq 2$. For p not 1, 2, or ∞ the range of a contractive projection is again isometric to some L_p space. This is due to Ando and to Tzafriri and can be found in [**9**, p. 152]. The Lamperti characterization of isometries in L_p spaces [**14**] is particularly easy in finite L_p spaces. Not because the space has few extreme points but rather because it has few isometries, the argument goes through as before. Some power of $T^N = R$ must correspond to the identity permutation and the multiplier for this power is ± 1-valued. Thus $T^{2N} = R^2$ is the identity. The L_∞ case can be argued again from the known properties of the range of contractive projections in Stonian spaces. The next theorem includes the L_∞ case via a Gelfand representation.

(2) In the case of an L_1 Markov operator we will describe $\mathbf{X}_1$ in greater detail in the next section.

Theorem 3. *If T is a strongly constricted contraction on a real $C(X)$ space, then T is periodic on X_1 and T is asymptotically periodic in the sense that*

$$||T^n f - T^n \pi f|| \to 0$$

where π is the deLeeuw–Glicksberg projection onto $\mathbf{X}_1$.

Proof. The argument is essentially the same as that of the previous theorem with the exception that the structure of the range of a contractive projection in $C(X)$ is not as well known. We will develop the necessary structure in the corresponding weak constrictor theorem of the next section. □

2. We will say that F is a *weak constrictor* for a contraction T if F is weakly compact and for each x in $B(\mathbf{X})$, we have in the norm topology

$$\operatorname{dist}(T^n x, F) \to 0.$$

A contraction is *weakly almost periodic* if for each x in $\mathbf{X}$, the orbit $\{T^n x : n \text{ in } \mathbf{Z}_+\}$ is precompact in the weak topology. There is a (far deeper) deLeeuw–Glicksberg decomposition for weakly almost periodic contractions. As before, we have a reducing pair

$$\mathbf{X} = \mathbf{X}_0 \oplus \mathbf{X}_1$$

and a commuting contractive projection π on $\mathbf{X}_1$. The description of $\mathbf{X}_1$ remains the same. $\mathbf{X}_0$ is characterized by the fact that 0 is in the weak closure of the orbit for all x in $\mathbf{X}_0$.

Theorem 4. *Let T be a weakly constricted contraction, then T is weakly almost periodic and $\mathbf{X}_1$ is reflexive.*

Proof. The weak compactness of F and the distance condition together imply that $\{T^n x : n \text{ in } \mathbf{Z}_+\}$ is weakly compact. Now recurrence in $\mathbf{X}_1$ and the distance condition imply that $B(\mathbf{X}_1)$ is contained in F. Thus $\mathbf{X}_1$ is reflexive as its unit ball is weakly compact. □

Remark . We are missing a great deal here. We do not know a necessary and sufficient condition for a contraction to be weakly constricted. The condition we do have tells us nothing on a reflexive space where every contraction is weakly almost periodic. As before, the lack of convexity is easily fixed by replacing F with its weakly closed convex hull. The lack of invariance is another matter however. The external construction may not work. The author is grateful to Paul Milnes for showing him how an example of Ching Chou can be used to produce a weakly compact set and a weakly almost periodic contraction so that the orbit is not weakly precompact. The internal approach is no better. We have precompact orbits in the weak topology so there is a nonempty invariant set Ω contained in F but this set could be so much smaller than F that the distance condition fails for Ω. We will be able to obtain some positive results in certain spaces using the reflexivity of $\mathbf{X}_1$ together with the fact that it is the range of a contractive projection.

Theorem 5. *Let T be a weakly constricted contraction on real L_1 or $C(X)$. Then $\mathbf{X}_1$ is finite dimensional and T is periodic on $\mathbf{X}_1$.*

Proof. As we mentioned earlier, the range of a contractive projection in L_1 is known to be isometric to another L_1 space. Now this space is reflexive if and only if its dual is reflexive. But that clearly implies that $\mathbf{X}_1$ must be finite dimensional.

We now consider the $C(X)$ case. Grothendieck [**5**] has shown that the only complemented reflexive subspaces of $C[0,1]$ are finite dimensional. In our setting, a reflexive range of a contractive projection, we can give a fairly direct elementary argument. Let M denote $\mathbf{X}_1$. The basic idea is to put an isometric copy of ℓ_1 into M^*. But M^* is also reflexive so cannot contain ℓ_1. Now if $\{\phi_i\}$ is a sequence of functionals in M^* and

$b = \{b_n\}$ is a member of ℓ_1 we certainly obtain a member of M^* by $\Sigma b_n \phi_n$. The problem is to control the norm of the functional by careful choice of $\{\phi_i\}$. Suppose ϕ is in $\operatorname{ext} B(M^*)$. Let μ be an extreme Hahn Banach extension of ϕ to all of $C(X)$. A standard convexity argument shows that μ is an extreme point of $B(C(X)^*)$. Hence, this ϕ has representing measures which are $+$ or $-$ Dirac measures on X. Let S_+ be the set of points x in X so that $\delta(x)$ represents $\phi(\operatorname{mod} M)$, and let S_- be the set of points x so that $-\delta(x)$ represents $\phi(\operatorname{mod} M)$. Both sets are closed, they are clearly disjoint, and at least one is nonempty. If f is in $B(M)$, then f must have a constant value $-1 \le k \le 1$ on S_+ and is then the constant $-1 \le -k \le 1$ on S_-. For any point y in S_+ and any f in M we have $(f, \phi) = (f, \delta(y)) = (\pi f, \delta(y)) = (f, \pi^* \delta(y))$. Thus, the measure $\pi^*\delta(y)$ represents ϕ. But we already know the extreme representing measures of ϕ, so $\pi^*\delta(y)$ is a norm one measure with positive part supported on S_+ and negative part supported on S_-. From these observations we see that if f is in $C(X)$ and has constant value k on S_+ and constant value $-k$ on S_-, then πf has the same property. Now suppose that $\{\phi_n\}$ is a collection of independent extreme points of $B(M^*)$. If any two sets of the corresponding collection of sets, $\{S_\pm(n)\}$, have a point in common, the sets have the same index and the same sign. Given a finite collection of these sets and a corresponding finite collection of constants so that the constants are antisymmetric on the sets we can certainly find f in $C(X)$ which interpolates the constants on the sets. We enlarge our finite collection of sets so that for any $S_+(n)$ in the collection, $S_-(n)$ is included (if not empty) and for any $S_-(n)$ in the collection, $S_+(n)$ is included (again if not empty). We can still find g in $C(X)$ of norm at most one which interpolates. By the remarks above, πg, which is in M, also interpolates. This shows that $b \to \Sigma b_n \phi_n$ embeds isometrically ℓ_1 into M^* if $B(M^*)$ has an infinite set of independent extreme points. If M^* were infinite dimensional it would have an infinite collection of independent extreme points in its unit ball. Thus, M is finite dimensional. We may also show that M is isometric to some $C(Y)$ where Y is obtained as a maximal collection on independent extremes in $B(M^*)$, again by transferring interpolation in $C(X)$ to interpolation in M.

In either the L_1 or the $C(X)$ case, the $\mathbf{X}_1$ space is finite dimensional and has a finite number of extreme points in its unit ball. Thus, the restricted operator is the composition of a multiplier and a permutation.

For some N we see that $R = T^N$ is an isometric multiplication operator so R^{2N} is the identity.

Remarks. (1) For the L_p, $p \neq 2$, case it can be shown that $\mathbf{X}_1$ is isometric to some L_p which decomposes into pieces on each of which T is represented by an invertible measure preserving transformation with Lebesgue spectrum (at least in the separable case—otherwise a field automorphism must be used in place of the point map of the Lamperti representation).

(2) In the complex case some power of T will be a unimodular multiplier. If T is a positive operator (or just real) on the complex space the multiplier must be real and the square of the power is again the identity.

Corollary 6. *If $\{T^t : t \geq 0\}$ is a weakly constricted contraction semigroup on the real space L_1 or $C(X)$, then $\mathbf{X}_1$ consists of invariant functions and T is the identity on $\mathbf{X}_1$.*

An L_1 Markov operator has a Hopf decomposition $X = C \cup D$ [**8**]. It is straightforward using the properties of this decomposition to show that

$$||1_D T^n f|| \to 0$$

for any f.

In our setting we also can show that the support of $\mathbf{X}_1$ is C and that the extreme points of the unit ball of $\mathbf{X}_1$ which are permuted correspond to a decomposition of C into a finite number of disjoint pieces $C_1, \ldots, C_n$ which are permuted by the operator. Each C_k is the support of a unique extreme invariant probability μ_k for $T^{2N} = R$. This N need be no larger than $n!$

Lasota and Yorke [**12**] showed that a weakly constricted L_1 Markov operator is strongly constricted if the operator is the dual of a deterministic nonsingular L_∞ transformation. This can be obtained fairly directly from Lin [**13**, Corollary 4.1]. However, Lin's criterion for convergence of iterates in norm seems difficult to apply in the general situation. Komornik [**7**] has shown weak implies strong for an arbitrary L_1 Markov operator essentially from first principles. We will examine what can be obtained for weakly constricted contractions on L_1 and $C(X)$ from the deLeeuw–Glicksberg machinery.

Theorem 7. *Let T be a weakly constricted contraction on L_1 or $C(X)$. Then for x in $\mathbf{X}_0$, the iterates $T^n x$ converge weakly to zero.*

Proof. For any operator S which is a weak operator cluster point of $\{T^n\}$ we clearly have for each x in $B(\mathbf{X})$ that Sx is in F. Thus, S is a weakly compact operator and S itself gives a splitting of $\mathbf{X} = \mathbf{X}_1(S) + \mathbf{X}_0(S)$ where $\mathbf{X}_1(S)$ is finite dimensional with S periodic on $\mathbf{X}_1(S)$. Suppose y is in $\mathbf{X}_0(T)$ and $S^n y = y$. A weakly compact operator on either L_1 or $C(X)$ maps weakly compact sets to norm compact sets [**4**, pp. 498, 508]. Let $\{n^i\}$ be arbitrary and look at

$$T^{n_i} y = T^{n_i} S^n y = S^n(T^{n_i} y).$$

Now S^n takes the weakly precompact set $\{T^{n_i} y\}$ to a norm precompact set $\{T^{n_i} y\}$. But 0 is a weak cluster point of $\{T^n y\}$, so 0 must be a norm cluster point as well. Since the norm is monotone decreasing on iterates, it follows that $||T^n y|| \to 0$. Thus $Sy = 0$. We then have the asserted claim and we see that the weak operator cluster points of $\{T^n\}$ is the finite set $\{\pi, \pi T, \ldots, \pi T^{n-1}\}$, where π is the deLeeuw–Glicksberg projection.

Remark . In light of Komornik's result this is of no interest for an L_1 Markov operator.

We will examine the $C(\mathbf{X})$ case in further detail and will assume that T is Markov on $C(\mathbf{X})$ for the rest of the section. The invariant functions for T generate an upper semicontinuous decomposition of the state space into closed sets. Each stochastically closed ergodic set supports exactly one invariant probability. There are at most a finite number of such sets. If f is in $C(\mathbf{X})$ with support disjoint from the finite number of decomposition sets which do support invariant probabilities, then

$$||T^n f|| \to 0.$$

These results together with a description of the sample path behavior can be found in [**6, 15**, and **17**]. Each ergodic invariant probability is supported on an irreducible stochastically and topologically closed set. We can restrict the process to such a set, and for this restriction the following result of Jamison applies. See [**8**] for a discussion of this result.

Proposition 8 (Jamison). *An irreducible weakly almost periodic Markov operator on $C(X)$ is strongly almost periodic.*

If T is uniquely ergodic on $C(X)$ we can identify the invariant support of the single invariant probability to a point to study the behavior of the iterate of a function of f which is zero on that support set. We will now look at two examples of such a set up.

Example 9. Let $\phi(x) = 2x \bmod (1)$ on $J = [0,1]$. Let $p = .1010010001\ldots$. Then the orbit closure of p is a countable compact subset X of J. Moreover, it can be shown that ϕ is a continuous map of X into X. It can also be shown that the induced deterministic Markov operator T is weakly almost periodic with $\mathbf{X}_1$ being the space of constants. But it can also be shown that $\{T^n\}$ has weak operator cluster points other than the deLeeuw–Glicksberg projection. From Theorem 7 it follows that T is not weakly constricted. Further details of some of the above assertions can be found in [**16**].

Example 10. Here we will take all of $[0,1]$ as the state space but identify the points 0 and 1. The operator is induced by the point map $x \to x^2$. This gives us a surjective Markov isometry. $\mathbf{X}_1$ is the one-dimensional space of constants, and clearly T cannot be strongly constricted. We will show that T is weakly constricted. To this end, it will be convenient to change variables. The process is equivalent to the translation $x \to x-1$ on R where the B-space is the continuous functions on R which have equal limits at $-\infty$ and $+\infty$. An obvious density argument shows we need only worry about a continuous function g with compact support and a bound on the derivative.

Now we define

$$F_n = \{f \in B(\mathbf{X}) : \text{supp} f \subset [-n^2, -n] \text{ and } ||f'|| \leq n\}.$$

Then F_n is norm precompact and it is not hard to see that

$$F = \cup F_n$$

is weakly precompact. For any g as above, we have

$$T^n g \in F, \qquad n \geq N,$$

where N is clearly dependent on g. Thus, T is weakly constricted. Thus, we cannot have a Komornik type theorem for $C(X)$.

3. A number of the properties of strongly constricted linear contractions remain true for nonlinear nonexpansive maps. A necessary and sufficient condition for such a map to be strongly constricted is that each point of the domain have norm precompact iterates and that the union of the ω-limit sets be norm compact. This set up was investigated in [**18**]. An investigation of asymptotic periodicity of nonexpansive maps in finite dimensional polyhedral B-spaces is carried out in [**21**]. If the union of the ω-limit sets is convex, then it can be shown that the asymptotic action is actually affine [**20**].

There seems to be little more to be said about weakly constricted nonexpansive maps. Bruck [**2**] has given some positive results in Hilbert space for the structure of the weak ω-limit sets. In [**19**] it is shown that the limit of weakly-convergent iterates need not be invariant.

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Added in Proof. Related work is to be found in my paper, *Weakly Constricted Operators and Jamison's Convergence Theorem*, Proc. Amer. Math. Soc., **106** (1989), 751–755, and in Rainer Wittman's paper, *Schwach Irreduzible Markoff-Operatoren*, Monat. Math., **105** (1988), 319–334.

REFERENCES

1. Wojciech Bartoszek, *Asymptotic periodicity of the iterates of positive contractions on Banach lattices*, preprint.

2. R.E. Bruck, *On the almost convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak omega-limit set*, Israel J. Math. **29** (1978), 1–16.

3. R.G. Douglas, *Contractive projections on an L_1 space*, Pacific J. Math. **15** (1965), 443–462.

4. N. Dunford and J.T. Schwartz, *Linear operators*, Wiley, New York, 1958.

5. A. Grothendieck, *Sur les applications lineares faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. **5** (1953), 129–173.

6. B. Jamison and R. Sine, *Sample path convergence of stable Markov operators*, Z. Wahr. **28** (1974), 173–177.

7. J. Komornik, *Asymptotic periodicity of the iterates of weakly constructive Markov operators*, Tohoku Math. J. **38** (1986), 15–27.

8. U. Krengel, *Ergodic Theorems*, deGruyter, Berlin, 1985.

9. H.E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, New York, 1974.

10. A. Lasota, T.Y. Li and J.A. Yorke, *Asymptotic periodicity of the iterates of Markov operators*, Trans. Amer. Math. Soc. **286** (1984), 751–764.

11. A. Lasota and M.C. Mackey, *Probabilistic properties of deterministic systems*, Cambridge University Press, Cambridge, 1985.

12. A. Lasota and J.A. Yorke, *Statistical Periodicity of Deterministic Systems*, Casopis Pest. Mat. **111** (1986), 1–13.

13. Michael Lin, *Mixing for Markov Operators*, Z. Wahr. **19** (1971), 231–242.

14. H.L. Royden, *Real Analysis*, Macmillan, New York, 1963.

15. R. Sine, *Geometric theory of a single Markov operator*, Pacific J. Math. **27** (1968), 155–166.

16. ———, *Convergence theorems for weakly almost periodic Markov operators*, Israel J. Math. **19** (1974), 246–255.

17. ———, *Sample path convergence of stable Markov operators* II, Indiana Univ. J. Math. **25** (1976), 23–43.

18. ———, *Structure and asymptotic behavior of abstract dynamical systems*, Nonlinear Anal. **2** (1978), 119–127.

19. ———, *On the counter-example of Alspach*, Proc. St. John's Conference, in *Nonlinear analysis*, M. Dekker, New York (1981).

20. ———, *Rigidity properties of nonexpansive mappings*, Nonlinear Anal. **11** (1987), 777–794.

21. ———, *A nonlinear Perron–Frobenius theorem*, to appear.

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