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EXTENSIONS OF MODULES CHARACTERIZED BY FINITE SEQUENCES OF LINEAR FUNCTIONALS

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ABSTRACT. Let S be an algebra over an algebraically closed field, K. If S is different from K, then it contains $K^2 = K \oplus K$ as a K-vector subspace, e.g., $S = K[\zeta]$, the polynomial ring in one variable over K. Then any S-module M gives rise to a pair of K-vector spaces $\mathbf{M} = (M, M)$ and a K-bilinear map from $K^2 \times M$ to M. This makes M a right module over the matrix ring, $R = \begin{bmatrix} K & K^2 \\ 0 & K \end{bmatrix}$. An R-module isomorphic to $\mathbf{M} = (M, M)$ where M is a $K[\zeta]$-module is said to be nonsingular; an R-module is torsion-free if it is isomorphic to a submodule of $\mathbf{M} = (M, M)$ where M is a torsion-free $K[\zeta]$-module. In this paper it is shown that extensions X of finite-dimensional torsion-free R-modules U by nonsingular R-modules are characterized by finite sequences of linear functionals. This provides an upper bound on the dimension of the vector space of extensions of U by V. Questions about such extensions become questions on the existence of linear functionals with appropriate properties. In particular, when $V = (K(\zeta), K(\zeta))$, where $K(\zeta)$ is the $K[\zeta]$-module of rational functions the setup provides a fertile source of indecomposable infinite-dimensional R-modules. We describe extensions, X, of U by V, with the property that the endomorphism ring of X is an integral domain. Moreover, X shares an infinite-dimensional indecomposable submodule with V.

Introduction. We fix a field K which we assume to be algebraically closed, and, unless otherwise stated, we let all vector spaces, linear and bilinear maps be over K. That K is algebraically closed is often dispensable in the paper, but it is convenient. For instance, the set $B = \{1/(\zeta - \theta)^n : \theta \in K, n = 1, 2, \dots\} \cup \{\zeta^n : n = 0, 1, 2, \dots\}$ is a K-basis for $K(\zeta)$. If the set of positive prime numbers is replaced by the set $\{1/(\zeta - \theta) : \theta \in K\}$, then one sees that a characterization of the $K[\zeta]$-submodules of the $K[\zeta]$-module $K(\zeta)$ is given in Section 85 of [**7**]. With this characterization as a point of departure, many attempts have been made to classify other torsion-free $K[\zeta]$-modules, see Section 93

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of [**7**]. Progress in this direction would also be useful in linear algebra as can be seen in Chapter VI of [**9**]. A linear operator T_1 on a vector space can be "perturbed" on some subspace to a new operator T_2. To have a framework for such perturbations Aronszajn and Fixman studied K^2-systems in [**1**]. By regarding one of the operators as the identity operator, the new framework subsumes the case of a single linear operator. By viewing pairs of matrices over K as K^2-systems the classical result of Kronecker on pencils of matrices are recovered in [**1**]. The term *Kronecker module* for K^2-systems is due to Ringel. As is pointed out below, $K[\zeta]$-modules are also Kronecker modules. As a Kronecker module, $K(\zeta)$ has finite-dimensional submodules. We shall give a description of these submodules below in the form they will be used here. One way to obtain new families of modules from a family of modules is to take extensions. Since the ultimate goal is a classification, it is best to start from modules with tractable characterizations. Therefore, we concentrate on extensions of finite-dimensional submodules, U, of $K(\zeta)$ by $K[\zeta]$-submodules of $K(\zeta)$ regarded as Kronecker modules. (Reversing the order in the extensions, or replacing U by a finite-dimensional torsion module, results in a split extension—as can be deduced from [**4**].) Many of the difficulties encountered in the study of infinite-dimensional modules are already manifested in these extensions. The extensions can be constructed from linear functionals. We shall be dealing mostly with linear functionals on subspaces of $K(\zeta)$ given by subsets of B—the basis of $K(\zeta)$, given above. If S is a subset of a vector space, $[S]$ denotes the subspace spanned by S. We now illustrate the above concepts with an easy example.

Fix a basis (a,b) of K^2. Let ℓ be a K-linear functional on $K(\zeta)$. It gives rise to a K-bilinear map

$$\begin{gathered}\circ : K^2 \times K(\zeta) \longrightarrow K \oplus K(\zeta)\\ \circ(e,f) = (\alpha\ell(f)\cdot 1, (\alpha+\beta\zeta)f)\end{gathered} \tag{1}$$

where $e = \alpha a + \beta b$. This makes the pair of vector spaces $(K(\zeta), K \oplus K(\zeta))$ a Kronecker module: a pair of vector spaces $V = (V_1, V_2)$ is said to be a Kronecker module if there is a K-bilinear map $\circ : K^2 \times V_1 \to V_2$. Call V_1 the *domain space*, V_2 the *range space*, $\circ$ the *system operation in V*. For $e \in K^2$, $v \in V_1$, $e \circ v$ will denote the image $\circ(e,v)$. When it is necessary to keep track of the system operation $\circ$ in V

we write $V = (V_1, V_2, \circ)$. When we let G be a module, then G_1 and G_2 are automatically the domain space and range space, respectively. "Module" always means Kronecker module.

A module U is a *submodule* of X if U_i is a subspace of X_i, $i = 1, 2$, and the system operation in U is the restriction of that in X. In that case we can form the *quotient module* $X/U = (X_1/U_1, X_2/U_2)$ with system operation

$$e \circ (x_1 + U_1) = e \circ x_1 + U_2. \tag{2}$$

In (2) the element $e \circ x_1$ on the right is from the system operation in X. The module in (1) is an extension of $(0, K)$ by $(K(\zeta), K(\zeta))$. A *homomorphism* $\phi = (\phi_1, \phi_2) : X \to V$ is a pair of K-linear maps $\phi_1 : X_1 \to V_1$, $\phi_2 : X_2 \to V_2$ such that

$$\begin{gathered} e \circ \phi_1(x_1) = \phi_2(e \circ x_1) \\ \text{for all } x_1 \in X_1, e \in K^2. \end{gathered} \tag{3}$$

The category of Kronecker modules is equivalent to the category of right R-modules where R is $\begin{bmatrix} K & K^2 \\ 0 & K \end{bmatrix}$. This category behaves in many ways like the category of modules over a commutative ring, see [**5**] for details.

Let

$$E : 0 \longrightarrow U \overset{(\kappa,\lambda)}{\longrightarrow} X \overset{(\sigma,\tau)}{\longrightarrow} V \longrightarrow 0 \tag{4}$$

be a short exact sequence. It gives rise to a factor set (or "factor system"). A factor set usually involves two functions, one describing the additive structure of the extension, the other the way that scalars from the ring act on it. (See, e.g., [**11**, p. 69 ff].) However, in (4) the exact sequences of vector spaces

$$0 \longrightarrow U_1 \longrightarrow X_1 \longrightarrow V_1 \longrightarrow 0$$

and

$$0 \longrightarrow U_2 \longrightarrow X_2 \longrightarrow V_2 \longrightarrow 0$$

split. Hence, the first mentioned function of the factor set can be taken to vanish identically and is, therefore, superfluous. We wind up with a single function as in the factor sets of p. 83 of [**8**] for *inessential* extensions. Since the domain and range spaces of X can be taken to be direct sums of those of U and V, not just as abelian groups, but as vector spaces, the factor sets of [**8**] simplify further. Instead of being maps from R to $\mathrm{Hom}(V,U)$ they can be taken to be linear maps of K^2 into $\mathrm{Hom}(V_1,U_2)$. Thus, a (V,U)-*factor set* is a module $(V_1,U_2,\star)$ where $\star$ is the system operation. For simplicity we refer to $\star$ as the factor set. The *zero factor set* is the factor set $\star$ with $e\star v=0$ for all $e\in K^2$, $v\in V_1$. In Section 1 we explain the relation between extensions of our modules and factor sets *ab initio*, showing that $\mathrm{Ext}(V,U)$ is naturally isomorphic to a vector space of equivalence classes of factor sets (Theorem 1.1).

Let $U=(U_1,U_2,\circ_1), V=(V_1,V_2,\circ_2)$. A (V,U)-factor set $\star$ is said to be equivalent to another factor set $\bar\star$ if there exist linear transformations $S:V_1\longrightarrow U_1$ and $T:V_2\longrightarrow U_2$ such that, for all $e\in K^2$, $v\in V_1$,

$$e\mathbin{\bar\star} v - e\star v = Te\circ_2 v - e\circ_1 Sv. \tag{5}$$

The set of (V,U)-factor sets forms a vector space $F(V,U)$ with a subspace $F_0(V,U)$ consisting of the factor sets equivalent to the zero factor set.

Every $K[\zeta]$-module M may be considered a Kronecker module $M=(M_1,M_2)$, where $M_1=M_2=M$ with $a\circ x=x$, $b\circ x=\zeta x$ for all $x\in M,(a,b)$ a fixed basis of K^2. We say that M *comes from* the $K[\zeta]$-module, M. This gives rise to a subcategory of our module category (depending on (a,b)) which is equivalent to the category of $K[\zeta]$-modules. We now describe the extensions that we classify in Corollary 1.6 up to congruence. Let P be the Kronecker module that comes from the $K[\zeta]$-module, $K[\zeta]$. For each positive integer m, let P_m be the subspace of $K[\zeta]$ spanned by polynomials of degree strictly less than m. Let P_0 be the zero subspace. Restricting the system operation in P to $\mathbf{P}_{m-1}=(P_{m-1},P_m)$ we see that $P=\cup_{m=1}^{\infty}\mathbf{P}_{m-1}$. (It follows from the version of Kronecker's theorem in [**1**] that every indecomposable finite-dimensional Kronecker module is a quotient of $\mathbf{P}_{m-1}$ for some m.) If, in (4), $U=\oplus_{j=1}^{r}\mathbf{P}_{m_j-1}$, where $r,m_1,\dots,m_r$ are arbitrary positive integers, and V is a module that comes from a $K[\zeta]$-module, then extensions of U by V are classified up to congruence by sequences of linear functionals, $(\ell_j)_{j=1}^{r}$, in $V_1^\star$, the vector space of linear functionals on V_1.

Let $V = (V_1, V_2)$ be a module. The dual module $V^\star = (V_2^\star, V_1^\star)$ has system operation given as follows: Let $e \in K^2$, $\ell \in V_2^\star$. We want $e \circ \ell$ in $V_1^\star$. For $v \in V_1$, set

$$(e \circ \ell)(v_1) = \ell(e \circ v_1), \tag{6}$$

where $e \circ v_1 \in V_2$ is from the system operation in V. We need dual modules in the statement of Proposition 1.5 from which Corollary 1.6 is obtained. With these results, indecomposable extensions of U by V are constructed with facility. The emphasis here is on the facility as there are quite sophisticated methods available for constructing indecomposable modules over algebras, see for instance [**2, 3, 6, 10,** and **16**]. It is in the nature of things that no one approach can account for all infinite-dimensional indecomposable modules.

1. Factor sets and linear functionals. We begin by establishing a natural isomorphism between $\operatorname{Ext}(V, U)$ and $F(V, U)/F_0(V, U)$. Consider the extension E of (4). Let $\circ_1, \circ$, and $\circ_2$ be the system operations in U, X, and V, respectively. Let μ, ν be splittings as vector spaces in the domain and range spaces of E, i.e., $\mu : V_1 \to X_1$ and $\nu : V_2 \to X_2$ are linear and

$$\sigma\mu = 1_{V_1} \quad \text{and} \quad \tau\nu = 1_{V_2} \tag{7}$$
$$\text{where } 1_{V_1}, 1_{V_2} \text{ are the identity maps on } V_1, V_2.$$

(We shall be sparing in the use of parentheses, e.g., μv_1 in place of $\mu(v_1)$ when no confusion results.) Since $(\sigma, \tau) : X \to V$ is a homomorphism it follows from (3) that for any $e \in K^2$, $v_1 \in V_1$, $\tau(e \circ \mu(v_1)) = e \circ_2 \sigma\mu(v_1)$. So by (7), $\tau(e \circ \mu - \nu e \circ_2)(v_1) = 0$, i.e., $(e \circ \mu - \nu e \circ_2)(v_1) \in \operatorname{Ker} \tau = \operatorname{Im} \lambda$ for all $v_1 \in V_1$. As λ is monic, there exists a unique element, denoted by $e \star v_1$, in U_2 such that

$$\lambda(e \star v_1) = (e \circ \mu - \nu e \circ_2)(v_1). \tag{8}$$

Due to the linearity of all the maps involved, $(e, v_1) \mapsto e \star v_1$ is a bilinear map from $K^2 \times V_1$ to U_2. Hence, $(V_1, U_2, \star)$ is a factor set.

We now show that congruent extensions of U by V give rise to equivalent factor sets. This implies in particular that the equivalence class of a factor set attached to a given extension does not depend on the choice of the splittings μ, ν. Let $\overline{E}$ be another extension of U by V. Maps and systems operations arising from $\overline{E}$ will be decorated with $^-$. The analogues for $\overline{E}$ of (7) and (8) are

$$\bar{\sigma}\bar{\mu} = 1_{V_1} \quad \text{and} \quad \bar{\tau}\bar{\nu} = 1_{V_2} \tag{9}$$

$$\bar{\lambda}(e\bar{\star}v_1) = (e\bar{\circ}\bar{\mu} - \bar{\nu}e\circ_2)(v_1). \tag{10}$$

Suppose $(\beta,\gamma) : (X_1, X_2) \to (\overline{X}_1, \overline{X}_2)$ is a homomorphism that renders E congruent to $\overline{E}$. So we have the following diagram of commutative squares which will be referred to subsequently as *the diagram*:

$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & U & \xrightarrow{(\kappa,\lambda)} & X & \xrightarrow{(\sigma,\tau)} & V & \longrightarrow & 0 \\
 & & \Big\Downarrow & & \Big\downarrow{\scriptstyle (\beta,\gamma)} & & \Big\Downarrow & & \\
\overline{E} : 0 & \longrightarrow & U & \xrightarrow{(\bar{\kappa},\bar{\lambda})} & \overline{X} & \xrightarrow{(\bar{\sigma},\bar{\tau})} & V & \longrightarrow & 0
\end{array}$$

Using $\bar{\lambda} = \gamma\lambda$, (8); $\gamma e\circ = e\bar{\circ}\beta$, i.e., (3) applied to the homomorphism (β,γ), we get that

$$\begin{aligned} \bar{\lambda}e\star = \gamma\lambda e\star &= \gamma e \circ \mu - \gamma\nu e\circ_2 \\ &= e\bar{\circ}\beta\mu - \gamma\nu e \circ_2 . \end{aligned} \tag{11}$$

Also, from (10), $\bar{\lambda}e\bar{\star} = e\bar{\circ}\bar{\mu} - \bar{\nu}e\circ_2$. So

$$\bar{\lambda}(e\bar{\star} - e\star) = (\gamma\nu - \bar{\nu})e \circ_2 - e\bar{\circ}(\beta\mu - \bar{\mu}). \tag{12}$$

From the diagram, (7) and (9), respectively, we get that $\bar{\tau}\gamma = \tau$, $\bar{\tau}\bar{\nu} = \tau\nu = 1_{V_2}$. Therefore, $(\gamma\nu - \bar{\nu})(v_2) \in \operatorname{Ker}\bar{\tau} = \operatorname{Im}\bar{\lambda}$ for all $v_2 \in V_2$. Since $\bar{\lambda}$ is monic there exists a unique element, denoted by $T(v_2)$, in U_2 such that $\bar{\lambda}T(v_2) = (\gamma\nu - \bar{\nu})(v_2)$. Hence, we have a linear map $T : V_2 \to U_2$.

From the diagram, (7) and (9) we get that $\bar{\sigma}\beta = \sigma, \bar{\sigma}\bar{\mu} = \sigma\mu = 1_{V_1}$. Therefore, $(\beta\mu - \bar{\mu})(v_1) \in \operatorname{Ker}\bar{\sigma} = \operatorname{Im}\bar{\kappa}$ for all $v_1 \in V_1$. Since $\bar{\kappa}$ is monic, this results in a linear map $S : V_1 \to U_1$ with $\bar{\kappa}S(v_1) = (\beta\mu - \bar{\mu})(v_1)$ for all $v_1 \in V_1$. Using S and T, (12) becomes $\bar{\lambda}(e\bar{\star} - e\star) = \bar{\lambda}Te\circ_2 - e\bar{\circ}\bar{\kappa}S = \bar{\lambda}(Te \circ_2 - e \circ_1 S)$, because $e\bar{\circ}\bar{\kappa} = \bar{\lambda}e\circ_1$, by (3) applied to $(\bar{\kappa}, \bar{\lambda})$. Since

$\bar{\lambda}$ is monic, we conclude that $e\bar{\star} - e\star = Te \circ_2 - e \circ_1 S$. Hence, $\bar{\star}$ and $\star$ are equivalent. Therefore, we have a well-defined map

$$f : \mathrm{Ext}\,(V,U) \longrightarrow F(V,U)/F_0(V,U).$$

f is one-to-one: Suppose $f(E) = \star$ is equivalent to $\bar{\star} = f(\overline{E})$, $\star$ and $\bar{\star}$ defined by (8) and (10). We have to define a homomorphism $(\beta,\gamma) : (X_1, X_2) \to (\overline{X}_1, \overline{X}_2)$ that makes E equivalent to $\overline{E}$. Let S and T be the maps that render $\star$ and $\bar{\star}$ equivalent as defined in (5).

As vector spaces, $X_1 = \kappa U_1 \dot{+} \mu V_1$, $X_2 = \lambda U_2 \dot{+} \nu V_2$, μ, ν as defined in (7). Hence, for every $x_1 \in X_1$ and $x_2 \in X_2$,

$$\begin{aligned} x_1 &= \kappa u_1 + \mu v_1 \\ x_2 &= \lambda u_2 + \nu v_2 \end{aligned} \tag{13}$$

for unique choices of u_1, v_1, u_2 and v_2. With $\bar{\mu}, \bar{\nu}$ as defined in (9) set

$$\begin{aligned} \beta x_1 &= \bar{\kappa} u_1 + (\bar{\kappa} S + \bar{\mu}) v_1 \\ \gamma x_2 &= \bar{\lambda} u_2 + (\bar{\lambda} T + \bar{\nu}) v_2. \end{aligned} \tag{14}$$

Using (7), (9), (13), (14), and $\sigma\kappa = \bar{\sigma}\bar{\kappa} = 0$, $\tau\lambda = \bar{\tau}\bar{\lambda} = 0$, one verifies that (β,γ) as defined in (14) makes the squares in the diagram commutative.

To prove that (β,γ) is a homomorphism we have to show, by (3), that $(e\bar{\circ}\beta - \gamma e\circ)x_1 = 0$. From (3), $e\bar{\circ}\bar{\kappa} = \bar{\lambda} e \circ_1$, $e \circ \kappa = \lambda e \circ_1$. So $(e\bar{\circ}\beta - \gamma e\circ)x_1 = e\bar{\circ}(\bar{\kappa} u_1 + (\bar{\kappa} S + \bar{\mu}) v_1) - \gamma e \circ (\kappa u_1 + \mu v_1) = \bar{\lambda} e \circ_1 u_1 + \bar{\lambda} e \circ_1 S v_1 + e\bar{\circ}\bar{\mu} v_1 - \gamma \lambda e \circ_1 u_1 - \gamma e \circ \mu v_1$. Since $\gamma\lambda = \bar{\lambda}$, the penultimate expression simplifies to $\bar{\lambda} e \circ_1 S v_1 + e\bar{\circ}\bar{\mu} v_1 - \gamma e \circ \mu v_1$. Therefore,

$$(e\bar{\circ}\beta - \gamma e\circ)x_1 = (\bar{\lambda} e \circ_1 S + e\bar{\circ}\bar{\mu} - \gamma e \circ \mu) v_1. \tag{15}$$

From (8) we get that $(e \circ \mu) v_1 = \lambda e \star v_1 + \nu e \circ_2 v_1 \in \lambda U_2 + \nu V_2$. Therefore, by (14), $\gamma(e \circ \mu v_1) = \bar{\lambda} e \star v_1 + (\bar{\lambda} T + \bar{\nu}) e \circ_2 v_1$. From (10), $e\bar{\circ}\bar{\mu} v_1 = (\bar{\lambda} e \bar{\star} + \bar{\nu} e \circ_2) v_1$. So (15) becomes: $(e\bar{\circ}\beta - \gamma e\circ)x_1 = \bar{\lambda}(e\bar{\star} - e \star - (Te \circ_2 - e \circ_1 S)) v_1$,which is 0, by (5), because S and T render $\star$ and $\bar{\star}$ equivalent. This completes the proof that f is one-to-one.

f is onto: Let $\star$ be a factor set. Let $X_1 = U_1 \oplus V_1$, $X_2 = U_2 \oplus V_2$. We make $X = (X_1, X_2)$ a module by

$$e \circ (u_1, v_1) = (e \circ_1 u_1 + e \star v_1, e \circ_2 v_1) \tag{16}$$

for all $e \in K^2$, $u_1 \in U_1$, $v_1 \in V_1$. The following is an exact sequence of modules.

$$E : 0 \longrightarrow U \overset{(\kappa,\lambda)}{\longrightarrow} X \overset{(\sigma,\tau)}{\longrightarrow} V \longrightarrow 0 \tag{17}$$

where κ, λ are the natural injections, σ, τ the natural projections. We now show that the factor set attached to E following the procedure that led to (8) is the factor set $\star$. Let μ, ν be the natural injections of V_1 into X_1 and V_2 into X_2, respectively. Then $(e \circ \mu - \nu e \circ_2) v_1 = e \circ (0, v_1) - (0, e \circ_2 v_1)$. By (16), $e \circ (0, v_1) = (e \star v_1, e \circ_2 v_1)$. Therefore, $(e \circ \mu - \nu e \circ_2) v_1 = (e \star v_1, 0) = \lambda(e \star v_1)$. Hence, (8) is satisfied. We have proved the essentials of the following theorem.

Theorem 1.1. *There is a natural isomorphism between the vector spaces* $\operatorname{Ext}(V, U)$ *and* $F(V, U)/F_0(V, U)$.

Corollary 1.2. *Let* $E : 0 \to U \to X \to V \to 0$ *be an extension of* U *by* V. *Then* E *is congruent to an extension, where* $X_1 = U_1 \oplus V_1$, $X_2 = U_2 \oplus V_2$ *and*

$$e \circ (u_1, v_1) = (e \circ_1 u_1 + e \star v_1, e \circ_2 v_1) \tag{18}$$

where $\star$ *is some* (V, U)*-factor set.*

Note. From now on, (a, b) is a fixed basis of K^2.

In Proposition 1.3 we shall be dealing with $\oplus_{j=1}^{r} P_{m_j - 1}$ for arbitrary positive integers $r, m_1, m_2, \dots, m_r$. It will be notationally convenient to use the following module in place of P_{m_j-1}. Let V_1 be a vector space with basis $\{v_{1j}, \dots, v_{m_j-1,j}\}$ (if $m_j = 1$, set $V_1 = 0$). Let W_1 have basis $\{w_{1j}, w_{2j}, \dots, w_{m_j j}\}$. We make $V = (V_1, W_1)$ a module by setting

$$\begin{aligned} a \circ v_{ij} &= w_{ij} \\ b \circ v_{ij} &= w_{i+1,j}, \qquad i = 1, \dots, m_j - 1. \end{aligned} \tag{19}$$

The maps $\varphi_1 : \zeta^i \mapsto v_{i+1,j}$, $i = 0, 1, \dots, m_j - 2$, and

$$\varphi_2 : \zeta^i \mapsto w_{i+1}, \qquad j, i = 0, 1, \dots, m_j - 1,$$

establish an isomorphism (φ_1, φ_2) between P_{m_j-1} and V, i.e., $\varphi_1 : P_{m_j-1} \to V_1$, $\varphi_2 : P_{m_j} \to V_2$ are isomorphisms of vector spaces and $\phi = (\phi_1, \phi_2)$ satisfies (3).

A module $V = (V_1, V_2)$ is *torsion-free* if for each nonzero e in K^2 the linear map

$$\begin{aligned} &T_e : V_1 \longrightarrow V_2 \\ &T_e(v_1) = e \circ v_1 \end{aligned} \tag{20}$$

is one-to-one. An extension of a torsion-free module by a torsion-free module is also torsion-free.

Proposition 1.3. *Let E be an extension of U by V. Suppose U is $P_{m_1-1} \oplus \cdots \oplus P_{m_r-1}$ for some positive integers $r, m_1, \dots, m_r$ and V is torsion-free. Then E is congruent to an extension in which the middle term is $(U_1 \oplus V_1, U_2 \oplus V_2)$ with the system operation given by*

$$\begin{aligned} a \circ (u_1, v_1) &= (a \circ_1 u_1, a \circ_2 v_1) \\ b \circ (u_1, v_1) &= \left(b \circ_1 u_1 + \sum_{j=1}^{r} \ell_j(v_1) e_j, b \circ_2 v_1 \right), \end{aligned} \tag{21}$$

where $\{e_j\}_{j=1}^r$ is the standard basis of K^r, and $\ell_j \in V_1^\star$, the space of linear functionals on V_1. (Note that U_2 contains K^r.)

Proof. We may replace P_{m_j-1} by the module in (19). So $\{e_j\}_{j=1}^r$ becomes $\{w_{1j}\}_{j=1}^r$ and U_1, U_2 have respective bases $\cup_{j=1}^r \cup_{i=1}^{m_j-1} \{v_{ij}\}$ and $\cup_{j=1}^r \cup_{i=1}^{m_j} \{w_{ij}\}$.

By Corollary 1.2, we may assume that the middle term $X = (X_1, X_2)$ of E is of the form $(U_1 \oplus V_1, U_2 \oplus V_2)$ and the system operation in X is given by

$$\begin{aligned} a \circ (u_1, v_1) &= (a \circ_1 u_1 + a \star v_1, a \circ_2 v_1) \\ b \circ (u_1, v_1) &= (b \circ_1 u_1 + b \star v_1, b \circ_2 v_1), \end{aligned}$$

where $\star$ is some (V,U)-factor set. So

$$a \star v_1 = \sum_{j=1}^{r} \sum_{i=1}^{m_j} f_{ij}(v_1) w_{ij}$$
$$b \star v_1 = \sum_{j=1}^{r} \sum_{i=1}^{m_j} g_{ij}(v_1) w_{ij},$$

where f_{ij}, g_{ij} are in $V_1^\star$. We shall define linear maps $S : V_1 \to U_1$ and $T : V_2 \to U_2$, which will give rise to a factor set $\bar\star$ equivalent to $\star$, and at the same time realize the conclusion of the proposition. For $v_1 \in V_1$, $v_2 \in V_2$, set

$$Sv_1 = \sum_{j=1}^{r} \sum_{i=1}^{m_j-1} h_{ij}(v_1) v_{ij} \tag{22}$$

$$Tv_2 = \sum_{j=1}^{r} \sum_{i=1}^{m_j} k_{ij}(v_2) w_{ij} \tag{23}$$

where h_{ij}, k_{ij} are to be determined. By (5) and (19) we get a factor set, $\bar\star$, equivalent to $\star$ if we put, for each $v_1 \in V_1$,

$$\begin{aligned} a \bar\star v_1 &= \sum_{j=1}^{r} \sum_{i=1}^{m_j} f_{ij}(v_1) w_{ij} + \sum_{j=1}^{r} \sum_{i=1}^{m_j} k_{ij}(a \circ_2 v_1) w_{ij} \\ &\quad - \sum_{j=1}^{r} \sum_{i=1}^{m_j-1} h_{ij}(v_1) w_{ij}; \\ b \bar\star v_1 &= \sum_{j=1}^{r} \sum_{i=1}^{m_j} g_{ij}(v_1) w_{ij} + \sum_{j=1}^{r} \sum_{i=1}^{m_j} k_{ij}(b \circ_2 v_1) w_{ij} \\ &\quad - \sum_{j=1}^{r} \sum_{i=1}^{m_j-1} h_{ij}(v_1) w_{i+1,j}. \end{aligned}$$

The plan now is to define h_{ij} and k_{ij} in terms of f_{ij} and g_{ij} to get (21). The coefficient of $w_{m_j j}$ in $a \bar\star v_1$ is $f_{m_j j}(v_1) + k_{m_j}(a \circ_2 v_1)$. Set $k_{m_j j}(a \circ_2 v_1) = -f_{m_j j}(v_1)$. Since V is torsion-free, $a \circ_2 v_1 = 0$ implies that $v_1 = 0$. Hence, $k_{m_j j}$ is well defined on $a \circ_2 V_1 = \{a \circ_2 v_1 : v_1 \in V_1\}$. For all $j = 1, \dots, r$, $i = 1, \dots, m_j$, set $k_{ij} \equiv 0$ on a vector space direct complement of $a \circ_2 V_1$ in V_2.

If $m_j \geq 2$, the coefficient of $w_{m_j j}$ in $b \bar{\star} v_1$ is $g_{m_j j}(v_1) + k_{m_j j}(b \circ_2 v_1)$

$-h_{m_j -1,j}(v_1)$. Set $h_{m_j-1,j}(v_1) = b_{m_j j}(v_1) + k_{m_j j}(b \circ_2 v_1)$. In this way we get that the coefficients of $w_{m_j j}$ in both $a \bar{\star} v_1$ and $b \bar{\star} v_1$ are zero. For $i \neq m_j$ we make the coefficient of w_{ij} in $a \bar{\star} v_1$ zero by setting $k_{ij}(a \circ_2 v_1) = -f_{ij}(v_1) + h_{ij}(v_1)$.

For $i \neq 1$, we make the coefficient of w_{ij} in $b \bar{\star} v_1$ zero by setting $h_{i-1,j}(v_1) = g_{ij}(v_1) + k_{ij}(b \circ_2 v_1)$. We now have that $a \bar{\star} v_1 = 0$ for all v_1 in V_1 while $b \bar{\star} v_1 = \sum_{j=1}^{r} (g_{1j}(v_1) + k_{1j}(b \circ_2 v_1)) w_{1j}$. The proposition now follows with $\ell_j(v_1) = g_{1j}(v_1) + k_{1j}(b \circ_2 v_1)$. □

Corollary 1.4. *Let U and V be as in Proposition* 1.3.

(a) *There is an onto linear map from $rV_1^\star = V_1^\star \oplus \cdots \oplus V_1^\star$ (r copies) to* $\mathrm{Ext}(V, U)$.

(b) $\dim \mathrm{Ext}(V, U) \leq r \dim V_1^\star$, *where* dim *is dimension as a K-vector space.*

Proof. (a). Given $(\ell_j)_{j=1}^r$ in $rV_1^\star$ we make $X = (X_1, X_2)$ a module by using (21) to define the system operation. This makes X an extension of U by V. In this way we get a map, f, from $rV_1^\star$ to $\mathrm{Ext}(V, U)$. By Theorem 1.1, f is a linear surjective map. Part (b) follows from Part (a). □

Given $(\ell_j)_{j=1}^r$ and $(\bar{\ell}_j)_{j=1}^r$ in $rV_1^\star$ the next proposition tells us when the extensions E and $\overline{E}$ that they give are congruent. By Theorem 1.1, Proposition 1.3 and (5), E is congruent to $\overline{E}$ if and only if there exist linear maps $S : V_1 \to U_1$, $T : V_2 \to U_2$ such that, for every v_1 in V_1, we have

$$Tb \circ_2 v_1 - b \circ_1 Sv_1 = \sum_{j=1}^{r} (\bar{\ell}_j(v_1) - \ell_j(v_1)) w_{1j} \tag{24}$$

$$Ta \circ_2 v_1 - a \circ_1 Sv_1 = 0. \tag{25}$$

Using (19) and the expressions for S and T from (22) and (23), (24) and (25) respectively become

$$\sum_{j=1}^{r}\sum_{i=1}^{m_j} k_{ij}(b \circ_2 v_1)w_{ij} - \sum_{j=1}^{r}\sum_{i=1}^{m_j-1} h_{ij}(v_1)w_{i+1,j} = \sum_{j=1}^{r}(\bar{\ell}_j(v_1) - \ell_j(v_1))w_{1j} \tag{26}$$

and

$$\sum_{j=1}^{r}\sum_{i=1}^{m_j} k_{ij}(a \circ_2 v_1)w_{ij} - \sum_{j=1}^{r}\sum_{i=1}^{m_j-1} h_{ij}(v_1)w_{1j} = 0. \tag{27}$$

Equating coefficients of w_{ij} in (26) and (27) leads to

$$k_{1j}(b \circ_2 v_1) = \bar{\ell}_j(v_1) - \ell_j(v_1), \tag{28}$$

$$k_{i+1,j}(b \circ_2 v_1) = h_{ij}(v_1), \qquad i = 1, \dots, m_{j-1}, \tag{29}$$

$$k_{ij}(a \circ_2 v_1) = h_{ij}(v_1), \qquad i = 1, \dots, m_{j-1}, \tag{30}$$

$$k_{m_j j}(a \circ_2 v_1) = 0. \tag{31}$$

Recalling the definition of the system operation in the dual module $V^\star = (V_2^\star, V_1^\star)$, see (6), we have proved the following proposition.

Proposition 1.5. *Suppose E and $\overline{E}$ are two extensions as in Proposition* 1.3, *given by $(\ell_j)_{j=1}^r, (\bar{\ell}_j)_{j=1}^r$. Then E is congruent to $\overline{E}$ if and only if, for each $j = 1, \dots, r$, $V_2^\star$ contains $k_{1j}, \dots, k_{m_j j}$ and $V_1^\star$ contains $h_{1j}, \dots, h_{m_j-1,j}$ such that, in $V^\star$, $b \circ_2 k_{1j} = \bar{\ell}_j - \ell_j, a \circ_2 k_{ij} = b \circ_2 k_{i+1,j} = h_{ij}, i = 1, \dots, m_j - 1, a \circ_2 k_{m_j j} = 0$.*

Corollary 1.6. *Suppose that in $V = (V_1, V_2), a \circ_2 V_1 = V_2$. Then E is congruent to $\overline{E}$ if and only if $\bar{\ell}_j = \ell_j$ for $j = 1, \dots, r$.*

Proof. If $a \circ_2 V_1 = V_2$, then from (31) we get that $k_{m_j j}$ is the zero map on V_2. From (29) and (30) we get that k_{ij}, $j = 1, \dots, r$, $i = 1, \dots, m_j$, are zero maps. From (28) we get that $\bar{\ell}_j = \ell_j$. □

Remark 1.7. In Proposition 1.3 it was not necessary that V be torsion-free, only that the linear map $T_a : V_1 \to V_2$, $T_a(v_1) = a \circ_2 v_1$, be one-to-one. Similarly, if T_b is one-to-one, the system operation in V can be simplified to

$$\begin{aligned} a \circ (u_1, v_1) &= \left(a \circ_1 u_1 + \sum_{j=1}^{r} \ell_j(v_1) w_{1j}, a \circ_2 v_1 \right) \\ b \circ (u_1, v_1) &= (b \circ_1 u_1, b \circ_2 v_1). \end{aligned} \tag{32}$$

Both forms of Proposition 1.3, (21) and (32), are needed in the study of infinite-dimensional modules. In this paper we use only (21).

Examples 1.8. Let M be a $K[\zeta]$-module. Then $\mathbf{M} = (M, M)$ is made a Kronecker module by setting, for all $x \in M$,

$$a \circ x = x, \qquad b \circ x = \zeta x. \tag{33}$$

In particular, $a \circ M = M$. So, with U as in Proposition 1.3 and M infinite-dimensional, the set of inequivalent extensions of U by $\mathbf{M}$ has the same cardinality as $M^\star$, the vector space of linear functionals on M. An important example of $\mathbf{M}$ is $\mathcal{R} = (K(\zeta), K(\zeta))$, where $K(\zeta)$ is the $K[\zeta]$-module of rational functions. It follows from Corollary 1.6, Lemma 1 and Theorem 2 in Chapter IX of [**9**] that $\operatorname{Ext}(\mathcal{R}, U)$ has dimension $2^{\text{card } K}$, both as a K-vector space and a $K(\zeta)$-vector space, as stated in [**12**, Proposition 1.7].

2. Constructing indecomposable extensions. In [**7**], Lemma 88.3 on *rigid systems* of groups is crucial in proving that various groups are indecomposable. If we were working inside the Kronecker module $\mathcal{R} = (K(\zeta), K(\zeta))$ one could imitate Section 88 of [**7**] to construct indecomposable submodules of $\mathcal{R}$ from rigid systems of infinite-dimensional submodules of $\mathcal{R}$, see [**15**]. (These indecomposable modules in [**15**] do not come from $K[\zeta]$-modules.) In [**15**] every element was required to have infinitely many divisors. It can be shown that the elements in the submodule U in Proposition 1.3 do not have this property. This explains why this requirement is assumed in a restricted form in Theorem 2.1. There are several references on the use of linear functionals to construct indecomposable Kronecker modules, e.g., [**13**] and [**14**]. The advantage of Theorem 2.1 over the others is that one also obtains direct information on endomorphism rings. We now give the details.

As in Section 1 (a, b) is a fixed basis of K^2. Let $X = (X_1, X_2)$ be a module. An element x_2 in X_2 is said to be *divisible in* X by $b - \theta a$ if for some x_1 in X_1 we have that

$$(b - \theta a) \circ x_1 = x_2. \tag{34}$$

Let H be a nonempty subset of K. To each $\theta \in H$ we attach either ∞ or a positive integer denoted in both cases by $h(\theta)$. And to $\{h(\theta) : \theta \in H\}$ we attach the following submodule V of $\mathcal{R}$. Let V_1 have basis

$$\{1/(\zeta - \theta)^t : \theta \in H, 0 < t < h(\theta) + 1\}, \tag{35}$$

and let V_2 have basis

$$\{1/(\zeta - \theta)^t : \theta \in H,\ 0 \le t < h(\theta) + 1\}. \tag{36}$$

Restricting the system operation in $\mathcal{R}$, given in (33), to V_1 makes $V = (V_1, V_2)$ a submodule of $\mathcal{R}$. We shall denote it by V_h. Its domain space is V_1 and its range space is V_2.

Theorem 2.1. *Let X be an extension of $U = P_{m_1-1} \oplus \cdots \oplus P_{m_r-1}$ by V_h with the system operation given as in* (21). *If every element (u_2, v_2), $v_2 \neq 0$, is divisible by $b - \theta a$ for infinitely many θ in K, then the endomorphism ring of X is an integral domain. In particular, X is indecomposable.*

Proof. Step 1. Let (φ, ψ) be an endomorphism of X. Then (φ, ψ) is the zero map if $\psi(0, v_2) = 0$ for all v_2 in V_2: Since $a \circ_2 v_2 = v_2$, we get from (3) and (21) that $\psi(0, v_2) = \psi(a \circ (0, v_2)) = a \circ \varphi(0, v_2)$. So, $a \circ \varphi(0, v_2) = 0$. As X is torsion-free this implies that $\varphi(0, v_2) = 0$ for all v_2 in V_2. For the rest of the proof of Step 1 we use $\psi(0, v_2) = \varphi(0, v_2) = 0$ for all v_2 in V_2. We shall use the notation in the proof of

Proposition 1.3. So, P_{m_j-1} is given as in (19). By hypothesis, $(w_{1j},1)$ is divisible by $b-\theta a$ for infinitely many θ in K. Choosing one such θ, we have, for some u_1 in $U_1, v_1 \in K(\zeta)$ that

$$(b-\theta a)\circ(u_1,v_1)=(w_{1j},1). \tag{37}$$

For appropriate scalars α_{ik}, $u_1=\sum_{k=1}^{r}\sum_{i=1}^{m_j-1} h_{ik}v_{ik}$. By (19), (21), and (33), $(b-\theta a)\circ(u_1,v_1)=(\sum_{k=1}^{r}\sum_{i=1}^{m_k-1}\alpha_{ik}w_{i+1,k}-\theta\sum_{k=1}^{r}\sum_{i=1}^{m_k-1}\alpha_{ik}w_{ik}+\sum_{k=1}^{r}\ell_k(v_1)w_{1k},(\zeta-\theta)v_1)$. Substituting this last expression in (37) and equating coefficients of w_{ik} leads to $u_1=0$ and $v_1=1/(\zeta-\theta)$.

Now, by (37) and (3), $\psi(w_{1j},1)=\psi((b-\theta a)\circ(0,1/(\zeta-\theta)))=(b-\theta a)\circ\varphi(0,1(\zeta-\theta))$. By the last paragraph, $\varphi(0,1/(\zeta-\theta))=0$. Hence, $\psi(w_{1j},1)=0$. Therefore, $\psi(w_{1j},0)=0$ because $\psi(0,1)=0$, by hypothesis. Again by (3) and (21), $a\circ\varphi(v_{ij},0)=\psi(a\circ(v_{ij},0))=\psi(w_{ij},0)$. Since X is torsion-free, $\psi(w_{ij},0)=0$ implies that $\varphi(v_{ij},0)=0$. From $\psi(b\circ(v_{ij},0))=\psi(w_{i+1,j},0)=b\circ\varphi(v_{ij},0)$, we deduce that $\varphi(v_{ij},0)=0$ implies that $\psi(w_{i+1,j},0)=0$. Therefore, $\psi(w_{1j},0)=0$ implies that (φ,ψ) vanishes on $P_{m_j-1}\oplus(0,0)$ for $j=1,\dots,r$. This completes the proof of the assertion in Step 1.

Let x be a nonzero element of X_2, the range space of X. Let θ be an element of K with x divisible in X by $b-\theta a$. So $(b-\theta a)\circ x_{11}=x$ for some element x_{11} in the domain space X_1 of X. We want to define the *height of* x *at* $\theta, h_x(\theta)$. To that end, set $x=x_{12}, a\circ x_{i1}=x_{i+1,2},(b-\theta a)\circ x_{i1}=x_{i2}$. Set $h_x(\theta)=t$ if t is the largest positive integer such that $a\circ x_{i1}=x_{i+1,2}$, $(b-\theta a)\circ x_{i1}=x_{i2}$, $i=1,\dots,t$, and $x_{t+1,2}$ is not divisible by $b-\theta a$. If there is no such t, put $h_x(\theta)=\infty$. To define $h_x(\infty)$, $x=x_{12}$, let $a\circ x_{i1}=x_{i2}$, $b\circ x_{i1}=x_{i+1,2}$. Then $h_x(\infty)$ is the largest positive integer t such that $b\circ x_{i1}=x_{i+1,2}$, $a\circ x_{i1}=x_{i2}$, $i=1,\dots,t$, and for no element x_1 in X_1 is $a\circ x_1=x_{t+1,2}$. If there is no such t, put $h_x(\infty)=\infty$. In what follows it is convenient to recognize the dependence of x_{ij}, $j=1,2$, on θ. So we shall replace x_{ij} by $x_{\theta ij}$. For each $\theta, x_{\theta 12}=x_{12}=x$. Since X is torsion-free, the $x_{\theta ij}$'s are uniquely determined.

With $x=x_{12}$ divisible by $b-\theta a$, let

$$X_{\theta 1}=[\{x_{\theta i1}:1\le i<h_x(\theta)+1\}],$$
$$X_{\theta 2}=[\{x_{\theta i2}:1\le i<h_x(\theta)+2\}].$$

The restriction of the system operation in X to $X_{\theta 1}$ makes $X_{\theta x} = (X_{\theta 1}, X_{\theta 2})$ a submodule of X. Let $X_x = \sum_{\theta \in K \cup \{\infty\}}$. We now define a submodule of $\mathcal{R}$ isomorphic to X_x. Let

$$\begin{aligned} V_{x1} &= [\{1/(\zeta-\theta)^t : 0 < t < h_x(\theta)+1; \theta \in K\}] \\ &\quad + [\{\zeta^t : 0 \le t < h_x(\infty)\}], \\ V_{x2} &= [\{1/(\zeta-\theta)^t : 0 \le t < h_x(\theta)+1, \theta \in K\}] \\ &\quad + [\{\zeta^t : 0 \le t < h_x(\infty)+1\}]. \end{aligned}$$

Restricting the system operation in $\mathcal{R}$, given in (33), to V_{x1} we see that $V_x = (V_{x1}, V_{x2})$ is a submodule of $\mathcal{R}$. The maps $\phi_1 : X_{x1} \to V_{x1}$, given by $\phi_1(x_{\theta i1}) = (\zeta-\theta)^{-i}$, $\phi_1(x_{\infty i1}) = \zeta^{i-1}$, and $\phi_2 : X_{x2} \to V_{x2}$, given by $\phi_2(x_{\theta i2}) = (\zeta-\theta)^{1-i}$, $\phi_2(x_{\infty i2}) = \zeta^i$, yield an isomorphism between X_x and V_x. It follows from [**4**, Thoerem 3.3] that ϕ_1 and ϕ_2 are well defined.

Step 2. *Properties of* X_x. The properties below can all be deduced from [**4**] and [**13**]. In fact, we duplicate some arguments there in several places.

(a). X/X_x is torsion-free: Let X' be the smallest submodule of X containing X_x with the property that X/X' is torsion-free. In the terminology of [**4**, Section 2], X' is a torsion-closed submodule of X of rank one. By [**4**, Theorem 3.3], $X' = X_x$.

(b). $X_x = X_f$ for every nonzero element f in the range space of X_x: By (a), X_x and X_f are rank one torsion-closed submodules of X. Since $X_x \cap X_f \neq 0$ one readily shows that $X_x = X_f$, see, e.g., [**13**, Lemma 4.1].

(c). Let f and x be two nonzero elements in the range space of X. Then $X_f \cap X_x = 0$ unless $X_f = X_x$: Let $X_f = (X_{f1}, X_{f2})$, $X_x = (X_{x1}, X_{x2})$. If $0 \neq x' \in X_{x1} \cap X_{f1}$, then $a \circ x' = w$ is a nonzero element in $X_{x2} \cap X_{f2}$ because X is torsion-free. By (b), $X_f = X_w = X_x$.

(d). If $\psi(w) = 0$ for any nonzero element w in X_{x2}, then $(\varphi, \psi) X_x = 0$: By (b), $X_w = X_x$. The argument at the end of Step 1 and the definition of X_w give that $\psi(w) = 0$ implies that $(\varphi, \psi) X_w = 0$.

(e). The endomorphism ring of X_x, $\mathrm{End}(X_x)$ is an integral domain. We prove this for V_x. Suppose $\psi(1) = f \in K(\zeta)$. Then by (3) and (33) we get that $\psi(1/(\zeta-\theta)^t) = f/(\zeta-\theta)^t = \varphi(1/(\zeta-\theta)^t)$ for all possible θ

and t. Hence, $(\varphi, \psi) = (f, f)$, multiplication by f in both the domain and range spaces of V_x.

Step 3. Let $(\varphi, \psi) \in \mathrm{End}(X)$. Then (φ, ψ) restricts to an element in $\mathrm{End}(X_x)$ for all $x = (0, v_2)$, $v_2 \neq 0$:

We shall need the following partial fraction expansions. Let n be any positive integer, θ and η two distinct elements in K. Then one has

$$\frac{\zeta^n}{\zeta - \theta} = \zeta^{n-1} + \theta\zeta^{n-2} + \cdots + \theta^{n-1} + \frac{\theta^n}{\zeta - \theta}. \tag{38}$$

$$\begin{aligned} \frac{1}{(\zeta - \theta)(\zeta - \eta)^n} &= \frac{(\eta - \theta)^{-1}}{(\zeta - \eta)^n} - \frac{(\eta - \theta)^{-2}}{(\zeta - \eta)^{n-1}} + \cdots \\ &\quad \pm \frac{(\eta - \theta)^{-n}}{\zeta - \eta} \pm \frac{(\eta - \theta)^{-n}}{\zeta - \theta}. \end{aligned} \tag{39}$$

If $\psi(x) = 0$, then by (d), $(\varphi, \psi)X_x = 0$. We may then suppose that $\psi(x) \neq 0$. Since (φ, ψ) is a homomorphism it follows from (3) and the definition of X_x that we have

$$h_{\psi(x)}(\theta) \geq h_x(\theta) \tag{40}$$

$$X_{\psi(x)} \supseteq (\varphi, \psi)X_x. \tag{41}$$

We claim that $\psi(x) = (u_2, f)$, $f \neq 0$. Suppose $f = 0$ and $(u_2, 0)$ is divisible by $b - \theta a$. Then for some (u_1, v_1), $(b - \theta a) \circ (u_1, v_1) = (u_2, 0)$. Then by (21), $v_1(\zeta - \theta) = 0$. Hence, $v_1 = 0$. Since U_1 is finite-dimensional and torsion-free it follows that $(u_2, 0)$ is divisible by $b - \theta a$ for only finitely many θ in K. However, by hypothesis, $x = (0, v_2)$, $v_2 \neq 0$ is divisible by $b - \theta a$ for infinitely many θ in K, i.e., $h_x(\theta) > 0$ for infinitely many θ in K. So, $f = 0$ contradicts (40). So, $\psi(x) = (u_2, f)$, $f \neq 0$.

Let $K_x = \{\theta \in K \cup \{\infty\} : x \text{ is divisible by } b - \theta a\}$. By hypothesis, K_x is infinite. Using (3) and (21), we see that the range space of $(\varphi, \psi)X_x$ is contained in the vector space $C = [\{f/(\zeta - \theta)^i : 0 \leq i < h_x(\theta) + 1, \theta \in K_x\}] + U_2$. (If $\theta = \infty, (\zeta - \theta)^{-i}$ is ζ^i.) Moreover, it is of finite codimension in C.

For ν in $K \cup \{\infty\}$, let $O_\nu(g)$ denote the order of the pole of g at ν. Recall that $x = (0, v_2)$, $0 \neq v_2 \in K(\zeta)$, $\psi(x) = (u_2, f)$, $f \neq 0$. Let

$S_f = \{1/(\zeta-\nu)^t : \nu$ a pole of f, $0 \le t \le O_\nu(f)\}$. S_{v_2} is defined similarly. Let $D = [\{1/(\zeta-\theta)^i : \theta \in K_x,\ 0 \le i < h_x(\theta)+1\}] + [S_f] + [S_{v_2}] + U_2$. From (38) and (39) we deduce that C is of finite codimension in D. Therefore, the range space of $(\varphi,\psi)X_x$ is of finite codimension in D. The same holds for X_x. Hence, Range$(X_x)\cap$ Range$(\varphi,\psi)X_x \neq 0$ because D is infinite-dimensional. By (41) and Step 2(c), $X_x = X_{\psi(x)}$. So, $X_x \supseteq (\varphi,\psi)X_x$, by (41).

Step 4. End(X) *is commutative.* Let $(\varphi,\psi),(\sigma,\tau)$ be two elements in End(X). For every $x = (0,v_2)$, with $v_2 \neq 0$, step 3 tells us that $(\varphi,\psi),(\sigma,\tau)$ restrict to elements of End(X_x), which is commutative by Step 2(e). Hence, $(\tau\psi-\psi\tau)(0,v_2)=0$. By Step 1, this implies that $(\sigma,\tau)(\varphi,\psi)-(\varphi,\psi)(\sigma,\tau)=0$.

Step 5. End(X) *is a domain.* Let (φ,ψ) and (σ,τ) be two nonzero elements in End(X). By Step 1, $\psi(0,f)\neq 0$ and $\tau(0,g)\neq 0$ for some f,g in $K(\zeta)$. We want to show that $\psi\tau$ is not the zero map. By Step 3, (φ,ψ) and (σ,τ) restrict to elements of End(X_f) and End(X_g), where $f=(0,f)$, $g=(0,g)$. If $\tau(f)\neq 0$ or $\psi(g)\neq 0$, then (φ,ψ) and (σ,τ) restrict to nonzero elements in End(X_f) or End(X_g) and we would be done by Step 2(e). Suppose $\tau(f)=0$ and $\psi(g)=0$; then $\tau(f+g)=\tau(g)\neq 0$, $\psi(f+g)=\psi(f)\neq 0$. So, (φ,ψ) and (σ,τ) restrict to nonzero maps in $X_{(f+g)}$. By Step 2(e) we are done with Step 5 and Theorem 2.1 is proved. □

Remark 2.2. Since Steps 4 and 5 are consequences of Steps 1 to 3 without further recourse to the nature of X, the conclusion of Theorem 2.1 is valid for any submodule of X for which Steps 1 to 3 can be proved.

The module X given as in (21) begets a submodule of itself and a submodule of V in the following way: let $X_{\ell 1} = \cap_{j=1}^{r}\mathrm{Ker}\,\ell_j \subset V_1 \subset X_1$ and $X_{\ell 2} = V_2 \subset X_2$. For v_1 in $X_{\ell 1}$, we have from (21) that $a\circ(0,v_1) = (0, a\circ_2 v_1) \in V_2$ and $b\circ(0,v_1) = (\sum_{j=1}^{r}\ell_j(v_1)e_j, b\circ_2 v_1) = (0, b\circ_2 v_1) \in V_2$. So the system operations in X and V agree when restricted to $X_{\ell 1}$ and they take $X_{\ell 1}$ to $X_{\ell 2}$. Therefore, $X_\ell = (X_{\ell 1}, X_{\ell 2})$ is both a submodule of X and a submodule of V. Let $(\varphi,\psi)\in$ End(X_ℓ). If $\varphi(0,v_1)\neq 0$, then $a\circ\varphi(0,v_1) = \psi(a\circ(0,v_1)) \neq 0$ because X is torsion-free. Hence, $(\varphi,\psi)=0$ if and only if $\psi = 0$. This is the analogue of Step 1 in the proof of Theorem 2.1. If X satisfies the hypotheses of

Theorem 2.1, then with X replaced by X_ℓ one gets the proofs of all the other steps in the proof of Theorem 2.1. Therefore, $\operatorname{End}(X_\ell)$ is an integral domain. The above discussion is summarized in Corollary 2.3.

Corollary 2.3. *Let X be an extension given as in* (21) *of Proposition* 1.3*. Then $X_\ell = (\cap_{j=1}^{r} \operatorname{Ker} \ell_j, V_2)$ is a submodule of both X and V. Moreover,* $\operatorname{End}(X_\ell)$ *is an integral domain whenever X satisfies the hypotheses of Theorem 2.1.*

Remark 2.4. The endomorphisms of X in Theorem 2.1 and X_l in Corollary 2.3 are in fact multiplications $(\alpha, \alpha), \alpha \in K$. We forego the details in favor of giving examples of extensions that satisfy the hypotheses of Theorem 2.1.

For r a positive integer let X^r denote the set of r-tuples of elements of X. In particular, $K(\zeta)^r$ is the set of r-tuples of rational functions. Since K is infinite, $\operatorname{Card} K = \operatorname{Card} K(\zeta) = \operatorname{Card} K(\zeta)^r$. Let H be any subset of K with $\operatorname{Card} H = \operatorname{Card} K$. We write H as a disjoint union of subsets indexed by $K(\zeta)^r$, where each subset has cardinality $\operatorname{Card} H$:

$$H = \dot{\bigcup_{f \in K(\zeta)^r}} H_f, \qquad \operatorname{card} H_f = \operatorname{card} H. \tag{42}$$

Let g_j, $j = 1, \dots, r$ be functions from H to K. These yield a function

$$\begin{aligned} &g : H^r \longrightarrow K^r, \\ &\text{where } g(\theta, \dots, \theta) = (g_1(\theta), \dots, g_r(\theta)) \\ &\text{and } g(x) = (0, \dots, 0) \text{ for all other elements} \\ &x \text{ in } H^r. \end{aligned} \tag{43}$$

Let $f = (f_1, \dots, f_r) \in K(\zeta)^r$ and let $\theta \in K$, θ not a pole of any f_j, $j = 1, \dots, r$. Set $f(\theta, \dots, \theta) = (f_1(\theta), \dots, f_r(\theta))$. By setting $f(x) = 0$ on all other elements of H^r we get a function, also denoted by f, from H^r to K^r. We say that f agrees with g in (43) at $\theta \in H$ if $f(\theta, \dots, \theta)$ is defined and $f_j(\theta) = g_j(\theta)$, $j = 1, \dots, r$.

Lemma 2.5. *Let H be a subset of K with* $\operatorname{card} H = \operatorname{card} K$*. Then there are functions $g_j : H \to K$, $j = 1, \dots, r$, such that $g : H^r \to K^r$*

given as in (43) *agrees with every element* f *in* $K(\zeta)^r$ *on an infinite subset of* H.

Proof. Express H as in (42). For $\theta \in H_f$, $f = (f_1, \dots, f_r)$, θ not a pole of any f_j, $j = 1, \dots, r$, set $g_j(\theta) = f_j(\theta)$. Set $g_j(\theta) = 0$ on all other elements θ in H_f. Now, use these g_j's to define $g : H^r \to K^r$ as in (43). Since H_f is infinite and the set of poles of f_j, $j = 1, \dots, r$, is finite, g has the required property. □

Let H be as in Lemma 2.5. To each element $\theta \in H$ we attach ∞ or a positive integer denoted in both cases by $h(\theta)$. And to $\{h(\theta) : \theta \in H\}$ we attach the submodule V_h, of $\mathcal{R}$ defined in (35) and (36). Let V_1 be the domain space of V_h. We define ℓ_j in $V_1^\star$ by letting $\ell_j(1/(\zeta - \theta)) = g_j(\theta)$, g_j as in Lemma 2.5. Set $\ell_j(x) = 0$ on all other elements x in the basis of V_1 given in (35). With $U = P_{m_1-1} \oplus \cdots \oplus P_{m_r-1}$ as in Proposition 1.3, we use the above ℓ_j's in (21) with $V = V_h$ to obtain an extension, X of U by V_h.

We claim that X satisfies the hypotheses of Theorem 2.1. To check that (u_2, v_2), $v_2 \neq 0$, is divisible in X by $b - \theta a$ for infinitely many θ in K we revert to the polynomial form of U. Denote elements of U_1 or U_2 by $p = (p_1, \dots, p_r)$. We recall that $a \circ_1 p_j = p_j$, $b \circ_1 p_j = \zeta p_j$.

Lemma 2.6. *Let* X *be an extension of* U *by* V_h. *An element* (p, v_2) *in* X_2 *is divisible by* $b - \theta a$ *if and only if*

$$\begin{aligned} &v_2/(\zeta - \theta) \text{ is in } V_1 \\ &\text{and} \quad \ell_j(v_2/(\zeta - \theta)) = p_j(\theta), \qquad j = 1, \dots, r. \end{aligned} \tag{44}$$

Proof. By (21), $(b - \theta a) \circ (q, v) = (p, v_2)$ if and only if $(((\zeta - \theta)q_1, \dots, (\zeta - \theta)q_r) + \sum_{j=1}^r \ell_j(v)e_j, (\zeta - \theta)v) = ((p_1, \dots, p_r), v_2)$; where $q = (q_1, \dots, q_r) \in U_1, v \in V_1$. Therefore,

$$(\zeta - \theta)q_j + \ell_j(v) = p_j$$

and

$$(\zeta - \theta)v = v_2.$$

Hence, $\ell_j(v) = p_j(\theta)$ and $v = v_2/(\zeta - \theta)$ as required. □

Example 2.7. Let $v_2 = 1/(\zeta - \eta)^n$. By (39), for $\theta \neq \eta$,

$$\frac{v_2}{\zeta - \theta} = \frac{(\eta - \theta)^{-1}}{(\zeta - \eta)^n} - \frac{(\eta - \theta)^{-2}}{(\zeta - \eta)^{n-1}} + \cdots \pm \frac{(\eta - \theta)^{-n}}{\zeta - \eta} \pm \frac{(\eta - \theta)^{-n}}{\zeta - \theta}.$$

With ℓ_j and p_j as in Lemma 2.6, $\ell_j(v_2/(\zeta - \theta)) = p_j(\theta)$ if and only if

$$\frac{1}{\eta - \theta}\ell_j\left(\frac{1}{\zeta - \eta}\right)^n - \frac{1}{(\eta - \theta)^2}\ell_j\left(\frac{1}{\zeta - \eta}\right)^{n-1} + \cdots$$
$$\pm \frac{1}{(\eta - \theta)^n}\ell_j\left(\frac{1}{\zeta - \eta}\right) \pm \frac{1}{(\eta - \theta)^n}\ell_j\left(\frac{1}{\zeta - \theta}\right) = p_j(\theta),$$

if and only if

(45)
$$\ell_j\left(\frac{1}{\zeta - \theta}\right) = \pm(\eta - \theta)^n\left\{\frac{1}{\eta - \theta}\ell_j\left(\frac{1}{\zeta - \eta}\right)^n - \frac{1}{(\eta - \theta)^2}\ell_j\left(\frac{1}{\zeta - \eta}\right)^{n-1} + \cdots \pm \frac{1}{(\eta - \theta)^n}\ell_j\left(\frac{1}{\zeta - \eta}\right) - p_j(\theta)\right\}.$$

With z as an indeterminate we get from (45) that $\ell_j(1/(\zeta - \theta)) = f_j(\theta)$, where $f_j(z)$ is the rational function

$$\pm(\eta - z)^n\left\{\frac{1}{\eta - z}\ell_j\left(\frac{1}{\zeta - \eta}\right)^n - \frac{1}{(\eta - z)^2}\ell_j\left(\frac{1}{\zeta - \eta}\right)^{n-1} + \cdots \pm \frac{1}{(\eta - z)^n}\ell_j\left(\frac{1}{\zeta - \eta}\right) - p_j(z)\right\}. \tag{46}$$

If $v_2 = \zeta^n$, we use (38) to obtain that the resulting rational function $f_j(z)$ is $(1/z^n)\{p_j(z) - \ell_j(\zeta^{n-1}) - z\ell_j(\zeta^{n-2}) - \cdots - z^{n-1}\ell_j(1)\}$ with $\ell_j(1/(\zeta - \theta)) = f_j(\theta)$.

For the rest of the verification that $(u_2, v_2), v_2 \neq 0$ is divisible in X by $b - \theta a$ for infinitely many θ in K, we shall restrict to θ *outside the finite set of zeros and poles of* v_2.

If $\theta \in H$, then, by (35) and (36), $(\zeta-\theta)^{-1} \in V_1 \cap V_2$. Since any v_2 in V_2 is a linear combination of $(\zeta-\eta)^{-n}$ for various η's in H and nonnegative integers n, it follows again from (35), (36) and (38), (39) that if $v_2 \in V_2$ and $\theta \in H$, then $v_2(\zeta-\theta)^{-1} \in V_1$. The computations in Example 2.7 show that $\ell_j(v_2(\zeta-\theta)^{-1}) = p_j(\theta)$ if and only if $\ell_j(\zeta-\theta)^{-1} = f_j(\theta)$ for a rational function $f_j(z)$ which is a linear combination of the functions in (46) and the line after (46) in Example 2.7. By choice, $\ell_j(\zeta-\theta)^{-1} = g_j(\theta)$, where the g_j's satisfy Lemma 2.5. Therefore, for infinitely many $\theta \in H$ we have $\ell_j(\zeta-\theta)^{-1} = g_j(\theta) = f_j(\theta)$. So from Example 2.7, for infinitely many $\theta \in H$, $\ell_j(v_2(\zeta-\theta)^{-1}) = p_j(\theta)$, $v_2(\zeta-\theta)^{-1} \in V_1$. By Lemma 2.6 the element (u_2, v_2), $v_2 \neq 0$, is divisible by $b-\theta a$ for infinitely many θ in $H \subset K$, as claimed. This completes the proof of the following proposition.

Proposition 2.8. *Let H be a subset of K with $\operatorname{card} H = \operatorname{card} K$. There is a submodule V_h of $\mathcal{R}$ and an extension X of $U = P_{m_1-1} \oplus \cdots \oplus P_{m_r-1}$ by V_h such that each element (u_2, v_2), $v_2 \neq 0$, is divisible by $b-\theta a$ for infinitely many θ in H.*

Remark 2.9. In order to include $\mathcal{R}$ among the modules V_H in Theorem 2.1 and the subsequent discussions we would take $H \subset K \cup \{\infty\}$. In case $\infty \in H$, the bases (35) and (36) would be supplemented respectively with $\{\zeta^t : 0 \leq t < h(\infty)\}$ and $\{\zeta^t : 0 < t < h(\infty)+1\}$, where $h(\infty)$ is ∞ or a positive integer.

Proposition 2.8 and the fact that $K = \cup_{k\in K} H_k$ (disjoint union) with $\operatorname{card} H_k = \operatorname{card} K$ for each $k \in K$ enables one to construct $\operatorname{card} K$ isomorphism classes of modules that satisfy the hypotheses of Theorem 2.1. In fact, if $k_1 \neq k_2$ the modules X_{k_1}, X_{k_2} corresponding to H_{k_1} and H_{k_2} in Proposition 2.8 have the property that the vector space of module homomorphisms from X_{k_1} to X_{k_2}, $\operatorname{Hom}(X_{k_1}, X_{k_2})$ is 0. This can be seen by observing that the elements of the form (u_2, f), $f \neq 0$, and $\psi(u_2, f)$, (φ, ψ) in $\operatorname{Hom}(X_{k_1}, X_{k_2})$ have nonzero height in disjoint sets. This forces $\psi(u_2, f)$ to be zero.

Remarks 2.10(a). The hypothesis in Theorem 2.1 that (u_2, v_2), $v_2 \neq 0$ is divisible by $b-\theta a$ for infinitely many θ in K implies the following property: Let $X' = (X_1', X_2')$ be the smallest submodule of X such that $(u_2, v_2) \in X_2'$ and X/X' is torsion-free. Then X' is infinite-dimensional.

There is a class of indecomposable extensions $X = (X_1, X_2)$ of $U = P_{m_1-1} \oplus \cdots \oplus P_{m_r-1}$ by V_h characterized by the opposite property: Let F be any finite subset of X_2 with $\operatorname{card} F \leq r$. Let $X' = (X_1', X_2')$ be the smallest submodule of X such that $F \subset X_2'$ and X/X' is torsion-free. Then X' is finite-dimensional. It follows that X is an extension of a finite-dimensional torsion-free module by V_h for some height function, h. Therefore, X is in the class of modules considered in Proposition 1.3. The module X is said to be *purely simple*. Given the easy characterization of torsion-free purely simple $K[\zeta]$-modules, see, e.g., [**7**, Section 85], torsion-free purely simple Kronecker modules are tantalizing. We refer to [**12-14**] for some of their properties. The divisibility hypothesis in Theorem 2.1 also implies that every finite-dimensional torsion-closed submodule of X is a submodule of U.

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